Synchronizing Finite Automata

Lecture V. Expansion Method

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1. Recap

Deterministic finite automata: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

$|Q\,.\,w| = 1$. Here $Q\,.\,v = \{\delta(q, v) \mid q \in Q\}$.

Any w with this property is a reset word for $\mathscr{A}$.

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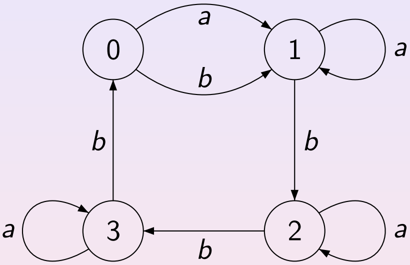

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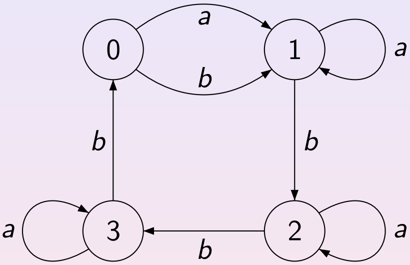

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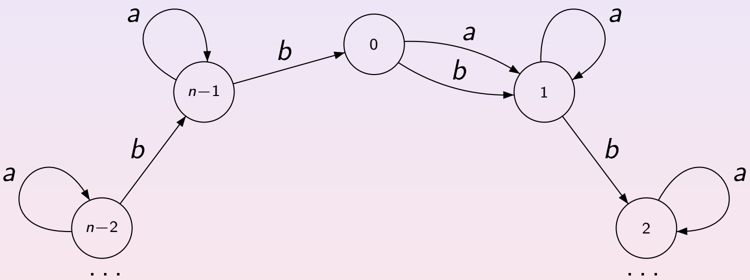

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$.

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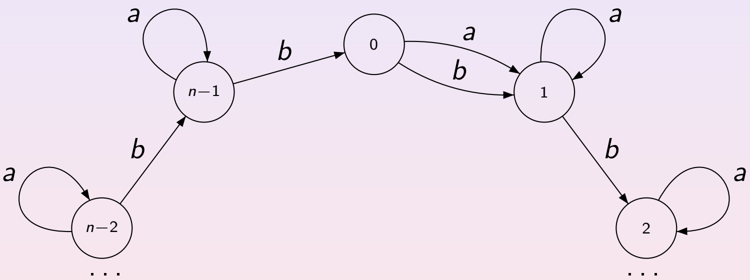

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$.

Define the Černý function $C(n)$ as the maximum reset threshold of all synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, $n = 2, 3, \ldots$, yields the inequality $C(n) \geq (n-1)^2$.

The Černý conjecture is the claim that in fact the equality $C(n) = (n-1)^2$ holds true.

Everything we know about the conjecture in general can be summarized in one line:

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Beyond the Černý series, the largest automaton that reaches the Černý bound is the 6-state automaton $\mathscr{K}_6$ found by Jarkko Kari (A counter example to a conjecture concerning synchronizing words in finite automata, EATCS Bull., 73 (2001) 146).

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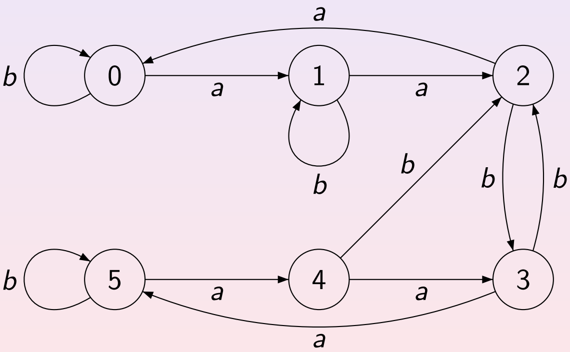

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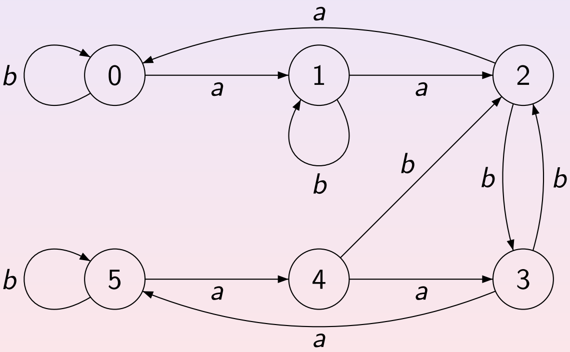

It has refuted several conjectures.

