Synchronizing Finite Automata

Lecture X. Synchronizing Automata and Primitive Matrices

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1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

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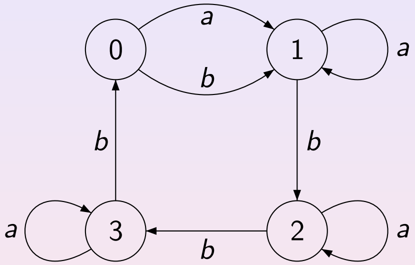

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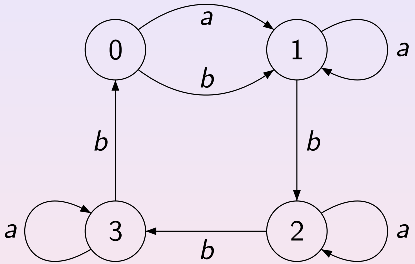

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3. Černý Conjecture

The Černý conjecture is the claim that every synchronizing automaton with n states possesses a reset word of length $(n-1)^2$. The validity of the conjecture is main open problem of the area.

Define the *Černý function* $C(n)$ as the maximum reset threshold for synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

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Why is the problem so surprisingly difficult?

One of the reasons: "slowly" synchronizing automata turn out to be extremely rare. Only one infinite series of n-state synchronizing automata with reset threshold $(n-1)^2$ is known (due to Černý, 1964), with a few (actually, 8) sporadic examples for $n \leq 6$.

As discussed in Lecture IV, finding n-automata with reset threshold close to n^2 requires an exhaustive search designed in a reasonable way. Specifying a 9-automaton with two input letters amounts to specifying a pair of function on a 9-element set. There are $9^{18} \approx 1.50 \times 10^{17}$ such pairs, and if one nanosecond is spent for calculating the reset threshold of each automaton, the exhaustive search takes around five years. Thus, one has to invoke some tricks to speed up the search and to use powerful parallel computing.

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Thus, the pattern is:

$(n-1)^2$ the first gap the "island" the second gap

The second gap first appears at 9 states and grows rather regularly with the number of states. The size of the island depends only on the parity of the number of states.

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The very same pattern appears in the distribution of exponents of non-negative matrices.

A non-negative matrix A is said to be primitive if some power A^k is positive. The minimum k with this property is called the exponent of A, denoted $\exp A$.

Helmut Wielandt proved in 1950 that for any primitive $n \times n$-matrix A, one has $\exp A \leq n^2 - 2n + 2 = (n-1)^2 + 1$, and this bound is tight. Possible exponents of $n \times n$-matrices were intensively studied in the 1960s, and it was discovered that two extreme values are each attained by a unique matrix, then there is a gap followed by an island followed by another gap. The sizes of the gaps and of the island perfectly match the sizes of the gaps and of the islands in possible reset thresholds of synchronizing automata with n states — basically one has the same picture shifted by 1. Clearly, this cannot be a mere coincidence.

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A directed graph (digraph) is a pair $D = \langle V, E \rangle$.

- V set of vertices
- $E \subseteq V \times V$ set of edges

This definition allows loops but excludes multiple edges.

The matrix of a digraph $D = \langle V, E \rangle$ is just the incidence matrix of the edge relation, that is, a $V \times V$-matrix whose entry in the row v and the column v' is 1 if $(v, v') \in E$ and 0 otherwise.

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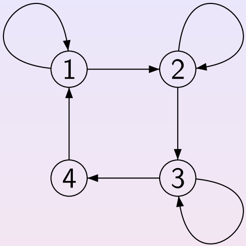
For instance, the matrix of the digraph



(with respect to the chosen numbering of its vertices) is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Conversely, given an $n \times n$-matrix $P = (p_{ij})$ with non-negative real entries, we assign to it a digraph $D(P)$ on the set $\{1, 2, \ldots, n\}$ as follows: (i, j) is an edge of $D(P)$ if and only if $p_{ij} > 0$.

This 'two-way' correspondence allows us to reformulate in terms of digraphs several important notions and results which originated in the classical Perron–Frobenius theory of non-negative matrices.

