Synchronizing Finite Automata

Lecture II: Algorithmic Issues

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1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

$|Q\,.\,w| = 1$. Here $Q\,.\,v = \{\delta(q, v) \mid q \in Q\}$.

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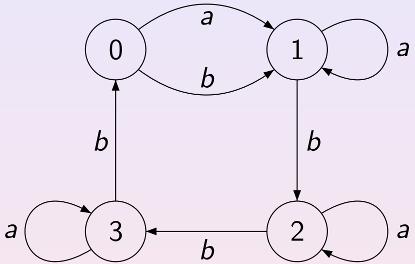

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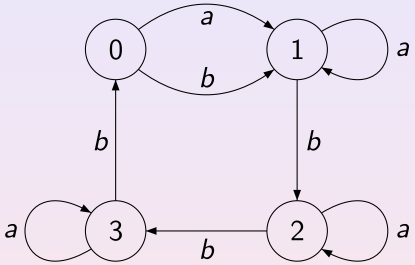

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Not every DFA is synchronizing. Therefore, the very first question is the following one: *given an automaton, how to determine whether or not it is synchronizing?* This question is easy, and a straightforward solution comes from the classic powerset construction by Rabin and Scott.

The *power automaton* $\mathcal{P}(\mathscr{A})$ of a given DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$:
- states are the non-empty subsets of Q,
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A $w \in \Sigma^*$ is a reset word for the DFA $\mathscr{A}$ iff w labels a path in $\mathcal{P}(\mathscr{A})$ starting at Q and ending at a singleton.

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Exercise: Write down a proof of this claim!

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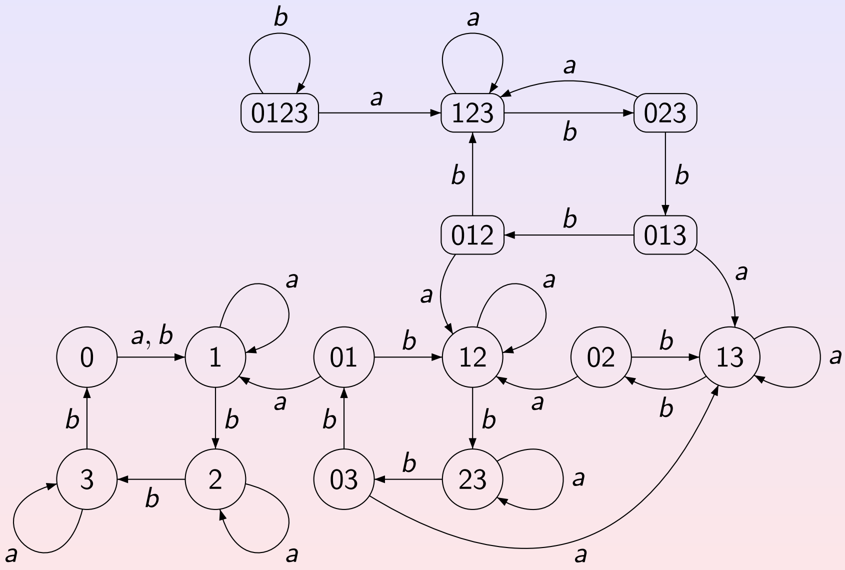

