Synchronizing Finite Automata

Lecture XI. Synchronizing Automata and Markov Chains

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Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

$|Q \,.\, w| = 1$. Here $Q \,.\, v = \{\delta(q, v) : q \in Q\}$.

Any w with this property is a reset word for $\mathscr{A}$.

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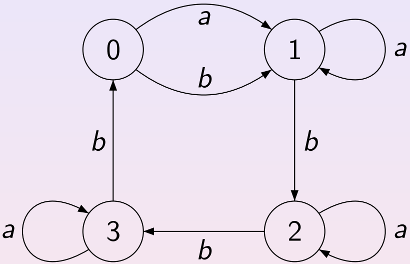

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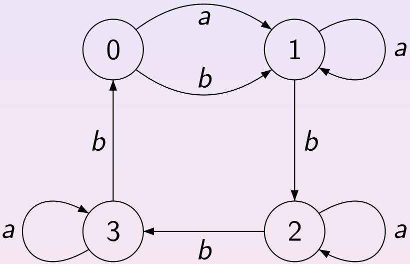

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Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

The Černý conjecture thus claims that in fact $C(n) = (n-1)^2$.

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How to get upper bounds for reset threshold?

For $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, a subset $P \subset Q$ is extensible if $P \supseteq R\,.\,w$ for some $w \in \Sigma^*$ of length at most $n = |Q|$ and some $R \subseteq Q$ with $|R| > |P|$. It was conjectured for some time that in strongly connected synchronizing automata every proper non-singleton subset is extensible. Observe that this would imply the Černý conjecture.

Indeed, some letter a should sent two states q, q' to the same state p. Let $P_0 = \{q, q'\}$ and, for $i > 0$, let P_i be such that $|P_i| > |P_{i-1}|$ and $P_{i-1} \supseteq P_i\,.\,w_i$ for some word w_i of length $\leq n$. Then in at most $n-2$ steps the sequence $P_0, P_1, P_2, \ldots$ reaches Q and

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that is, $w_{n-2}w_{n-3}\cdots w_1 a$ is a reset word. The length of this reset word is at most $n(n-2)+1 = (n-1)^2$.

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5. Example

For an illustration, consider the subset automaton of the Černý automaton $\mathscr{C}_4$.

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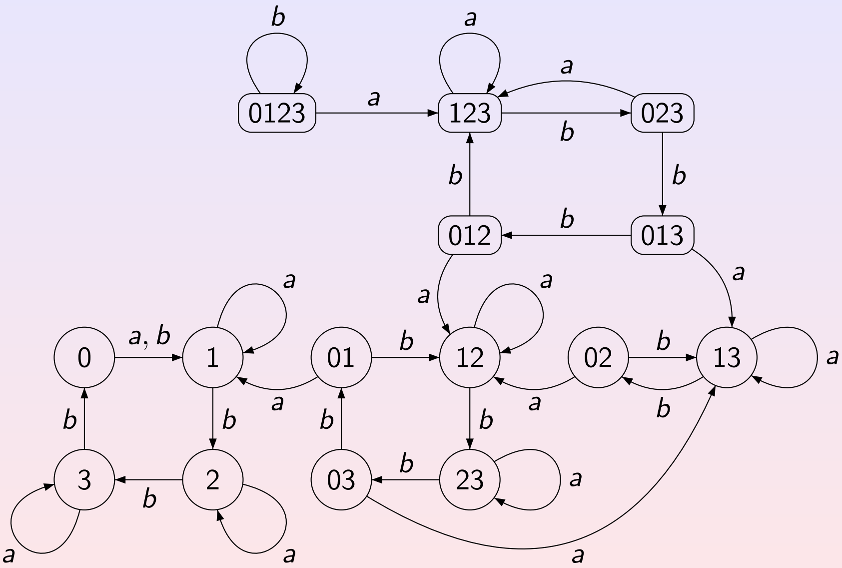