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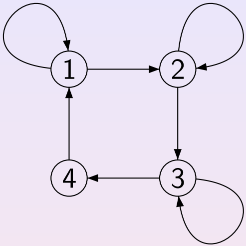

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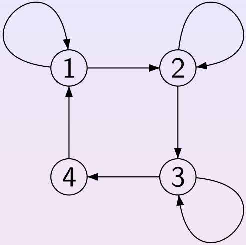

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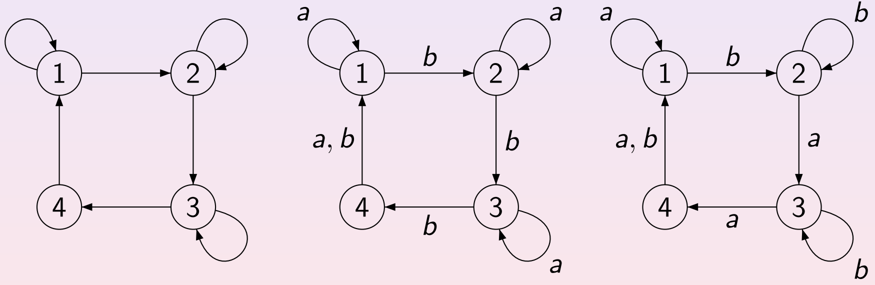

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A digraph D is primitive if D is strongly connected and the greatest common divisor of the lengths of all cycles in D is equal to 1.

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Underlying digraphs of strongly connected synchronizing automata are primitive.

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A digraph D is primitive iff there exists t such that for each pair of vertices there exists a path between them of length exactly t (proved by Frobenius in the language of matrices). (This is equivalent to saying that the t-th power of the matrix of D is positive.) The least t with this property is called the exponent of the digraph D and is denoted by $\gamma(D)$.

1950, Wielandt: The exponent of every primitive digraph on n vertices is not greater than $(n-1)^2+1$ and this bound is tight.

1964, Dulmage–Mendelsohn: There is exactly one primitive digraph on n vertices with exponent $(n-1)^2+1$ and exactly one primitive digraph on n vertices with exponent $(n-1)^2$.
If $n > 4$ is even, then there is no primitive digraph D on n vertices such that $n^2-4n+6 < \gamma(D) < (n-1)^2$.
If $n > 3$ is odd, then there is no primitive digraph D on n vertices such that $n^2-3n+4 < \gamma(D) < (n-1)^2$, or $n^2-4n+6 < \gamma(D) < n^2-3n+2$.

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A digraph D is primitive iff there exists t such that for each pair of vertices there exists a path between them of length exactly t (proved by Frobenius in the language of matrices). (This is equivalent to saying that the t-th power of the matrix of D is positive.) The least t with this property is called the exponent of the digraph D and is denoted by $\gamma(D)$.

1950, Wielandt: The exponent of every primitive digraph on n vertices is not greater than $(n-1)^2+1$ and this bound is tight.

1964, Dulmage–Mendelsohn: There is exactly one primitive digraph on n vertices with exponent $(n-1)^2+1$ and exactly one primitive digraph on n vertices with exponent $(n-1)^2$.

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13. Exponents vs Reset Lengths

Exponents of primitive digraphs with 9 vertices vs reset thresholds of 2-letter strongly connected synchronizing automata with 9 states

N	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51
# of primitive digraphs with exponent N	1	1	0	0	0	0	0	1	1	2	0	0	0	0	3
# of 2-letter synchronizing automata with reset threshold N	0	1	0	0	0	0	0	1	2	3	0	0	0	4	4

The Wielandt automaton $\mathscr{W}_n$ is a (unique) coloring of the Wielandt digraph W_n with $\gamma(W_n) = (n-1)^2 + 1$.