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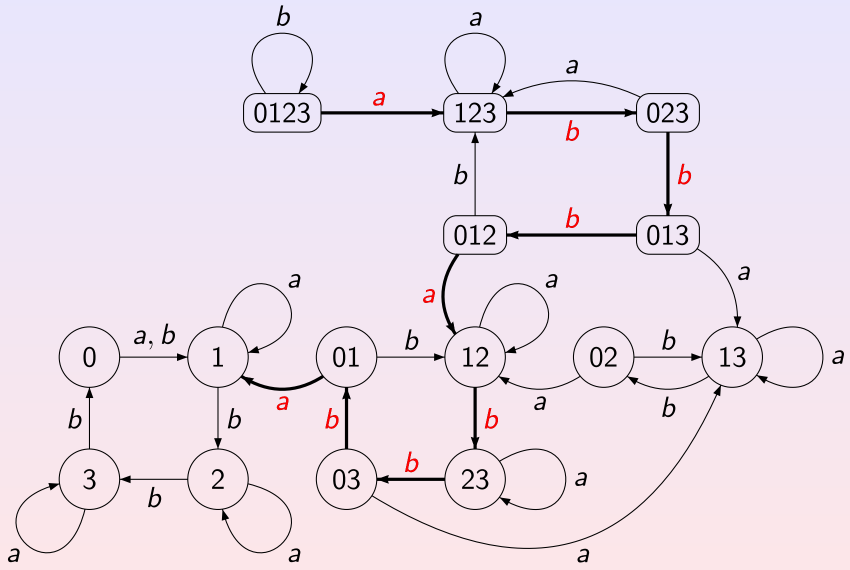

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Thus, the question of whether or not a given DFA $\mathscr{A}$ is synchronizing reduces to the following reachability question in the underlying digraph of the power automaton $\mathcal{P}(\mathscr{A})$: is there a path from Q to a singleton? The latter question can be easily answered by BFS. This algorithm is however exponential w.r.t. the size of $\mathscr{A}$.

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Proposition. *A DFA* $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ *is synchronizing iff for every* $q, q' \in Q$ *there exists a word* $w \in \Sigma^*$ *such that* $\delta(q, w) = \delta(q', w)$.

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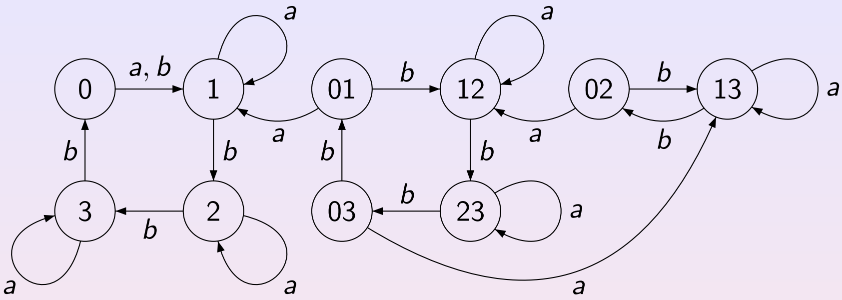

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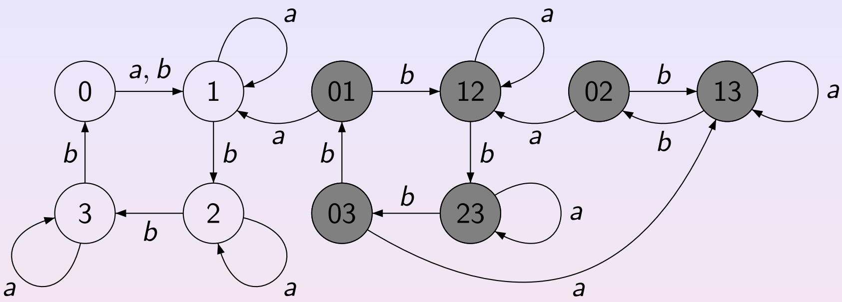
Observe that the reset word constructed this way is of length 10 while we know a reset word of length 9.



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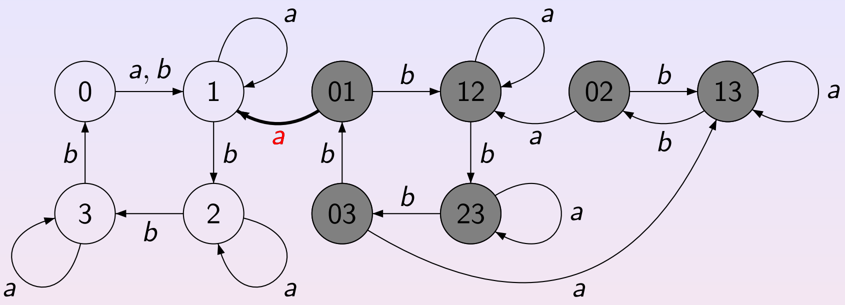

