Synchronizing Finite Automata

Lecture VI. Automata with Zero

Mikhail Volkov

Ural Federal University / Hunter College

1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

2. Algebraic Perspective

One can treat DFAs as unary algebras: each letter of the input alphabet defines a unary operation on the state set.

This allows us to apply to automata all standard algebraic notions, e.g., the notions of a subalgebra (subautomaton), a congruence, a quotient automaton.

Subautomata: if $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is a DFA, and $S \subseteq Q$ is such that $\delta(s, a) \in S$ for all $s \in S$ and $a \in \Sigma$, consider the DFA $\mathscr{S} := \langle S, \Sigma, \tau \rangle$ where $\tau = \sigma|_{S \times \Sigma}$. The latter equality means that $\tau(s, a) := \delta(s, a)$ for all $s \in S$ and $a \in \Sigma$.

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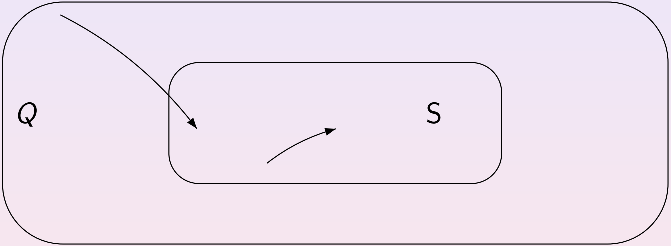

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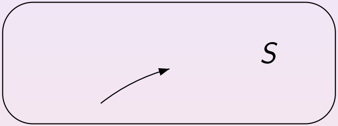

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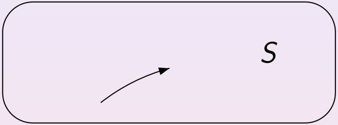

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Exercise: show that a DFA has no proper subautomata iff it is strongly connected.

A singleton subautomaton is normally called a sink state or just a sink. At a sink state each letter must have a loop.

We study synchronizing automata and, clearly, a synchronizing automaton may have at most one sink.

If a DFA has a unique sink state, this state is called a zero state or just a zero.

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4. Automata with Zero

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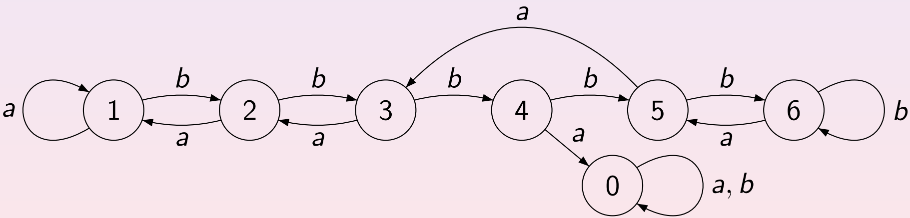

5. Congruences and Quotient Automata

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is called a congruence if $(p, q) \in \pi$ implies $\big(\delta(p, a), \delta(q, a)\big) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$. For π being a congruence, $[q]_\pi$ is the π-class containing the state q.

The *quotient* $\mathscr{A}/\pi$ is the DFA $\langle Q/\pi, \Sigma, \delta_\pi\rangle$ where $Q/\pi = \{[q]_\pi \mid q \in Q\}$ and the function δ_π is defined by the rule $\delta_\pi([q]_\pi, a) = [\delta(q, a)]_\pi$.

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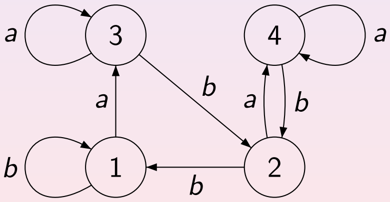

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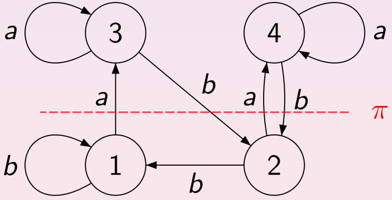