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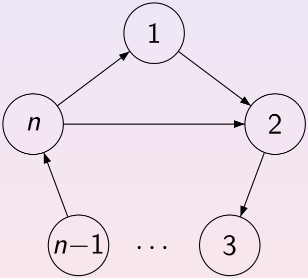

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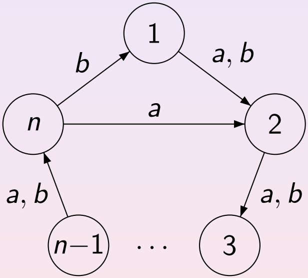

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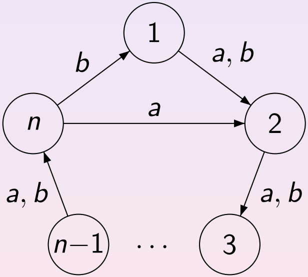

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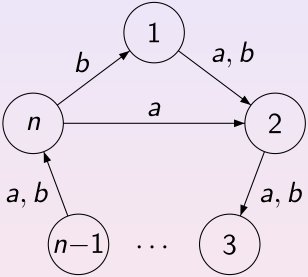

We shall show that the reset threshold of $\mathscr{W}_n$ is $n^2 - 3n + 3$.

In a similar way, every digraph with large exponent "generates" slowly synchronizing automata.

Observation

Let a strongly connected synchronizing automaton with n states and reset threshold t be a coloring of a digraph D. Then

$$\gamma(D) \leq t + n - 1.$$

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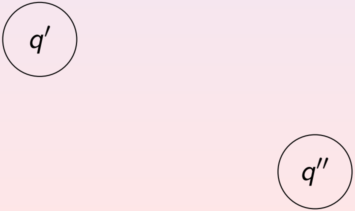

15. Colorings of Digraphs with Large Exponents

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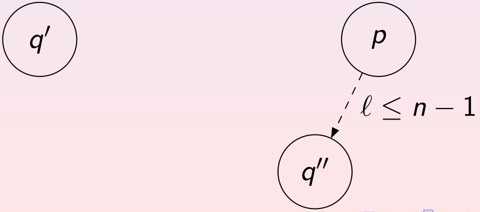

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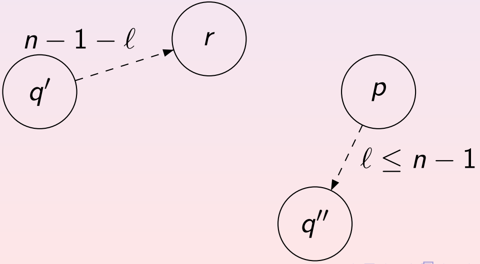

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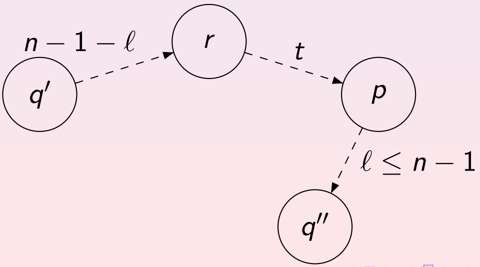

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Let a strongly connected synchronizing automaton with n states and reset threshold t be a coloring of a digraph D. Then

$$\gamma(D) \leq t + n - 1.$$

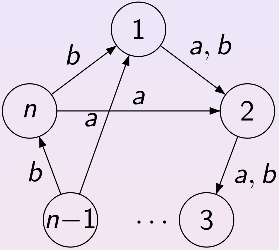
For instance, the reset threshold t of the Wielandt automaton $\mathscr{W}_n$ must satisfy

$$t \geq \gamma(W_n) - n + 1 = (n-1)^2 + 1 - n + 1 = n^2 - 3n + 3,$$

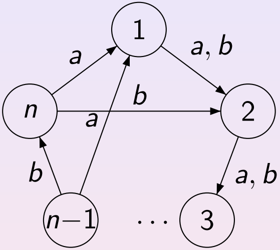
and it is easy to find a reset word of length $n^2 - 3n + 3$.

16. Further Automata

Colorings of the unique digraph with exponent $(n-1)^2$



$n^2 - 3n + 4$



$n^2 - 3n + 2$

Left: The slowest automaton after $\mathscr{C}_n$.
Right: None of the letters act as a cyclic permutation.

However, not every slowly synchronizing automaton we discovered can be obtained in such a way.

There are slowly synchronizing automata that cannot be obtained as colorings of a digraph with large exponent. For instance, the Černý automaton $\mathscr{C}_n$ has reset threshold $(n-1)^2$ while its underlying digraph has exponent $n-1$.