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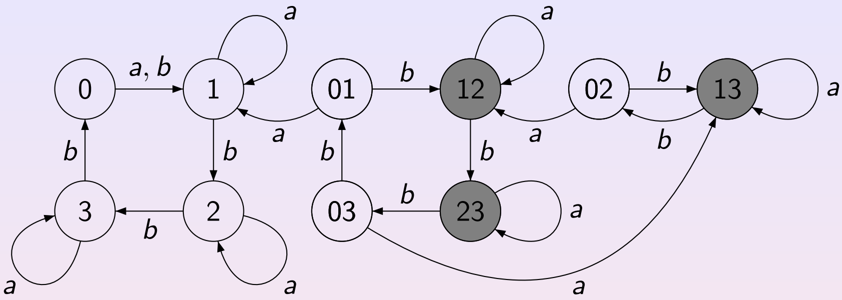

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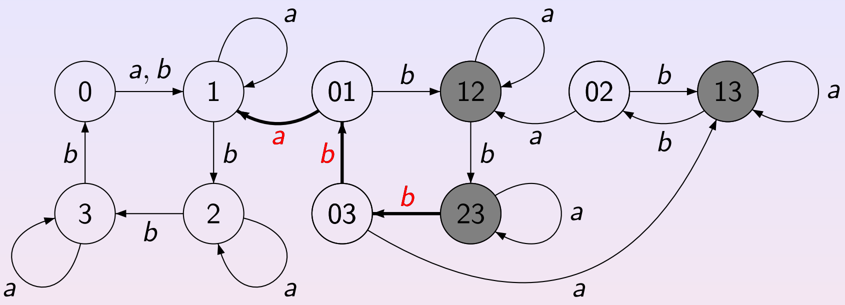

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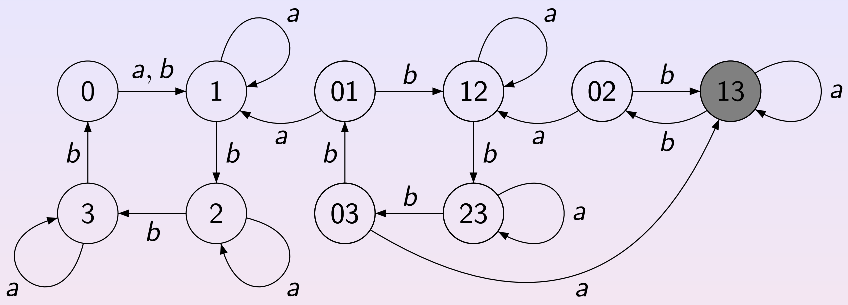

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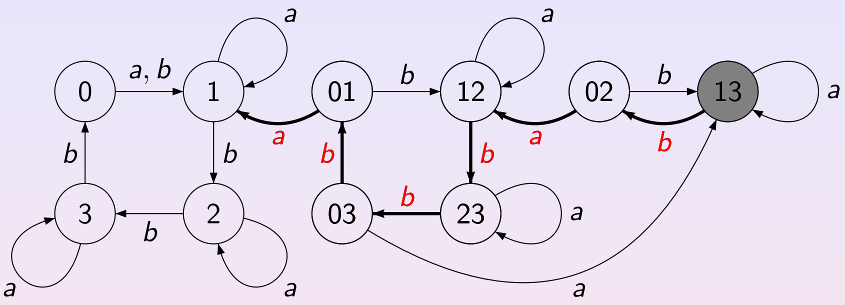

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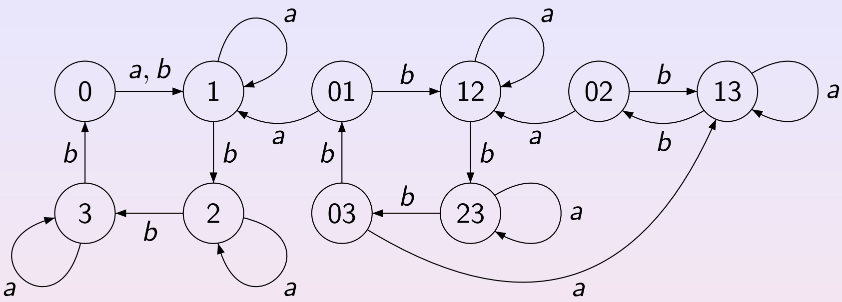

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Thus, recognizing synchronizability reduces to a reachability problem in the automaton whose states are the 2-subsets and the 1-subsets of Q. The latter can be solved by BFS in $O(n^2 \cdot |\Sigma|)$ time where $n = |Q|$.