In particular, by Kari's example has refuted the extensibility conjecture.

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. For $P \subseteq Q$ and $w \in \Sigma^*$,

$$Pw^{-1} := \{q \in Q \mid q \, . \, w \in P\}.$$

A subset $P \subset Q$ is extensible if there exists a word $w \in \Sigma^*$ of length at most $n = |Q|$ such that $|Pw^{-1}| > |P|$.

It was conjectured that in synchronizing automata every proper non-singleton subset is extensible.

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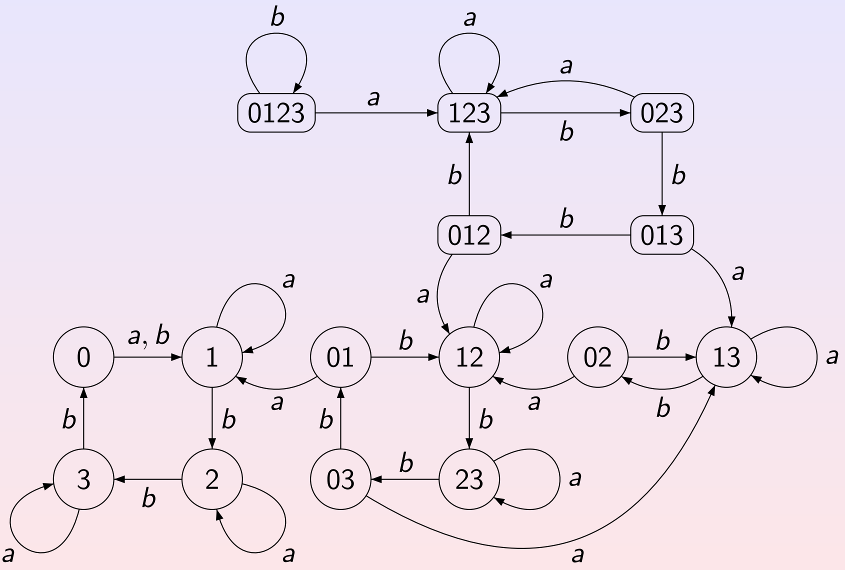