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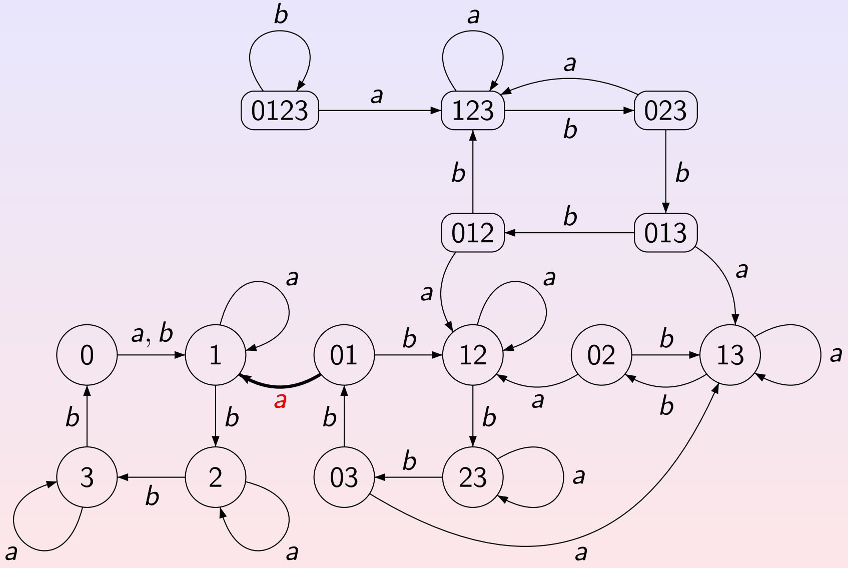

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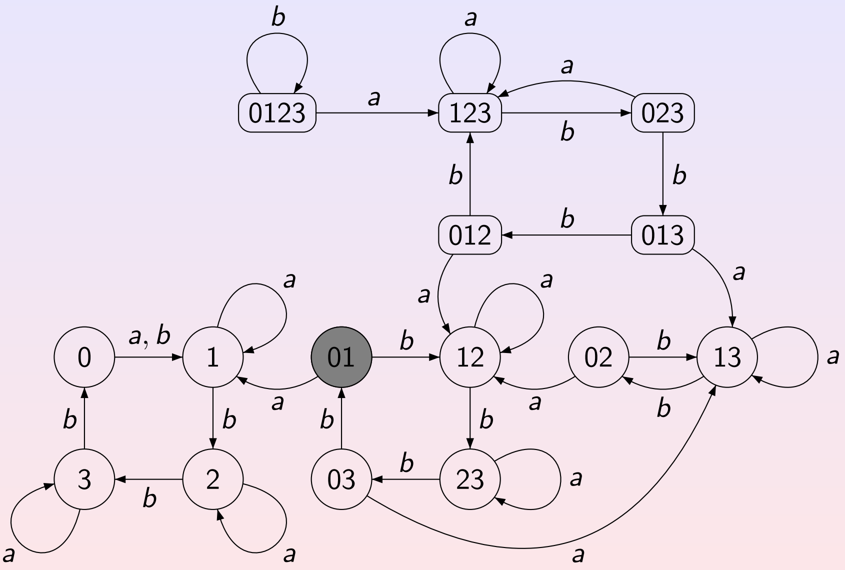

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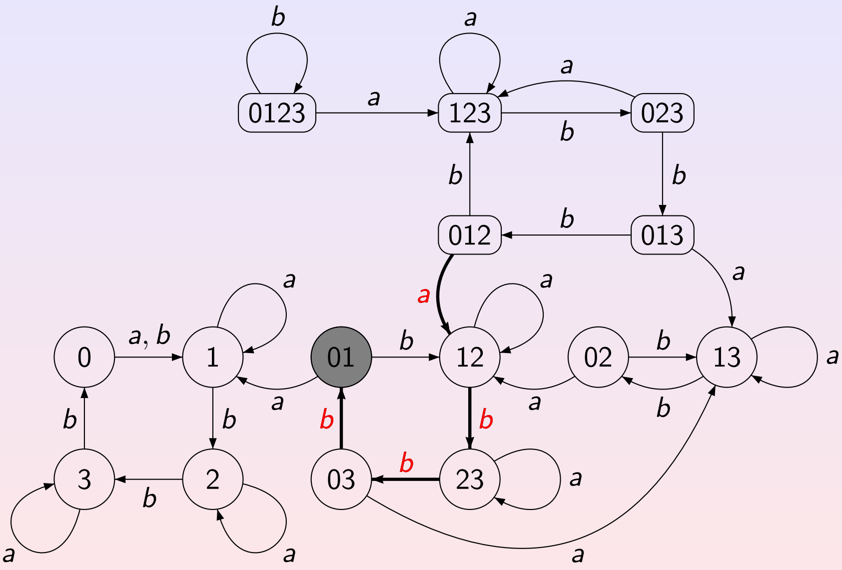