Can one do better? It is an open problem.

Recently, Mikhail Berlinkov has developed a (non-trivial) algorithm that checks whether or not an automaton with n states is synchronizing and spends time $O(n)$ on average. The worst case complexity of Berlinkov's algorithm is still quadratic.

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8. Results-II

In fact, the basic algorithm not only recognizes synchronizability but also returns a reset word provided that such exists.

If one also wants to produce a reset word, one need $O(n^3 + n^2 \cdot |\Sigma|)$ time. Why? One needs time to write down the word!

Clearly, the resulting reset word has length $O(n^3)$: the algorithm makes at most $n-1$ steps and the length of the segment added in the step when k states are still to be compressed ($n \geq k \geq 2$) is at most $1 + \#$ of blank 2-subsets, i.e., $1 + \binom{n}{2} - \binom{k}{2}$. This gives the upper bound close to $\frac{n^3 - n}{3}$.

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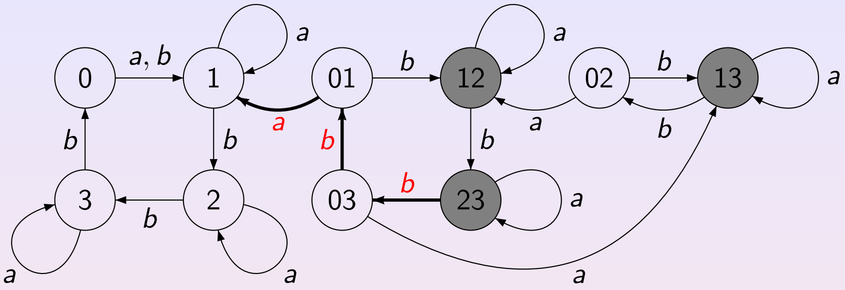
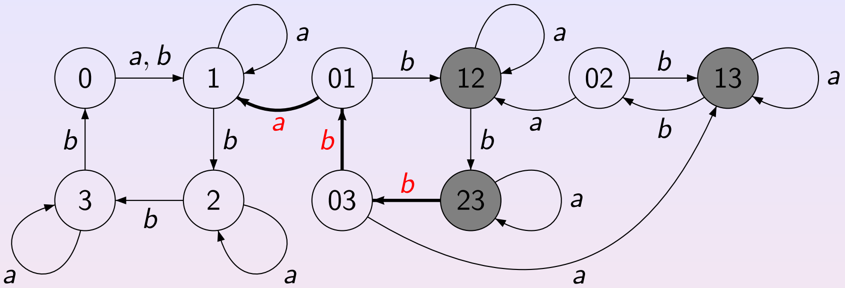
9. A Resource for Improvement



We see that the shortest path from a light-grey 2-subset to a singleton does not necessarily pass through all blank 2-subsets. Consider a generic step of the algorithm at which states to be compressed form a set P with $|P| = k > 1$ and let $v = a_1 \cdots a_\ell$ with $a_i \in \Sigma$, $i = 1, \ldots, \ell$, be a word of minimum length such that $|P\,.\,v| < k$.



