Synchronizing Finite Automata

Lecture VII. Aperiodic Automata

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Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

$|Q \,.\, w| = 1$. Here $Q \,.\, v = \{\delta(q, v) \mid q \in Q\}$.

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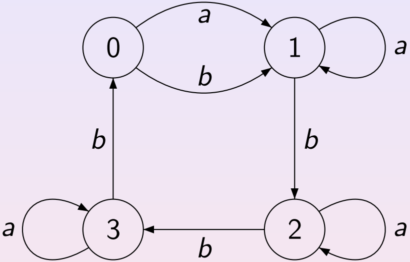

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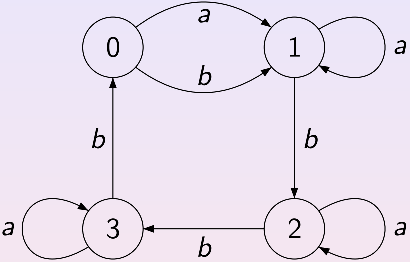

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3. Černý Conjecture

The Černý conjecture is the claim that every synchronizing automaton with n states possesses a reset word of length $(n-1)^2$. The validity of the conjecture is main open problem of the area.

Define the *Černý function* $C(n)$ as the maximum reset threshold of all synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

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Since the Černý problem has proved to be hard in general, a natural approach is to consider its restrictions to various special classes of automata.

There are several classes in which synchronizing automata have been investigated with (at least partial) success:

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A monoid is said to be aperiodic if all its subgroups are singletons.

A DFA is called aperiodic (or counter-free) if its transition monoid is aperiodic.

An equivalent 'elementary' formulation: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is aperiodic iff for every $q \in Q$ and every $w \in \Sigma^*$ there exists a positive integer m such that $q \,.\, w^m = q \,.\, w^{m+1}$.

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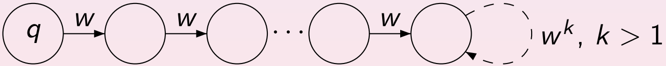

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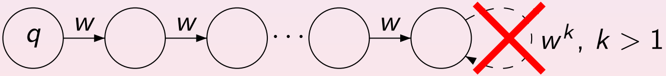

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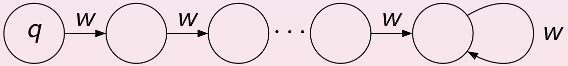
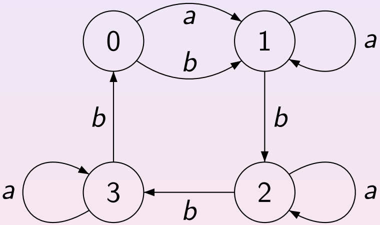


The Černý automaton $\mathscr{C}_4$ is not aperiodic since the letter b acts as a cyclic permutation of the states and thus generates a 4-element subgroup in the transition monoid of $\mathscr{C}_4$.



The following automaton is aperiodic:

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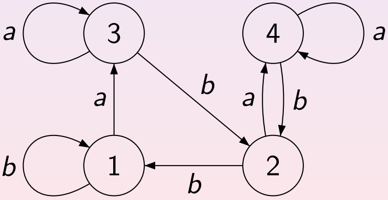

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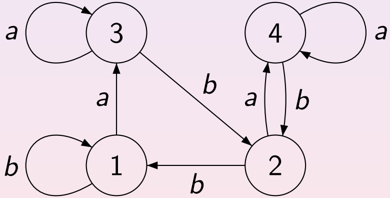

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a	3	4	3	4
b	1	1	2	2

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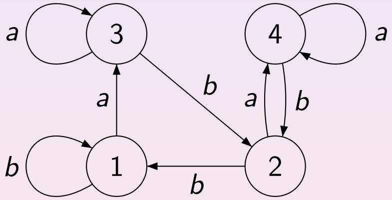

	1	2	3	4
a	3	4	3	4
b	1	1	2	2
ab	2	2	2	2
ba	3	3	4	4
b^2	1	1	1	1

$a^2 = a$

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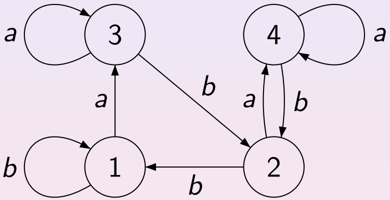

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aba	4	4	4	4
b^2a	3	3	3	3

$a^2 = a,\ ab^2 = b^2,\ bab = ab,\ b^3 = b^2$

7. Complexity

In general, there is no way to verify whether or not a given DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is aperiodic avoiding the calculation of its transition monoid and the cardinality of the monoid can reach $|Q|^{|Q|}$. The problem is known to be PSPACE-complete (Sang Cho and Dung T. Huynh, "Finite-automaton aperiodicity is PSPACE-complete", Theor. Comput. Sci. 88 (1991) 99–116).