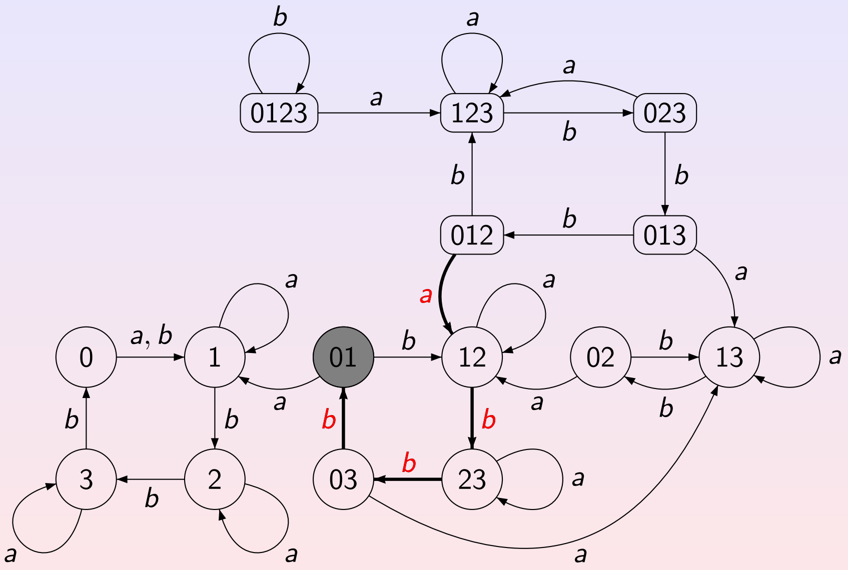

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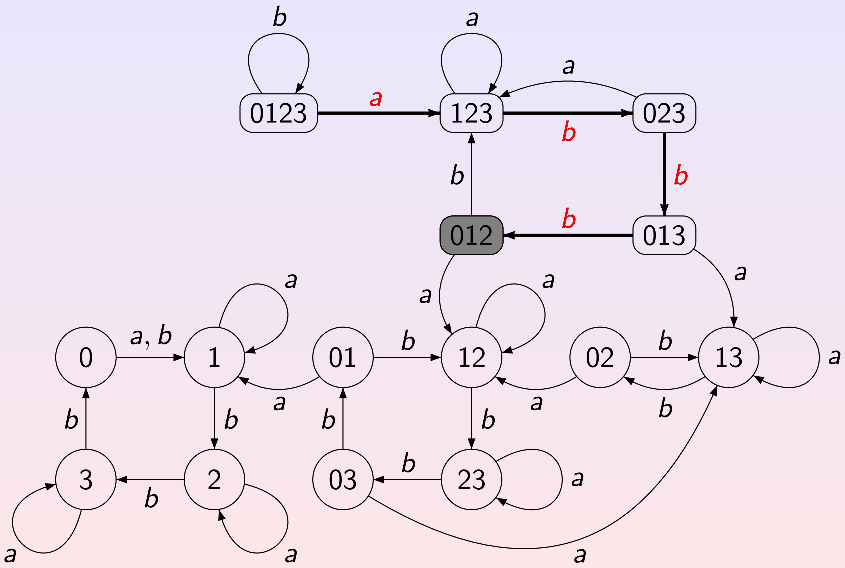

Observe that the extensibility conjecture implies the Černý conjecture.

Indeed, if $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ with $|Q| = n$ is synchronizing, then some letter $a \in \Sigma$ should sent two states $q, q' \in Q$ to the same state p.

Let $P_0 = \{q, q'\}$ and, for $i > 0$, let P_i be such that $|P_i| > |P_{i-1}|$ and $P_i = P_{i-1}w_i^{-1}$ for some word w_i of length $\leq n$.

Then in at most $n - 2$ steps the sequence $P_0, P_1, P_2, \dots$ reaches Q and

$$Q \, . \, w_{n-2}w_{n-3} \cdots w_1 a = \{p\},$$

that is, $w_{n-2}w_{n-3} \cdots w_1 a$ is a reset word.

The length of this reset word is at most $n(n-2) + 1 = (n-1)^2$.

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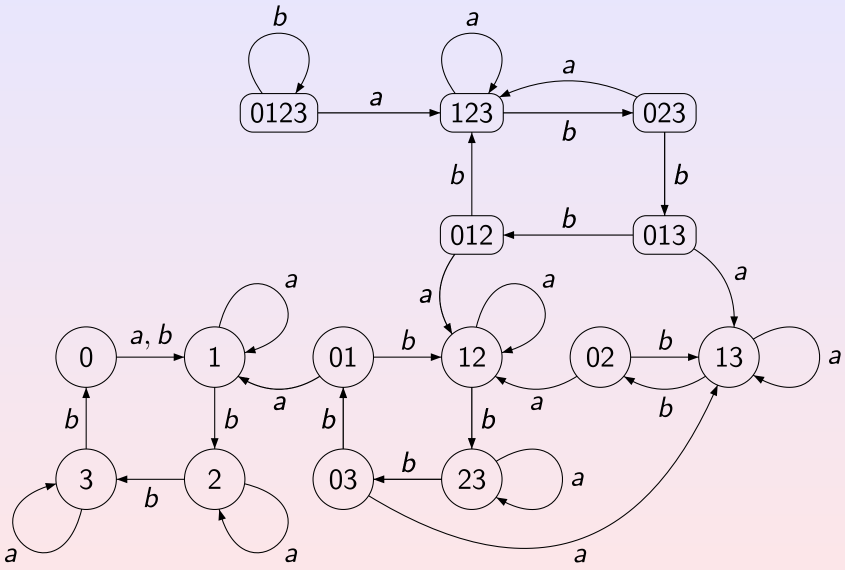