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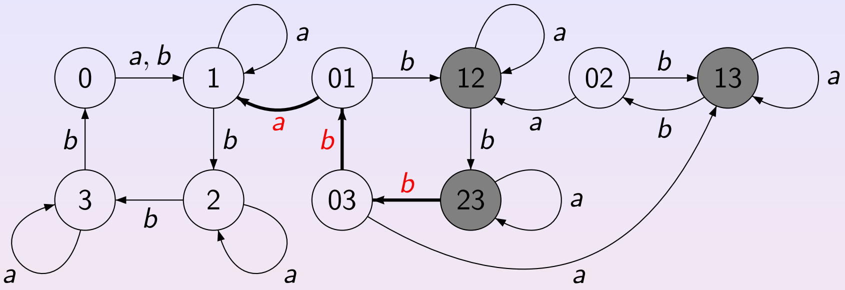

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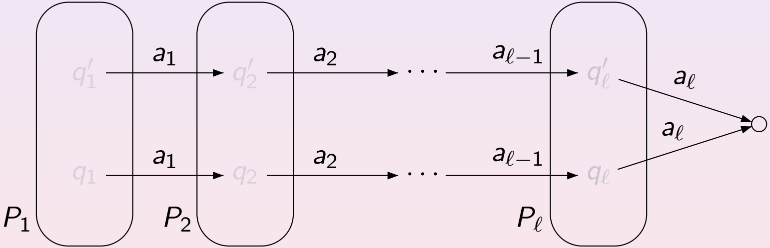
The sets $P_1 = P$, $P_2 = P_1 . a_1$, …, $P_\ell = P_{\ell-1} . a_{\ell-1}$ are k-subsets of Q. Since $|P_\ell . a_\ell| < |P_\ell|$, there exist two states $q_\ell, q'_\ell \in P_\ell$ such that $\delta(q_\ell, a_\ell) = \delta(q'_\ell, a_\ell)$. Now define 2-subsets $R_i = \{q_i, q'_i\} \subseteq P_i$, $i = 1, \ldots, \ell$, such that $\delta(q_i, a_i) = q_{i+1}$, $\delta(q'_i, a_i) = q'_{i+1}$ for $i = 1, \ldots, \ell - 1$.



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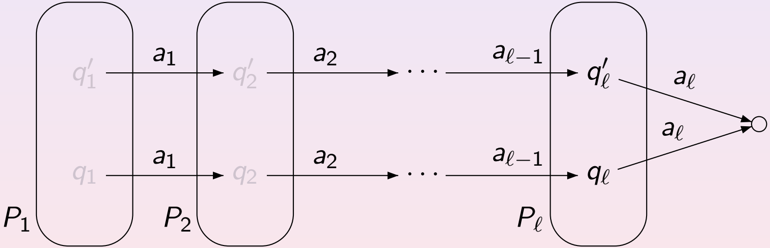

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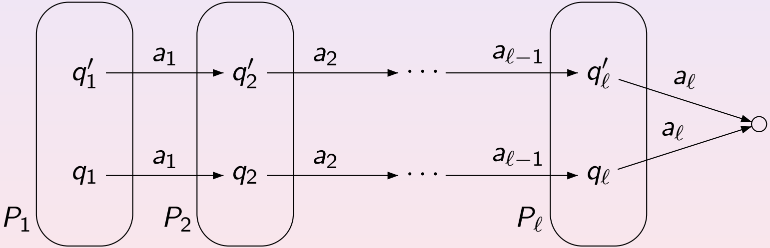
10. Studying Generic Step

The sets $P_1 = P$, $P_2 = P_1 . a_1$, $\dots$, $P_\ell = P_{\ell-1} . a_{\ell-1}$ are k-subsets of Q. Since $|P_\ell . a_\ell| < |P_\ell|$, there exist two states $q_\ell, q'_\ell \in P_\ell$ such that $\delta(q_\ell, a_\ell) = \delta(q'_\ell, a_\ell)$. Now define 2-subsets $R_i = \{q_i, q'_i\} \subseteq P_i$, $i = 1, \dots, \ell$, such that $\delta(q_i, a_i) = q_{i+1}$, $\delta(q'_i, a_i) = q'_{i+1}$ for $i = 1, \dots, \ell - 1$.