Also, the synchronization issues remain difficult when restricted to the class of aperiodic automata. Indeed, inspecting the reduction from SAT to SHORT-RESET-WORD shown in Lecture III, one can see that the construction results in an aperiodic automaton, and therefore, the question of whether or not a given aperiodic automaton admits a reset word whose length does not exceed a given positive integer is NP-complete.

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In some cases, however, aperiodicity is granted.

A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is monotonic if Q admits a linear order $\leq$ such that, for each $a \in \Sigma$, the transformation $\delta(\sqcup, a)$ preserves $\leq$:

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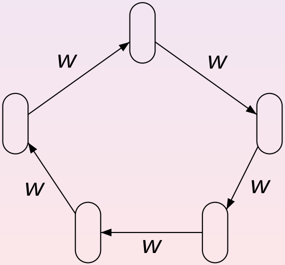

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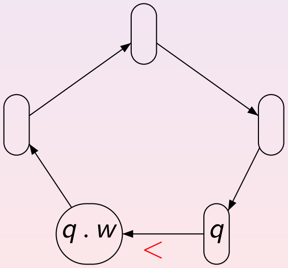

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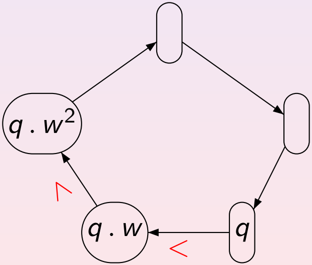

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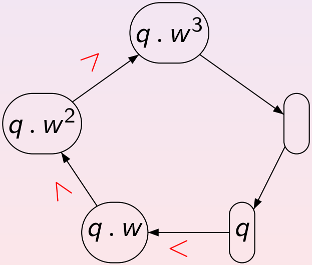

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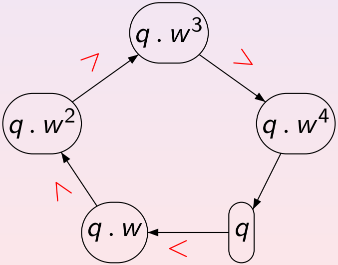
$$p \leq q \Rightarrow \delta(p, a) \leq \delta(q, a).$$

Monotonic automata are aperiodic (known and easy).



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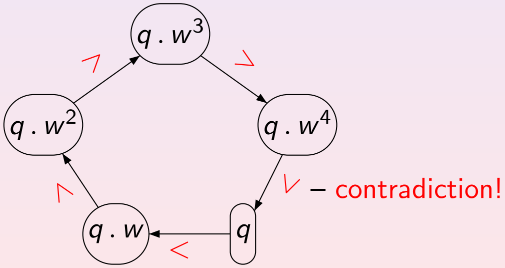

9. Generalized Monotonicity

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We call a DFA $\mathscr{A}$ generalized monotonic of level ℓ if it admits a strictly increasing chain of stable binary relations $\rho_0 \subset \rho_1 \subset \cdots \subset \rho_\ell$, satisfying the following conditions:

- ρ_0 is the equality;
- for each $i = 1, \dots, \ell$, the congruence π_{i-1} generated by ρ_{i-1} is contained in ρ_i and the relation ρ_i/π_{i-1} is a linear order on each π_i/π_{i-1}-class;
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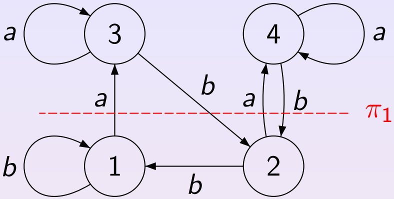

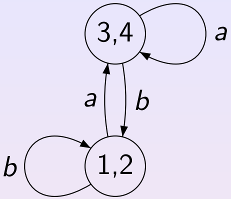

Endowing Q with the order $\leq_1$ such that $1 <_1 2$ and $3 <_1 4$, we get a linear order on each π_1-class. If we order Q/π_1 by letting $\{1,2\} <_2 \{3,4\}$, the quotient automaton becomes monotonic. It can be shown that the automaton is not monotonic. Moreover, it cannot be emulated by any monotonic automaton.