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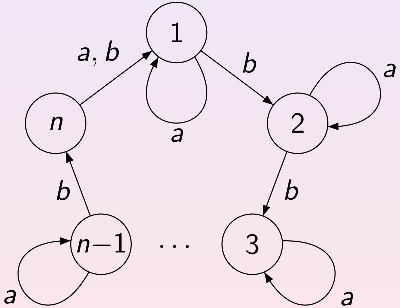

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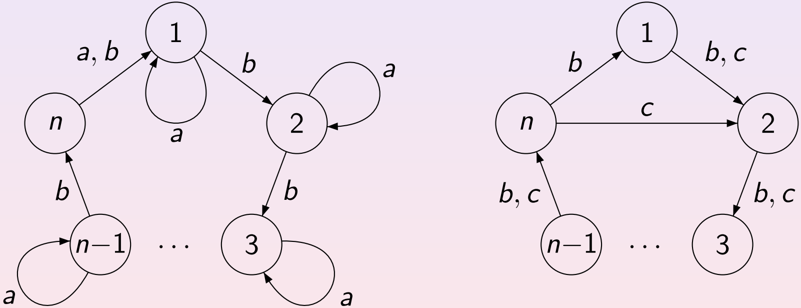

However, $\mathscr{C}_n$ becomes $\mathscr{W}_n$ under the action of b and $c = ab$.

Let w be a shortest reset word for $\mathscr{C}_n$. It must end with a and every other occurrence of a in w is followed by an occurrence of b. Thus, $w = w'a$ where w' can be rewritten into a word v over the alphabet $\{b, c\}$. Since w' and v act in the same way, the word vc is a reset word for $\mathscr{W}_n$. Hence $|v| \geq n^2 - 3n + 2$.

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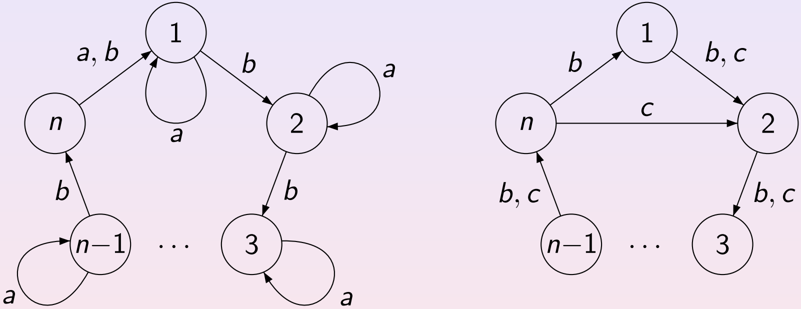

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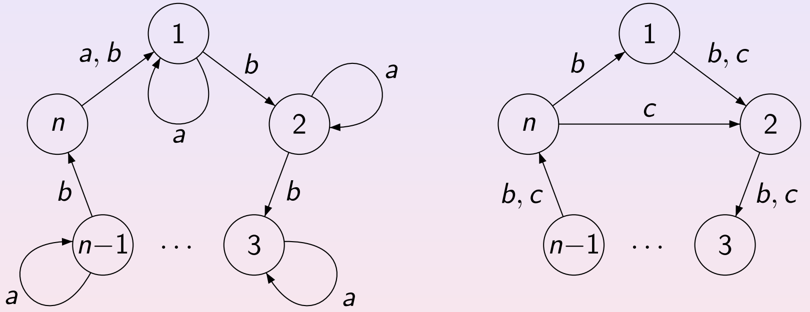

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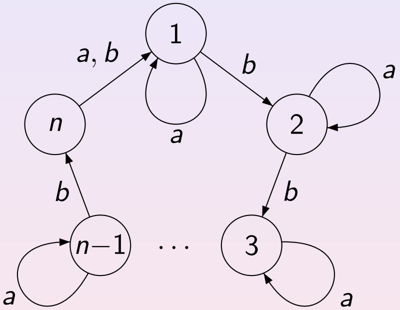