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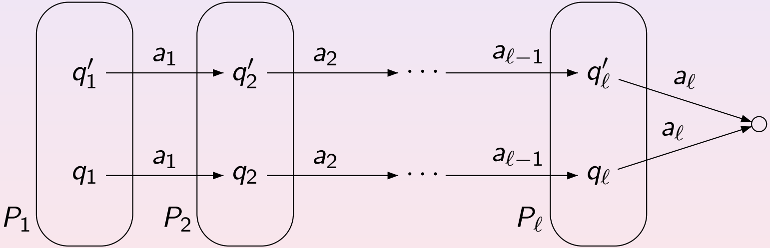

The condition that v is a word of minimum length with $|P \,.\, v| < |P|$ implies $R_i \nsubseteq P_j$ for $1 \le j < i \le \ell$.

11. Combinatorial Configuration

Our question reduces to the following problem in combinatorics of finite sets:

Let Q be an n-set, $P_1, \ldots, P_\ell$ a sequence of its k-subsets ($k > 1$) such that each P_i, $1 < i \leq \ell$, includes a "fresh" 2-subset that does not occur in any previous P_j ($1 \leq j < i$). How long can such renewing sequences be?

A construction: fix a $(k-2)$-subset W of Q, list all $\binom{n-k+2}{2}$ 2-subsets of $Q \setminus W$ and let T_i be the union of W with the i^{th} 2-subset in the list. This gives the renewing sequence $T_1, \ldots, T_s$ of length $s = \binom{n-k+2}{2}$. Is this the maximum?

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The question turned out to be very difficult and was solved (in the affirmative) by Peter Frankl (An extremal problem for two families of sets, Eur. J. Comb., 3 (1982) 125–127).

The proof uses linearization techniques which are quite common in combinatorics of finite sets. One reformulates the problem in linear algebra terms and then uses the corresponding machinery.

We identify Q with $\{1, 2, \ldots, n\}$ and assign to each k-subset $I = \{i_1, \ldots, i_k\}$ the following polynomial $D(I)$ in variables $x_{i_1}, \ldots, x_{i_k}$ over the field of reals.

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13. Linearization

$$I = \{i_1, \ldots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k\times k}$$

Then one proves that:

• the polynomials $D(P_1), \ldots, D(P_\ell)$ are linearly independent whenever the k-subsets $P_1, \ldots, P_\ell$ form a renewing sequence;

• the polynomials $D(T_1), \ldots, D(T_s)$ (derived from the "standard" sequence) generate the linear space spanned by all polynomials of the form $D(I)$.

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$$D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k\times k}$$

Suppose that k-subsets $P_1, \dots, P_\ell$ form a renewing sequence but $D(P_1), \dots, D(P_\ell)$ are linearly dependent. Then some polynomial $D(P_j)$ should be expressible as a linear combination of the preceding polynomials $D(P_1), \dots, D(P_{j-1})$. By the definition of a renewing sequence, P_j contains a couple $\{p, p'\}$ such that $\{p, p'\} \not\subseteq P_i$ for all $i < j$. If we substitute $x_p = p$, $x_{p'} = p'$ and $x_t = 0$ for $t \neq p, p'$ in each polynomial $D(P_1), \dots, D(P_j)$, then the polynomials $D(P_1), \dots, D(P_{j-1})$ vanish (since the two last columns in each of the resulting determinants become proportional) and so does any linear combination of the polynomials.

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15. Linearization, Step 1, completed

$$D(P_j) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & p & p^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & p' & (p')^2 \\ 1 & i_3 & i_3^2 & \cdots & i_3^{k-3} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & 0 & 0 \end{vmatrix}_{k\times k}$$

(For simplicity, here we assume that $i_1 = p$, $i_2 = p'$.)

The value of $D(P_j)$ is the determinant being the product of a Vandermonde $(k-2)\times(k-2)$-determinant with the 2×2-determinant $\begin{vmatrix} p & p^2 \\ p' & (p')^2 \end{vmatrix}$, whence this value is not 0.

Hence $D(P_j)$ cannot be equal to a linear combination of $D(P_1), \ldots, D(P_{j-1})$.