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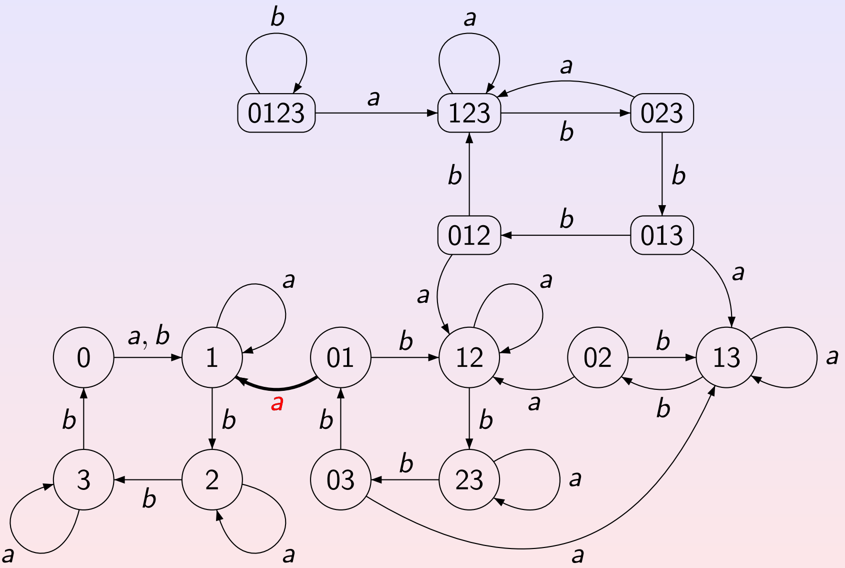
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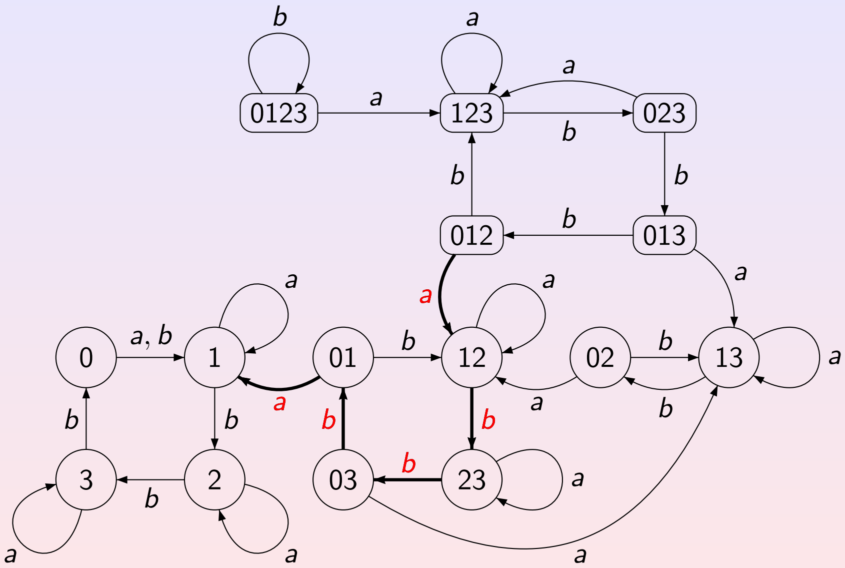

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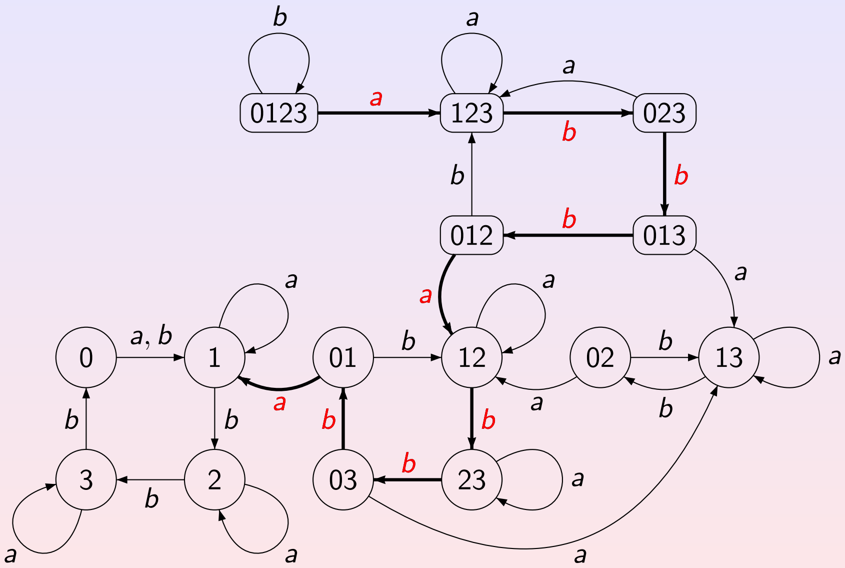

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Several important results confirming the Černý conjecture for various partial cases have been proved by verifying the extensibility conjecture for the corresponding automata. This includes:

- Louis Dubuc's result for automata in which a letter acts on the state set Q as a cyclic permutation of order $|Q|$ (Sur le automates circulaires et la conjecture de Černý, RAIRO Inform. Theor. Appl., 32 (1998) 21–34 [in French]).
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In this lecture, we present Kari's result.