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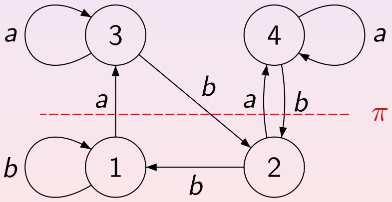

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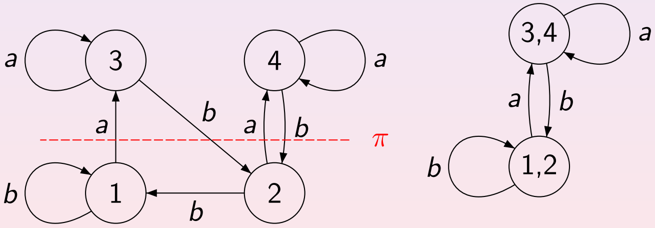

Suppose that $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is a DFA and $\mathscr{S} = \langle S, \Sigma, \tau \rangle$ is a subautomaton of $\mathscr{A}$.

The partition of Q into classes one of which is S and all others are singletons is a congruence of $\mathscr{A}$.

It is called the Rees congruence corresponding to $\mathscr{S}$ and is denoted by $\rho_{\mathscr{S}}$.

Clearly, in the quotient automaton $\mathscr{A}/\rho_{\mathscr{S}}$ the state S is a sink.

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7. Useful Observation

1. Any subautomaton of a synchronizing automaton is synchronizing, and every reset word for an automaton also serves as a reset word for any of its subautomata.

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8. A Reduction

Let **C** be any class of automata closed under taking subautomata and quotients, and let $\mathbf{C}_n$ stand for the class of all automata with n states in **C**.

Let $f : \mathbb{Z}^+ \to \mathbb{N}$ be any function such that

$$f(n) \geq f(n-m+1) + f(m) \text{ whenever } \; n \geq m \geq 1.$$

If each synchronizing automaton in $\mathbf{C}_n$ which either is strongly connected or possesses a zero has a reset word of length $f(n)$, then the same holds true for all synchronizing automata in $\mathbf{C}_n$.

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9. A Reduction: Proof

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a synchronizing automaton in $\mathbf{C}_n$.
Consider the set S of all states to which the automaton $\mathscr{A}$ can be reset and let $m = |S|$.
If $q \in S$, then there exists a reset word $w \in \Sigma^*$ such that $Q.w = \{q\}$.
Then wa also is a reset word and $Q.wa = \{\delta(q, a)\}$ whence $\delta(q, a) \in S$.
This means that, restricting the transition function δ to $S \times \Sigma$, we get a subautomaton $\mathscr{S}$ with the state set S.
Since $\mathscr{S}$ is synchronizing and strongly connected and since the class $\mathbf{C}$ is closed under taking subautomata, we have $\mathscr{S} \in \mathbf{C}$.
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10. A Reduction: End of the Proof

Now consider the Rees congruence $\rho_{\mathscr{S}}$ of the automaton $\mathscr{A}$.

The quotient $\mathscr{A}/\rho_{\mathscr{S}}$ is synchronizing, has S as a zero, and has $n-m+1$ states.

Since the class **C** is closed under taking quotients, we have $\mathscr{A}/\rho_{\mathscr{S}} \in \mathbf{C}$.

Hence $\mathscr{A}/\rho_{\mathscr{S}}$ has a reset word u of length $f(n-m+1)$.

Since $Q.u \subseteq S$ and $S.v$ is a singleton, we conclude that also $Q.uv \subseteq S.v$ is a singleton.

Thus, uv is reset word for $\mathscr{A}$, and the length of this word does not exceed $f(n-m+1)+f(m) \le f(n)$ according to the condition imposed on the function f.

It is easy to check that the function $f(n)=(n-1)^2$ satisfies the condition above.

We see that it suffices to prove the Černý conjecture for strongly connected automata and for automata zero.