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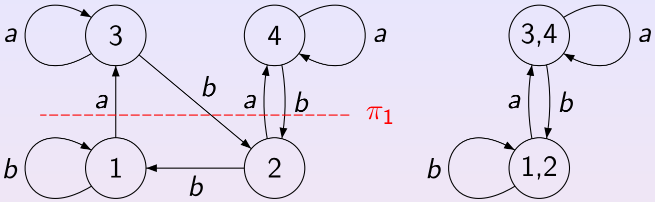

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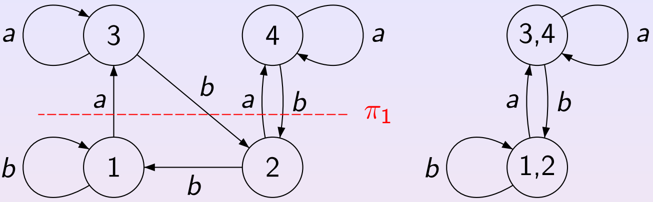

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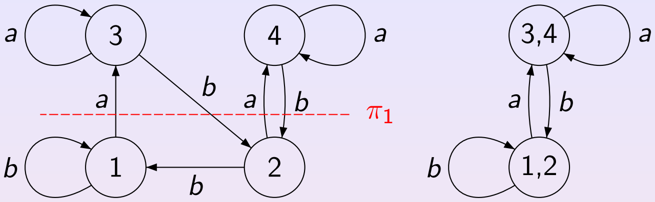

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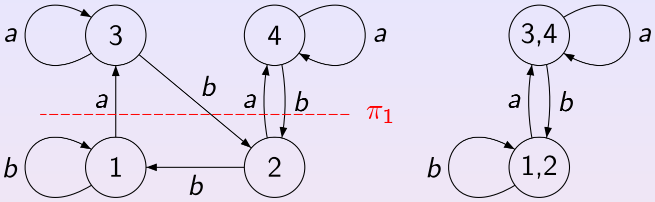

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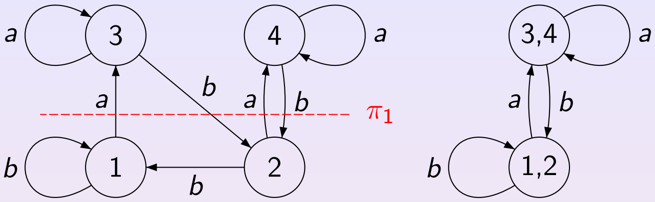

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It is remarkable that each of these directions has led to a major open problem, and the 3 problems play nowadays a central role in the theory of finite automata.

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By Kleene's theorem every regular language can be described by a regular expression, say, $((a+ba)^*ab)^*(b+aa)^*$. Here words denote corresponding singleton languages, $+$ stands for union, concatenation means product and * is the Kleene star (iteration).

The Kleene star is clearly the most 'infinite' operation. One cannot eliminate it because neither union nor product can produce infinite languages from finite ones. However, one can use also complement (the class of regular languages is closed under complement by Kleene's theorem). An extended regular expression is built from words by using union, product, Kleene star, and complement, say, $((a+ba)^C ab)^*(b+(aa)^C)^*$.

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$$(ab)^*$$

Here $E_1 \setminus E_2 = E_1 \cap E_2^C$ can be expressed as $(E_1^C + E_2)^C$ by De Morgan's law.
To understand the formula, observe that $a + a^C = \Sigma^*$.

However, for the language $(a^2)^*$ that looks alike $(ab)^*$ we would not be able to construct a star-free extended regular expression.

How can one distinguish between regular languages that need star and 'star-free' languages?

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Schützenberger's Theorem, 1964

A regular language L admits a star-free extended regular expression iff the minimal automaton of L is aperiodic.

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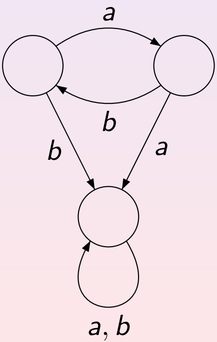
For instance, for $(ab)^*$ and $(a^2)^*$ the minimal automata are



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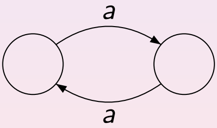

By the (extended) star height of a regular language L we mean the minimum number of nested stars over all (extended) regular expressions representing L.