A (directed) graph is strongly connected if for every pair of its vertices, there exists a (directed) path from one to the other.

A graph is Eulerian if it is strongly connected and each of its vertices serves as the tail and as the head for the same number of edges.

A DFA is said to be Eulerian if so is its underlying graph.

Since in any DFA the number of edges starting at a given state is the same (the cardinality of the input alphabet), in an Eulerian DFA the number of edges ending at a any state is the same.

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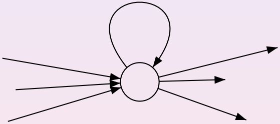

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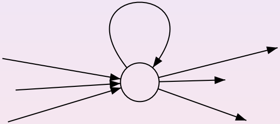

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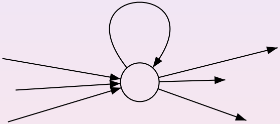

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Now suppose that $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is an Eulerian synchronizing automaton with $|Q| = n$ and $|\Sigma| = k$. Then for every $P \subseteq Q$, the equality

$$\sum_{a \in \Sigma} |Pa^{-1}| = k|P| \qquad (*)$$

holds true since the left-hand side is the number of edges in the underlying graph of $\mathscr{A}$ with ends in P.

The equality (*) readily implies that for each $P \subseteq Q$, one of the following alternatives takes place:
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$$|Pa^{-1}| = |P| \text{ for all letters } a \in \Sigma$$

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13. Our Aim

Assume that a subset $S \subseteq Q$ and a word $u \in \Sigma^+$ are such that $|Su^{-1}| \neq |S|$ and u is a word of minimum length with this property. We write $u = aw$ for some $a \in \Sigma$ and $w \in \Sigma^*$ and let $P = Sw^{-1}$. Then $|P| = |S|$ by the choice of u and $Pa^{-1} = Su^{-1}$ whence $|Pa^{-1}| \neq |P|$.

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14. Linearization

Assume that $Q = \{1, 2, \dots, n\}$. Assign to each subset $P \subseteq Q$ its characteristic vector $[P]$ in the linear space $\mathbb{R}^n$ of n-dimensional row vectors over $\mathbb{R}$ as follows: i-th entry of $[P]$ is 1 if $i \in P$, otherwise it is equal to 0.

For instance, $[Q]$ is the all ones row vector and the vectors $[1], \dots, [n]$ form the standard basis of $\mathbb{R}^n$.

Observe that for any vector $x \in \mathbb{R}^n$, the inner product $\langle x, [Q] \rangle$ is equal to the sum of all entries of x. In particular, for each subset $P \subseteq Q$, we have $\langle [P], [Q] \rangle = |P|$.

Assign to each word $w \in \Sigma^*$ the linear operator φ_w on $\mathbb{R}^n$ defined by $\varphi_w([i]) = [iw^{-1}]$ for each $i \in Q$. It is then clear that $\varphi_w([P]) = [Pw^{-1}]$ for each $P \subseteq Q$.

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15. A Reformulation

The inequality $|Su^{-1}| \neq |S|$ that we look for can be rewritten as $\langle \varphi_u([S]), [Q] \rangle \neq \langle [S], [Q] \rangle$ or $\langle \varphi_u([S]) - [S], [Q] \rangle \neq 0$.

Let $x = [S] - \frac{|S|}{n}[Q]$. Then $x \neq 0$ as $S \neq Q$ and $\langle x, [Q] \rangle = 0$. Since $Qu^{-1} = Q$ for every word u, we have $\varphi_u([Q]) = [Q]$. Hence

$$\langle \varphi_u([S]) - [S], [Q] \rangle = \langle \varphi_u(x + \frac{|S|}{n}[Q]) - (x + \frac{|S|}{n}[Q]), [Q] \rangle =$$

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Thus, a word u satisfies $|Su^{-1}| \neq |S|$ if and only if the vector $\varphi_u(x)$ lies beyond the subspace U of all vectors orthogonal to $[Q]$. We aim to bound the minimum length of such word u but first we explain why words sending x beyond U exist.