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Hence $\mathscr{A}/\rho_{\mathscr{S}}$ has a reset word u of length $f(n - m + 1)$.
Since $Q.u \subseteq S$ and $S.v$ is a singleton, we conclude that also $Q.uv \subseteq S.v$ is a singleton.
Thus, uv is reset word for $\mathscr{A}$, and the length of this word does not exceed $f(n - m + 1) + f(m) \leq f(n)$ according to the condition imposed on the function f.

It is easy to check that the function $f(n) = (n - 1)^2$ satisfies the condition above.

We see that it suffices to prove the Černý conjecture for strongly connected automata and for automata zero.

10. A Reduction: End of the Proof

Now consider the Rees congruence $\rho_{\mathscr{S}}$ of the automaton $\mathscr{A}$.
The quotient $\mathscr{A}/\rho_{\mathscr{S}}$ is synchronizing, has S as a zero, and has $n-m+1$ states.
Since the class **C** is closed under taking quotients, we have $\mathscr{A}/\rho_{\mathscr{S}} \in \mathbf{C}$.
Hence $\mathscr{A}/\rho_{\mathscr{S}}$ has a reset word u of length $f(n-m+1)$.
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Thus, uv is reset word for $\mathscr{A}$, and the length of this word does not exceed $f(n-m+1)+f(m) \leq f(n)$ according to the condition imposed on the function f.

It is easy to check that the function $f(n) = (n-1)^2$ satisfies the condition above.
We see that it suffices to prove the Černý conjecture for strongly connected automata and for automata zero.

11. Automata with Zero

If a synchronizing automaton with k states has a zero, then it has a reset word of length $\leq \frac{k(k-1)}{2}$.

The algorithm makes at most $k-1$ steps and the length of the segment added in the step when t states still hold coins $(k-1 \geq t \geq 1)$ is at most $k-t$. The total length is $\leq 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$.

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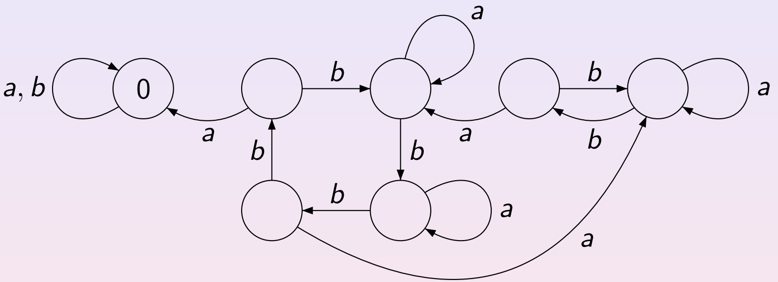

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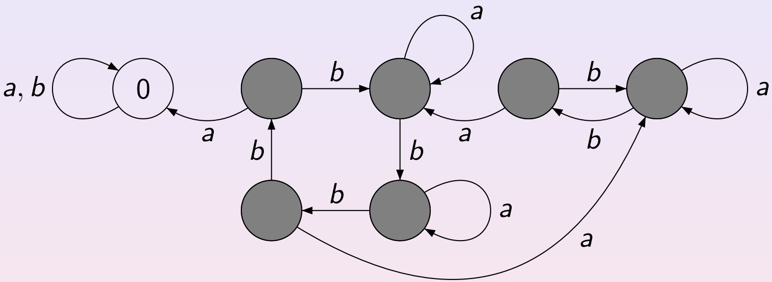

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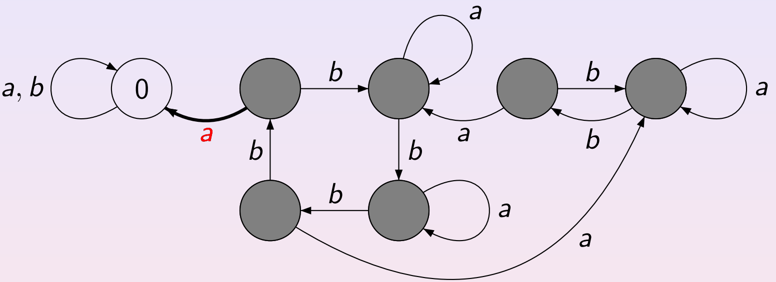