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16. Linearization, Step 2

Now we aim to prove that the polynomials $D(T_1), \dots, D(T_s)$ (derived from the "standard" sequence) generate the linear space spanned by all polynomials of the form $D(I)$. Take an arbitrary k-element subset $I = \{i_1, \dots, i_k\}$ of Q. We claim that the polynomial $D(I)$ is a linear combination of $D(T_1), \dots, D(T_s)$.

We induct on the cardinality of the set $I \setminus W$. If $|I \setminus W| = 2$, then I is the union of W with some couple from $Q \setminus W$, whence $I = T_i$ for some $i = 1, \dots, s$. Thus, $D(I) = D(T_i)$ and our claim holds true.

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17. Linearization, Step 2, continued

If $|I \setminus W| > 2$, there is $i_0 \in W \setminus I$. Let $I' = I \cup \{i_0\}$. There exists a polynomial $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \cdots + \alpha_{k-3} x^{k-3}$ over $\mathbb{R}$ such that $p(i_0) = 1$ and $p(i) = 0$ for all $i \in W \setminus \{i_0\}$. Consider the determinant

$$\Delta = \begin{vmatrix} p(i_0) & 1 & i_0 & i_0^2 & \cdots & i_0^{k-3} & x_{i_0} & x_{i_0}^2 \\ p(i_1) & 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ p(i_2) & 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p(i_k) & 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{(k+1)\times(k+1)}.$$

Clearly, $\Delta = 0$ as the first column is the sum of the next $k-2$ columns with the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-3}$.

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Clearly, $\Delta = 0$ as the first column is the sum of the next $k-2$ columns with the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-3}$.

17. Linearization, Step 2, continued

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18. Linearization, Step 2, completed

Expanding Δ by the first column gives the identity

$$\sum_{j=0}^{k}(-1)^j p(i_j)D(I' \setminus \{i_j\}) = 0.$$

Since $p(i_0) = 1$ and $I' \setminus \{i_0\} = I$, the identity rewrites as

$$D(I) = \sum_{j=1}^{k}(-1)^{j+1} p(i_j)D(I' \setminus \{i_j\}),$$

and since $p(i) = 0$ for all $i \in W \setminus \{i_0\}$, all the non-zero summands in the right-hand side are such that $i_j \notin W$.

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and since $p(i) = 0$ for all $i \in W \setminus \{i_0\}$, all the non-zero summands in the right-hand side are such that $i_j \notin W$. For each such i_j, we have

$$(I' \setminus \{i_j\}) \setminus W = I' \setminus (W \cup \{i_j\}) = (I \cup \{i_0\}) \setminus (W \cup \{i_j\}) = (I \setminus W) \setminus \{i_j\},$$

whence $|(I' \setminus \{i_j\}) \setminus W| = |I \setminus W| - 1$ and by the inductive assumption, the polynomials $D(I' \setminus \{i_j\})$ are linear combinations of $D(T_1), \ldots, D(T_s)$.

19. Results

Thus, in the step when k states are still to be compressed, the compression can always be achieved by applying a suitable word of length $\leq \binom{n-k+2}{2}$.

Summing up over $k = n, \ldots, 2$, we see that the greedy algorithm always returns a reset word of length $\leq \frac{n^3-n}{6}$:

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} =$$

$$\binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} =$$

$$\binom{4}{3} + \binom{4}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} = \cdots = \binom{n+1}{3} = \frac{n^3-n}{6}.$$

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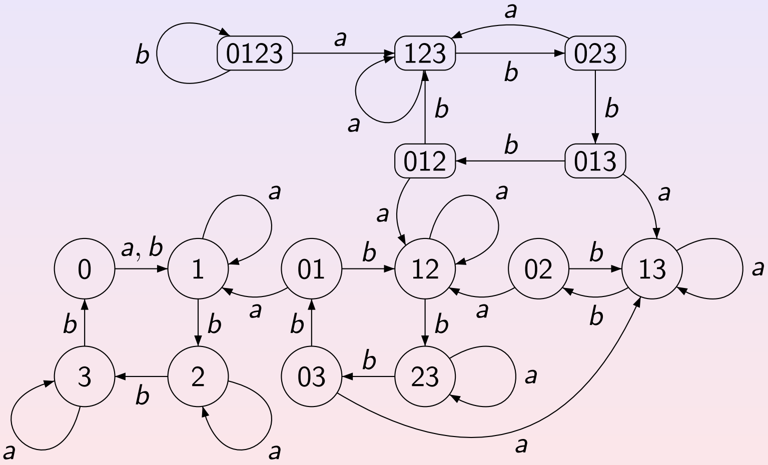
20. Greedy Algorithm