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16. How to Leave a Subspace

Since the automaton $\mathscr{A}$ is synchronizing and strongly connected, there exists a word $w \in \Sigma^*$ such that $Q\,.\,w \subseteq S$—one can first synchronize $\mathscr{A}$ to a state q and then move q into S by applying a word that labels a path from q to a state in S. Then

$$\varphi_w(x) = \varphi_w([S] - \frac{|S|}{n}[Q]) = \varphi_w([S]) - \frac{|S|}{n}\varphi_w([Q]) = (1 - \frac{|S|}{n})[Q] \neq 0.$$

Now consider the chain of subspaces $U_0 \subseteq U_1 \subseteq \dots$, where U_j is spanned by all vectors of the form $\varphi_w(x)$ with $|w| \le j$. Clearly, if $U_{j+1} = U_j$ for some j, then $\varphi_a(U_j) \subseteq U_j$ for all $a \in \Sigma$ whence $U_i = U_j$ for every $i \ge j$. Let ℓ be the least number such that $\varphi_u(x) \notin U$ for some word u of length ℓ, that is, the smallest ℓ such that $U_\ell \not\subseteq U$. Then in the chain $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_\ell$ all inclusions are strict.

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Hence

$$1 = \dim U_0 < \dim U_1 < \cdots < \dim U_{\ell-1} < \dim U_\ell$$

and, in particular, $\dim U_{\ell-1} \geq \ell$. But by our choice of ℓ we have $U_{\ell-1} \subseteq U$ whence $\dim U_{\ell-1} \leq \dim U$. Since U is the orthogonal complement of a 1-dimensional subspace, $\dim U = n-1$, and we conclude that $\ell \leq n-1$.

Thus, we have proved that for every proper subset $S \subset Q$, there exists a word $u \in \Sigma^*$ of length at most $n-1$ such that $|Su^{-1}| \neq |S|$. Recall that for Eulerian automata this implies that every proper subset can be extended by a word of length at most $n-1$ whence $\mathscr{A}$ has a reset word of length at most $(n-2)(n-1)+1 = n^2-3n+3 < (n-1)^2$.

17. Upper Bound

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Kari's upper bound $(n-2)(n-1)+1 = n^2 - 3n + 3$ is far from being tight.

The best theoretical lower bounds for the restriction of the Černý function to the class of Eulerian synchronizing automata known so far are of magnitude $\frac{n^2}{2}$ (Pavel Martyugin, Vladimir Gusev, Vojtěch Vorel, Marek Szykuła).

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Define the automaton $\mathscr{M}_n$ (from Matricaria) on the state set $\{1, 2, \dots, n\}$, where $n \geq 5$ is odd, in which a and b act as follows:

$$k \,.\, a = \begin{cases} k & \text{if } k \text{ is odd,} \\ k+1 & \text{if } k \text{ is even;} \end{cases} \qquad k \,.\, b = \begin{cases} k+1 & \text{if } k \neq n \text{ is odd,} \\ k & \text{if } k \text{ is even,} \\ 1 & \text{if } k = n. \end{cases}$$

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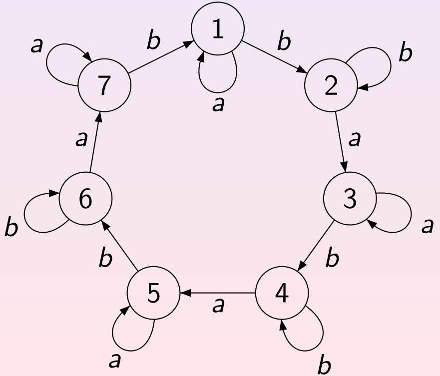

Observe that $\mathscr{M}_n$ is Eulerian. One can verify that the word $b(b(ab)^{\frac{n-1}{2}})^{\frac{n-3}{2}}b$ of length $\frac{n^2-3n+4}{2}$ is a reset word for $\mathscr{M}_n$.

Now let w be a reset word of minimum length for $\mathscr{M}_n$. The action of aa is the same as the action of a. Therefore aa could not be a factor of w. (Otherwise reducing this factor to just a results in a shorter reset word.) So every occurrence of a, maybe except the last one, is followed by b. If we let $c = ab$, then either w or wb (if w ends with a) can be rewritten into a word u over the alphabet $\{b, c\}$. The actions of b and c induce a new automaton on the state set of $\mathscr{M}_n$ and u is easily seen to be a reset word for this new automaton.