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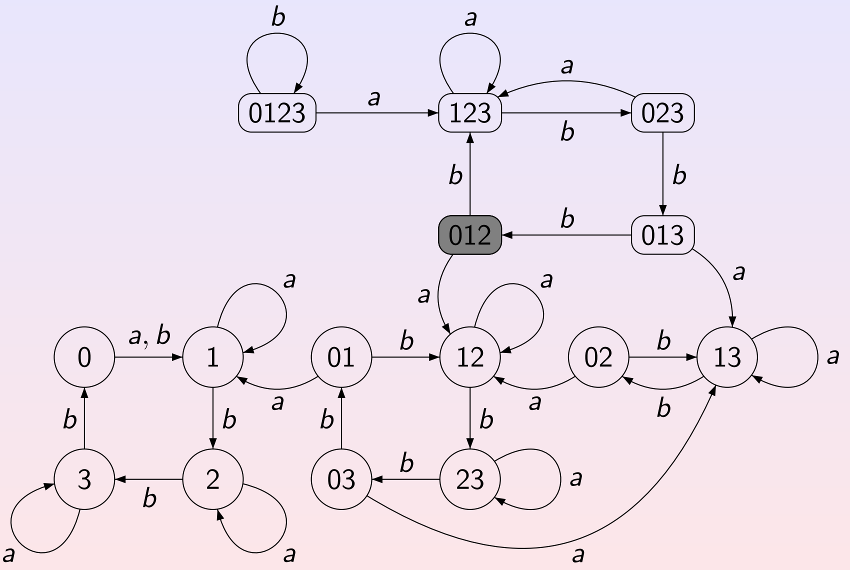

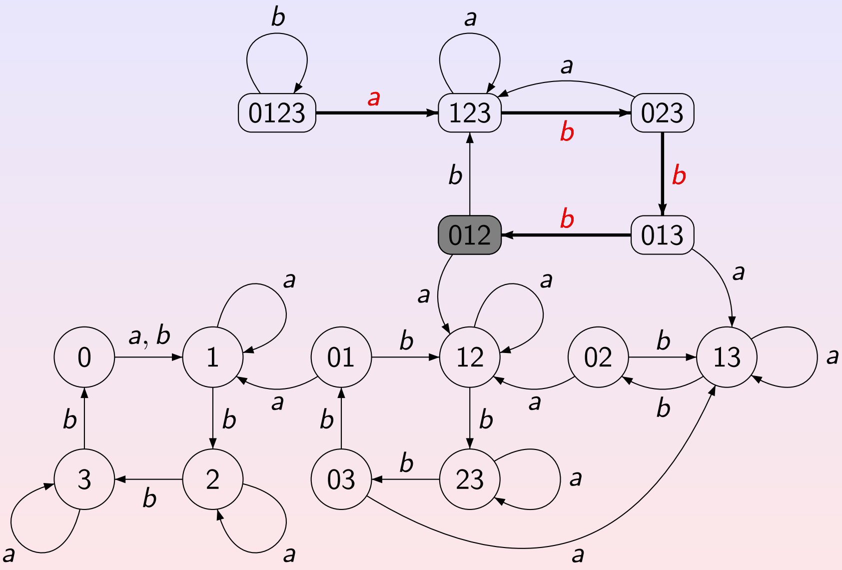
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6. Applications of Extensibility

Several important results confirming the Černý conjecture for various partial cases have been proved by verifying the extensibility conjecture for the corresponding automata. This includes:

- Louis Dubuc's result for automata in which a letter acts on the state set Q as a cyclic permutation of order $|Q|$ (Sur le automates circulaires et la conjecture de Černý, RAIRO Inform. Theor. Appl., 32 (1998) 21–34 [in French]).
- Jarkko Kari's result for automata with Eulerian digraphs (Synchronizing finite automata on Eulerian digraphs, Theoret. Comput. Sci., 295 (2003) 223–232).
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The first (implicit) example was a 6-state automaton found by Jarkko Kari in 2001, and Mikhail Berlinkov has constructed for every $\alpha < 2$ an infinite series of synchronizing automata $\mathscr{B}_\alpha = \langle Q, \Sigma, \delta \rangle$ such that there is a proper non-singleton subset $P \subset Q$ that cannot be extended by any word of length $< \alpha|Q|$ (On a conjecture by Carpi and D'Alessandro, Int. J. Found. Comput. Sci. 22 (2011) 1565–1576).