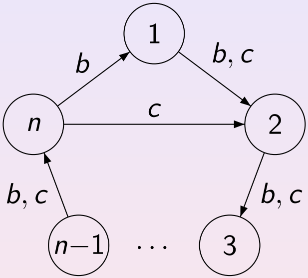

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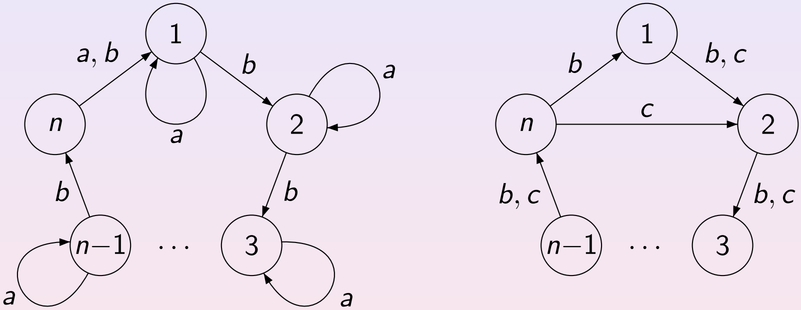

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Further, v contains at least $n - 2$ occurrences of c. Since each occurrence of c in v corresponds to an occurrence of ab in w', we conclude that $|w'| \geq n^2 - 3n + 2 + n - 2 = n^2 - 2n$.

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Thus, it is the Wielandt digraph that stays behind the Černý automaton!

In a similar manner it is easy to recover every known slowly synchronizing automaton from a suitable digraph with large exponent.

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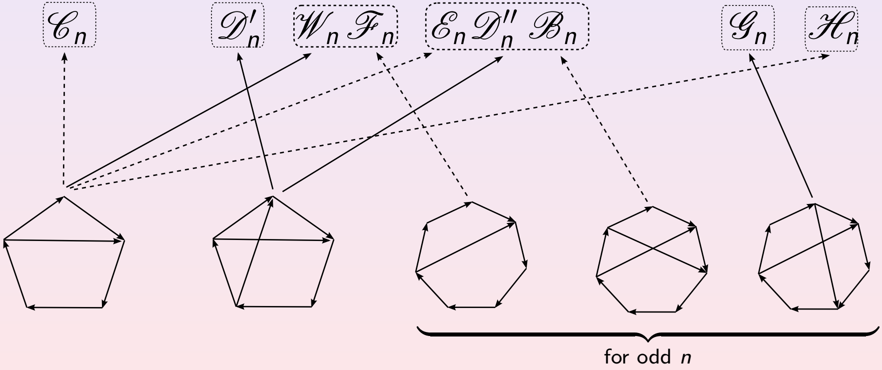
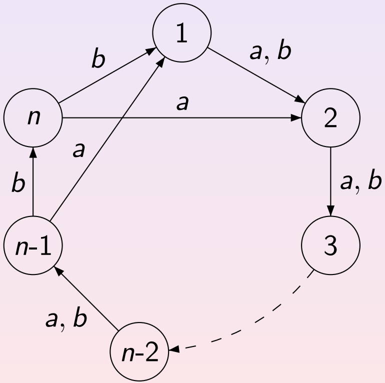


17. Further Examples

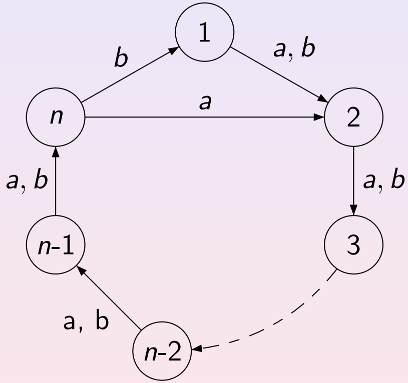
All other automata from the ‘island’ can be explained via the same trick.