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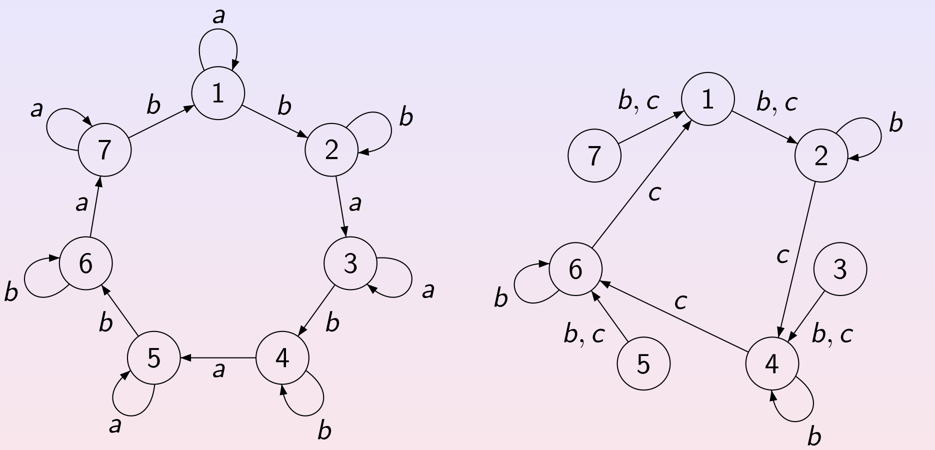

The automaton $\mathscr{M}_7$ and the automaton induced by the actions of b and $c = ab$

22. Induced Automaton

After applying the first letter of u it remains to synchronize the subautomaton on the set of states $S = \{1\} \cup \{2k \mid 1 \leq k \leq \frac{n-1}{2}\}$, and this subautomaton is isomorphic to $\mathscr{C}_{\frac{n+1}{2}}$. Thus, if $u = xu'$ for some letter x, then u' is a reset word for $\mathscr{C}_{\frac{n+1}{2}}$ and it can be shown that u' has at least $(\frac{n+1}{2})^2 - 3(\frac{n+1}{2}) + 2 = \frac{n^2-4n+3}{4}$ occurrences of c and at least $\frac{n-1}{2}$ occurrences of b. Since each occurrence of c in u' corresponds to an occurrence of the factor ab in w, we conclude that the length of w is at least $1 + 2\frac{n^2-4n+3}{4} + \frac{n-1}{2} = \frac{n^2-3n+4}{2}$.

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(Martyugin, unpublished) $\lfloor \dfrac{n^2-5}{2} \rfloor \leq C_E(n) \leq n^2-3n+3$ (Kari, 2003).

23. Extensibility vs Kari's Example

Back to extensibility, in $\mathscr{K}_6$ there exists a 2-subset that cannot be extended to a larger subset by any word of length 6 (and even by any word of length 7).

Thus, the extensibility conjecture fails, and the approach based on it cannot prove the Černý conjecture in general.

However, studying the extensibility phenomenon in synchronizing automata appears to be worthwhile: if there is a linear bound on the minimum length of words extending non-singleton proper subsets of a synchronizing automaton, then there is a quadratic bound on the minimum length of reset words for the automaton.

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GreedyExtenstion($\mathscr{A}$)

```
if |qa^{-1}| = 1 for all q ∈ Q and a ∈ Σ then
    return Failure
else
    w ← a such that |qa^{-1}| > 1          ▷ Initializing the current word
    P ← qa^{-1} such that |qa^{-1}| > 1     ▷ Initializing the current set
while |P| < |Q| do
    if |Pu^{-1}| ≤ |P| for all u ∈ Σ* then
        return Failure
    else
        take a word v ∈ Σ* of minimum length with |Pv^{-1}| > |P|
    w ← vw                                 ▷ Updating the current word
    P ← Pv^{-1}                            ▷ Updating the current set
return w
```

25. α-Extensibility

In contrast to Compression Algorithm, it is not clear whether Extension Algorithm admits a polynomial-time implementation.

Moreover, in general we know no non-trivial bound on the length of the words v that the main loop of Extension Algorithm appends to the current word. However, one can isolate some cases in which rather strong bounds on $|v|$ do exist.

Let α be a positive real number. An automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is α-extensible if for any subset $P \subset Q$ there is $w \in \Sigma^*$ of length at most $\alpha|Q|$ such that $|Pw^{-1}| > |P|$.

An α-extensible automaton with n states has a reset word of length $\alpha n^2 + O(n)$.