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8. Linearization

We associate a natural linear structure with each automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$. Assume that $Q = \{1, 2, \dots, n\}$ and assign to each subset $K \subseteq Q$ its characteristic vector $[K] \in \mathbb{R}^n$ (the space of n-dimensional column vectors): the i-th entry of $[K]$ is 1 if $i \in K$, otherwise the entry is 0.

For each word $w \in \Sigma^*$, its action on Q gives rise to a linear transformation of $\mathbb{R}^n$; we denote by $[w]$ the matrix of this transformation in the standard basis $[1], \dots, [n]$ of $\mathbb{R}^n$.

Clearly, the matrix $[w]$ has exactly one non-zero entry in each column and this entry is equal to 1.

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$$([a]^T[q], p_1) < ([w_1 a]^T[q], p_1)$$
$$\dots$$
$$([w_{d-1} \cdots w_2 w_1 a]^T[q], p_d) < ([w_d \cdots w_2 w_1 a]^T[q], p_d) = 1.$$

Adaptive vector (p_i); unpredictable increment, whence no upper bound on d; however, $|w_i| \leq n$.

Assume that $\Sigma = \{a_1, a_2, \dots, a_k\}$. Each positive stochastic vector $\pi \in \mathbb{R}_+^k$ defines a probability distribution on Σ. Consider a process in which an agent randomly walks on the underlying graph of $\mathscr{A}$, choosing for each move an edge labeled a_i with probability $p(a_i)$. This is a Markov chain with the transition matrix

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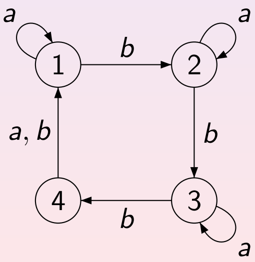

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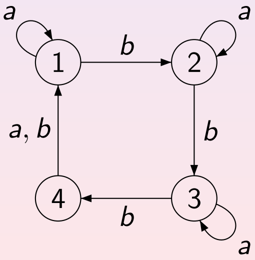

$$[a] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [b] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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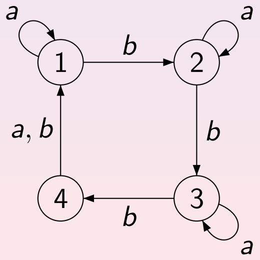

If $p(a) = \frac{1}{3}$, $p(b) = \frac{2}{3}$, then $S = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 \end{pmatrix}$

13. Stationary Distributions

We may assume that $\mathscr{A}$ is strongly connected.

If $\mathscr{A}$ is synchronizing, then the matrix $S = S(\mathscr{A}, \pi)$ is primitive since its digraph is such.

By basic properties of Markov chains, there exists the stationary distribution $\alpha \in \mathbb{R}^n_+$ of this Markov chain, that is, a unique positive stochastic vector satisfying $S\alpha = \alpha$.

Indeed, since S is a column stochastic we have $S^T\mathbf{1}_n = \mathbf{1}_n$. Thus, 1 is an eigenvalue of S^T, but then 1 is also an eigenvalue of S. Then S has an eigenvector α belonging to 1. Since S is primitive, the Perron–Frobenius theorem applies, telling us that this eigenvector α is positive and unique up to a positive scalar, so that one can make it be stochastic.

It is known that for every initial distribution $\beta \in \mathbb{R}^n_+$, the sequence $S^n\beta$ tends to α as $n \to +\infty$ (ergodic theorem).

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Theorem (Berlinkov, 2012)

Let $\mathscr{A}$ be a strongly connected synchronizing automaton with n states and k letters, $\pi \in \mathbb{R}^k_+$ a positive stochastic vector, and α the stationary distribution of the Markov chain with the transition matrix $S(\mathscr{A}, \pi) = \sum_{i=1}^{k} p(a_i)[a_i]$. Then there exist a state q, a letter a, and a sequence of words $w_1, w_2, \dots, w_d$ of length at most $n - 1$ such that

$$([q], \alpha) < ([a]^T[q], \alpha) < ([w_1 a]^T[q], \alpha) < \dots$$
$$\dots < ([w_d \cdots w_2 w_1 a]^T[q], \alpha) = 1.$$

15. Another Comparison

Approach	'Classic'	Jungers	Berlinkov

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Vector	Constant ($\mathbf{1}_n$)	Depends on both $\mathscr{A}$ and the subset to be extended	Depends on $\mathscr{A}$ and probability distribution but not on the subset

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$\lvert w_i \rvert$	Hard to predict	At most n	At most $n-1$

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An immediate application: a new proof of the Černý conjecture for automata with Eulerian digraphs. In this case the matrix $S(\mathscr{A}, \pi)$ is doubly stochastic whence the uniform vector $\mathbf{1}_n$ is its stationary distribution and $d \leq n - 2$.