It is known that there exist regular languages of any given star height and that, given a language, its star height can be decided. However analogous problems are open for extended star height.

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Extended Star Height Problem

Is there a regular language of extended star height > 1?

Is the class of languages of extended star height 1 decidable?

16. Krohn–Rhodes Theorem

A DFA is said to be a group automaton if every letter acts as a permutation of the state set. Group automata are just Cayley graphs of groups and are antipodes of aperiodic automata.

Thus, $\mathscr{A}$ decomposes into counter (=group) and non-counter (=aperiodic) components. Group components can be further decomposed into cascade compositions of Cayley graphs of simple groups while aperiodic components are cascade compositions of flip-flops and their 1-letter subautomata.

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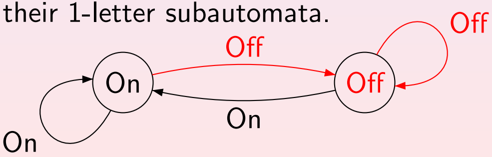

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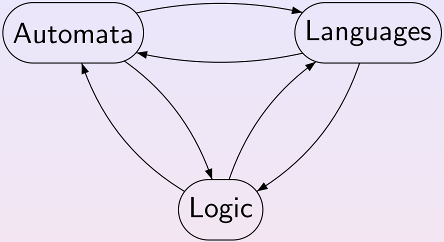
Group Complexity Problem

Given a finite automaton $\mathscr{A}$, can one decide the group complexity of $\mathscr{A}$?
In particular, can we decide if the group complexity of $\mathscr{A}$ is 1?

18. Logic for Words

There is a magic triangle



Logic for words has first order variables (positions) that take values in $\{1, 2, \dots\}$, second order variables (sets of positions) whose values are subsets of $\{1, 2, \dots\}$, the usual connectives and quantifiers, the predicate symbol $<$ with the usual meaning (and maybe some additional numerical predicates), and a special predicate Q_a for each letter a with the meaning: $Q_a x$ is true iff the position x holds the letter a.

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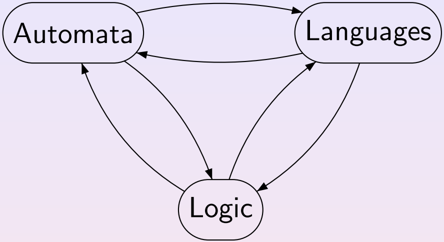

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Any closed formula of this logic defines a language.

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Monadic second order formulas define precisely regular languages (Büchi, 1960), but we would not be able to construct a first order formula defining $(a^2)^*$.

Can one distinguish between 'second' and 'first' order languages?

20. McNaughton's Theorem

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Dot-depth 1 and dot-depth 2 are known to be decidable (Knast, 1980, for 1 and Place-Zeitoun, 2014, for 2)

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As discussed in Lecture VI, we may restrict to strongly connected automata.

Here we encounter a small surprise: *every strongly connected aperiodic automaton* $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ *is synchronizing.*

Indeed, take any $q, q' \in Q$. Since $\mathscr{A}$ is strongly connected, there exists $w \in \Sigma^*$ such that $q \,.\, w = q'$. On the other hand, $\mathscr{A}$ is aperiodic whence there exists a positive integer m such that $q \,.\, w^m = q \,.\, w^{m+1}$. Applying w^m to the equality $q \,.\, w = q'$, we get $q \,.\, w^{m+1} = q' \,.\, w^m$ whence $q \,.\, w^m = q' \,.\, w^m$. Thus, every pair of states can be synchronized, and by Černý's criterion, this ensures that $\mathscr{A}$ is synchronizing.

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Partial Order

Avraham Trahtman (The Černý conjecture for aperiodic automata, Discrete Math. Theor. Comp. Sci. 9(2) (2007) 3–10) has proved that every synchronizing aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$ (so less than $(n-1)^2$).

Recall that $\frac{n(n-1)}{2}$ is precisely Rystsov's bound for n-state synchronizing automata having a sink. Thus, it remains to prove that every strongly connected aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$.

The key observation by Trahtman is that every strongly connected aperiodic automaton admits a non-trivial stable partial order.

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Recall that $\frac{n(n-1)}{2}$ is precisely Rystsov's bound for n-state synchronizing automata having a sink. Thus, it remains to prove that every strongly connected aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$.