Several important classes of synchronizing automata are known to be 2-extensible, for instance, one-cluster automata (Marie-Pierre Béal, Mikhail Berlinkov, Dominique Perrin, A quadratic upper bound on the size of a synchronizing word in one-cluster automata, Int. J. Found. Comput. Sci., 22 (2011) 277–288).

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On the other hand, for any $\alpha < 2$ Mikhail Berlinkov (On a conjecture by Carpi and D'Alessandro, Int. J. Found. Comput. Sci. 22 (2011) 1565–1576) constructed a synchronizing one-cluster automaton that is not α-extensible.

For $n > \frac{3}{2-\alpha}$, this automaton is not α-extensible. In fact, the shortest word that extends the set $\{q_0, q_{n-1}\}$ is $a^{n-2}ba^{n-2}$.

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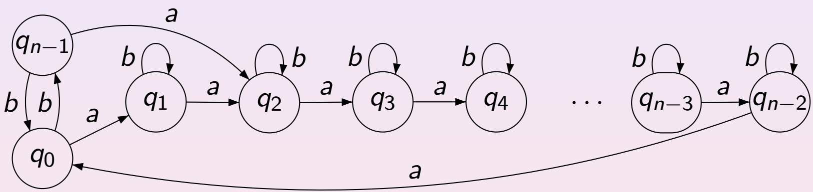

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26. Berlinkov's Series

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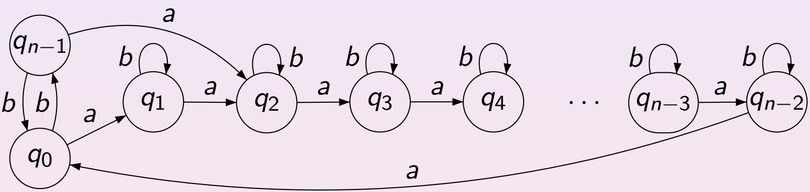

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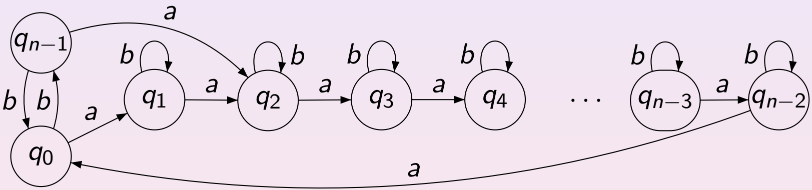

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27. Non-extensible Automata

Finally, Andrzej Kisielewicz and Marek Szykuła (Synchronizing automata with extremal properties, MFCS 2015, LNCS 9234 (2015) 331–343) constructed a series of synchronizing automata that are not α-extensible for any α.

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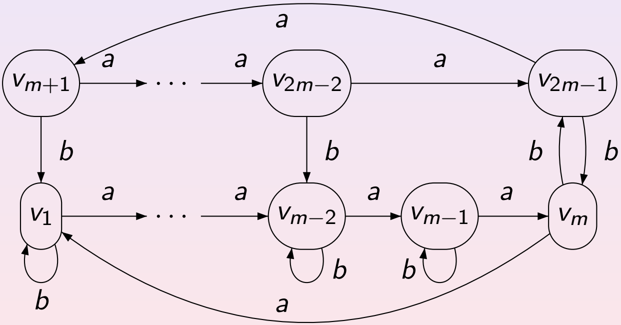

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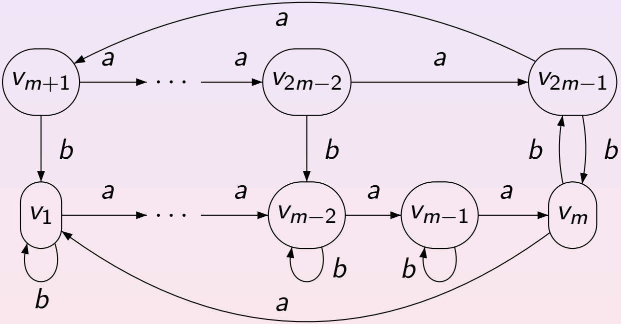

The automata in the series have subsets that require words of length as big as $m^2 + O(m)$ in order to be extended.

Open problem: to investigate the worst-case/average-case behaviour of the greedy extension algorithm.

Some experimental work that can be used in this direction has been done in a recent paper by Adam Roman and Marek Szykuła (Forward and backward synchronizing algorithms, Expert Systems with Applications, 42 (2015) 9512–9527).

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