1 0 0 0 0 0 1 2 3 0 0 0 4 4



Reset threshold $n^2 - 3n + 4$

1 0 0 0 0 0 1 2 3 0 0 0 4 4



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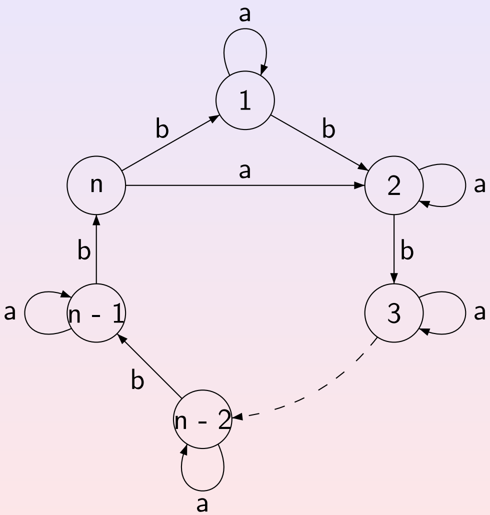
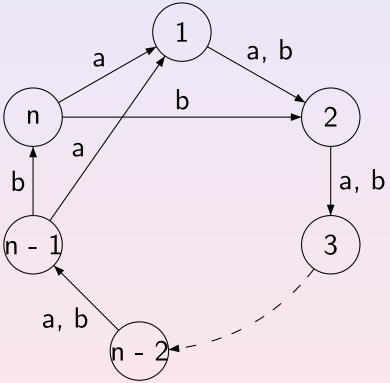
1 0 0 0 0 0 1 2 3 0 0 0 4 4



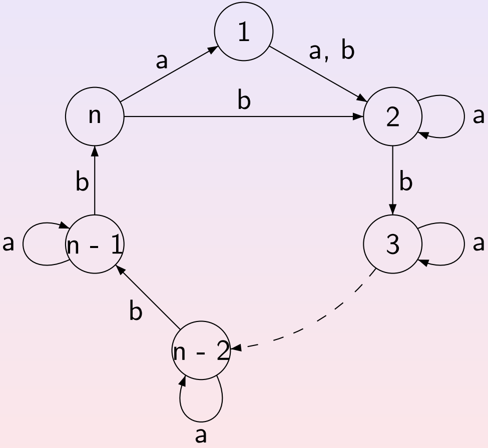
Reset threshold $n^2 - 3n + 3$, n odd

1 0 0 0 0 0 1 2 3 0 0 0 4 4



Reset threshold $n^2 - 3n + 2$

1 0 0 0 0 0 1 2 3 0 0 0 4 4



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22. Further Examples

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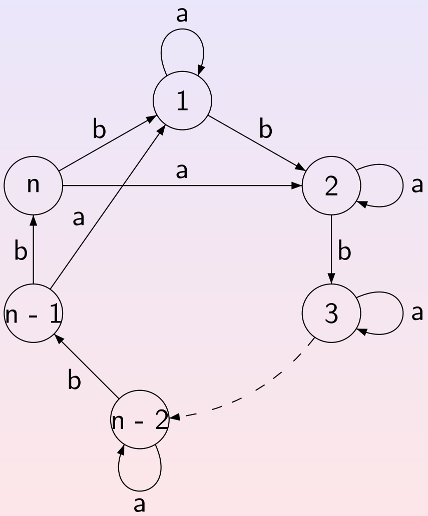

Reset threshold $n^2 - 3n + 2$, n odd

23. New Conjectures

(a) (The Černý conjecture) The reset threshold of every synchronizing n-automaton does not exceed $(n-1)^2$.

(b) If $n > 6$, then there exists exactly one n-automaton with reset threshold $(n-1)^2$, namely, $\mathscr{C}_n$.

(c) If $n > 6$, then there exists no n-automaton whose reset threshold is greater than $n^2 - 3n + 4$ but less than $(n-1)^2$.

(d) If $n > 7$ is odd, then there exists exactly one n-automaton with reset threshold $n^2 - 3n + 4$, exactly two n-automata with reset threshold $n^2 - 3n + 3$, and exactly three n-automata with reset threshold $n^2 - 3n + 2$. There exists no n-automaton whose reset threshold is between $n^2 - 4n + 8$ and $n^2 - 3n + 1$.

(e) If $n > 8$ is even, then there exists exactly one n-automaton with reset threshold $n^2 - 3n + 4$, exactly one n-automaton with reset threshold $n^2 - 3n + 3$, and exactly two n-automata with reset threshold $n^2 - 3n + 2$. There exists no n-automaton whose reset threshold is between $n^2 - 4n + 7$ and $n^2 - 3n + 1$.

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