Steinberg's result about pseudo-Eulerian automata follows as well; here an automaton is $\mathscr{A} = (Q, \Sigma)$ is pseudo-Eulerian if there is a probability distribution π on Σ such that the matrix $S(\mathscr{A}, \pi)$ is doubly stochastic.

Several new results where a quadratic bound on the reset threshold can be achieved.

Still one extra degree of freedom: the choice of probability distribution.

An 'obvious' choice: the distribution under which the random walk on $\mathscr{A}$ synchronizes in the shortest expected time.

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17. Proof

Recall the setting:

• $\mathscr{A} = (Q, \Sigma)$, a synchronizing automaton with $|Q| = n$ and $\Sigma = \{a_1, a_2, \dots, a_k\}$.

• $\pi \in \mathbb{R}^k_+$, a stochastic vector (a probability distribution on Σ).

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Take a vector $x \in \mathbb{R}^n$ with $(x, \alpha) = 0$ and let $v \in \Sigma^*$ be a word of minimum length such that $([v]^T x, \alpha) > 0$.

1. $\sum_{|u|=r} p(u)([u]^T x, \alpha) = 0$ for every $r > 0$.

2. If $u \in \Sigma^*$ is a word with $|u| < |v|$, then $([u]^T x, \alpha) = 0$.

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18. Claims 1 and 2

Since $S\alpha = \alpha$, we have $S^r\alpha = \alpha$ for every positive integer r.

Since $S^r = \sum_{|u|=r} p(u)[u]$, we have $\sum_{|u|=r} p(u)[u]\alpha = \alpha$.

Multiplying through by x, we obtain
$0 = (\alpha, x) = \left(\sum_{|u|=r} p(u)[u]\alpha, x\right) = \sum_{|u|=r} p(u)([u]\alpha, x) = \sum_{|u|=r} p(u)([u]^T x, \alpha)$. This proves claim 1.

By the choice of v, if $|u| < |v|$, then $([u]^T x, \alpha) \leq 0$.

But if $|u| = r$, then $\sum_{|u|=r} p(u)([u]^T x, \alpha) = 0$ by claim 1.

If non-positive summands sum up to 0, each of them should be 0.

This proves claim 2.

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But if $|u| = r$, then $\sum_{|u|=r} p(u)([u]^T x, \alpha) = 0$ by claim 1.
If non-positive summands sum up to 0, each of them should be 0.
This proves claim 2.

Since $S\alpha = \alpha$, we have $S^r\alpha = \alpha$ for every positive integer r.
Since $S^r = \sum_{|u|=r} p(u)[u]$, we have $\sum_{|u|=r} p(u)[u]\alpha = \alpha$.
Multiplying through by x, we obtain
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18. Claims 1 and 2

Since $S\alpha = \alpha$, we have $S^r\alpha = \alpha$ for every positive integer r.
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19. Claim 3

Suppose that $|v| \geq \dim\langle [u]\alpha : |u| \leq n-1\rangle := D$.

Claim 2 implies that $([u]^T x, \alpha) = (x, [u]\alpha) = 0$ for every word u such that $|u| < D$.

For $i = 1, 2, \ldots,$ let U_i be the subspace spanned by all vectors $[u]\alpha$ with $|u| \leq i-1$.

The chain

$$\langle \alpha \rangle = U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = \langle [u]\alpha : |u| \leq n-1\rangle$$

of non-zero subspaces in the n-dimensional space $\mathbb{R}^n$ must become constant at some $j \leq \dim(U_n) = D \leq |v|$.

The subspace U_j is invariant with respect to all the letters in Σ, whence, in particular, $[v]\alpha$ belongs to U_j. Since $(x, [u]\alpha) = 0$ for every u with $|u| \leq \dim(U_n) = \dim(U_j)$, we conclude that $(x, g) = 0$ for each $g \in U_j$. But $[v]\alpha$ belongs to U_j, whence $(x, [v]\alpha) = ([v]^T x, \alpha) = 0$, and this contradicts the condition $([v]^T x, \alpha) > 0$. This proves claim 3 and the theorem.

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