GreedyCompression($\mathscr{A}$)

```
w ← ε                                              ▷ Initializing the current word
P ← Q                                              ▷ Initializing the current set
while |P| > 1 do
    if |P . u| = |P| for all u ∈ Σ* then
        return  Failure
    else
        take a word v ∈ Σ* of minimum length with |P . v| < |P|
    w ← wv                                         ▷ Updating the current word
    P ← P . v                                      ▷ Updating the current set
return  w
```

21. Example Revisited

We have already seen that the greedy algorithm fails to find a reset word of minimum length.



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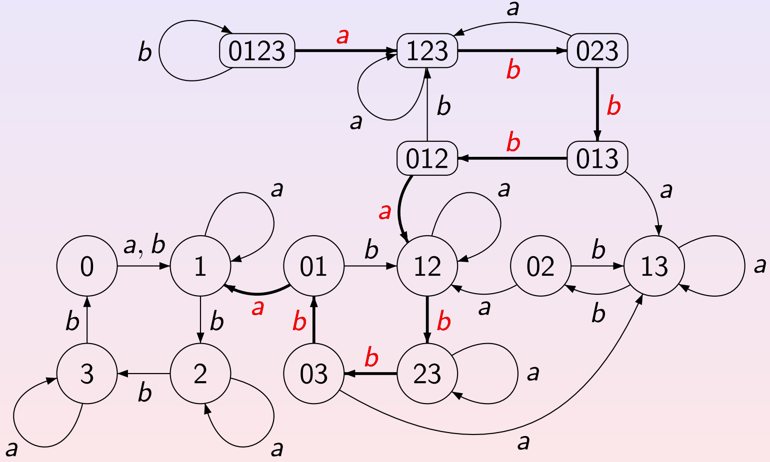

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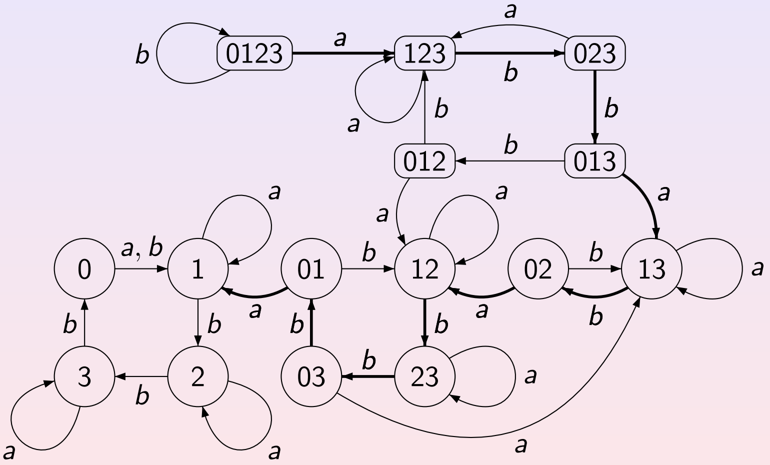

Actually, the gap between the minimum length of a reset word and the length of the word produced by the greedy algorithm may be arbitrarily large: for each $n > 1$ there exists a synchronizing automaton with n states whose shortest reset word has length $(n-1)^2$ while the greedy algorithm produces a reset word of length $\Omega(n^2 \log n)$.

Very recently, Dmitry Ananichev and Vladimir Gusev (in print) have provided a deep analysis of the worst case behaviour of all natural variants of the greedy algorithm.

The behaviour of the greedy algorithm on average is not yet understood; practically it behaves rather well.

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