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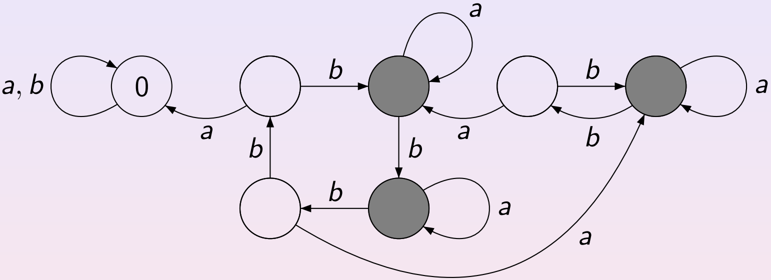

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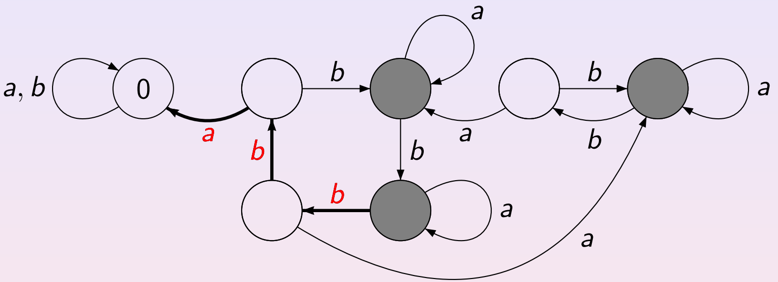

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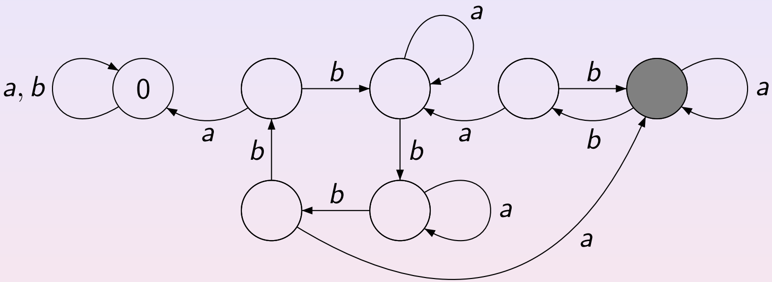

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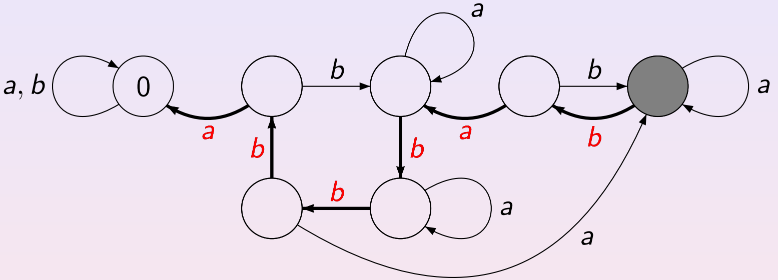

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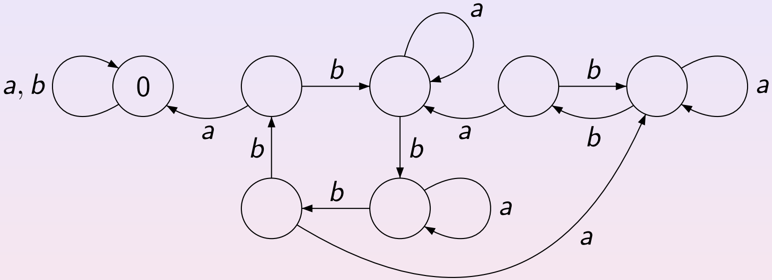

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