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23. Constructing Partial Order

Given a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, its square $\mathscr{A}^{[2]} = \langle Q \times Q, \Sigma, \delta^{[2]} \rangle$ is defined by $\delta^{[2]}\big((q,p),a\big) = \big(\delta(q,a), \delta(p,a)\big)$.
Warning: it is not quite the same as the automaton on all at most 2-element subsets that we considered in Lecture II.

If $\mathscr{A}$ is synchronizing and strongly connected, then $\mathscr{A}^{[2]}$ has a least strongly connected component $D = \{(q,q) \mid q \in Q\}$. Let K be a strongly connected component immediately following D in the natural order of strongly connected components.

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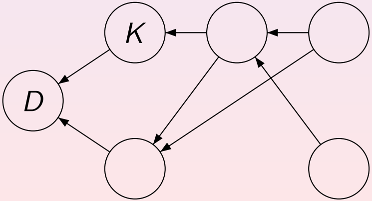

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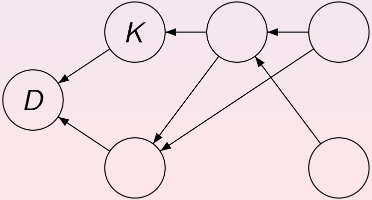

Then $K \cup D$ is a non-trivial stable reflexive relation on Q.

24. $\mathscr{A}$ is aperiodic

Let $\succeq_K$ be the transitive closure of $K \cup D$. It is clear that $\succeq_K$ is non-trivial, stable, reflexive and transitive.
Now we show that $\succeq_K$ is antisymmetric whenever $\mathscr{A}$ is aperiodic.

Suppose that there are $p, q \in Q$ such that $p \neq q$ and $p \succeq_K q \succeq_K p$. Then there is a sequence of $p_0, p_1, \ldots, p_k \in Q$ such that $k > 1$, $p_0 = p = p_k$, $q = p_j$ for some j, $0 < j < k$, and $(p_i, p_{i+1}) \in K$ for all $i = 0, 1, \ldots, k-1$. We choose the shortest such sequence $p_0, p_1, \ldots, p_k$ (over all possible 'obstacles' (p, q) to antisymmetry).

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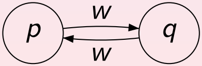

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Suppose that $k > 2$. Then p_0, p_1, p_2 are all distinct. By the definition of K, there exists $w \in \Sigma^*$ such that $(p_0, p_1) . w = (p_1, p_2)$, that is, $p_0 . w = p_1$, $p_0 . w^2 = p_1 . w = p_2$. Since $\mathscr{A}$ is aperiodic, there exists m such that $p_0 . w^{m+1} = p_0 . w^m$; we choose the least m with this property. Observe that $m > 1$ since $p_0 . w^2 \neq p_0 . w$.

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Now we apply w^{m-1} to each state in the sequence $p_0, p_1, \dots, p_k$. Since $K \cup D$ is stable, we get that for all $i = 0, 1, \dots, k-1$ either $(p_i \,.\, w^{m-1}, p_{i+1} \,.\, w^{m-1}) \in K$ or $p_i \,.\, w^{m-1} = p_{i+1} \,.\, w^{m-1}$. The choice of m ensures $p_0 \,.\, w^{m-1} \neq p_1 \,.\, w^{m-1} = p_0 \,.\, w^m$ whence the new sequence still contains an obstacle to antisymmetry. On the other hand, $p_1 \,.\, w^{m-1} = p_0 \,.\, w^m = p_0 \,.\, w^{m+1} = p_2 \,.\, w^{m-1}$, and therefore, if we retain in the new sequence only the first state from each group of adjacent equal states, the sequence becomes shorter. This contradicts the minimality of $p_0, p_1, \dots, p_k$.

26. Using the Order

Thus, $\succeq_K$ is a non-trivial partial order. How does it help?

We denote by π_K the symmetric closure of $\succeq_K$. Clearly, π_K is a congruence of the automaton $\mathscr{A}$.

The quotient automaton $\mathscr{A}/\pi_K$ has $< n$ states and so it has a short reset word v by the induction assumption (we induct on the number of states). In $\mathscr{A}$, the word v sends the whole state set into a single π_K-class, say, T. In order to quickly synchronize T choose a minimum state $q \in T$ and a maximum state $p \in T$ w.r.t. $\succeq_K$ ($p \neq q$ whenever $|T| > 1$). Since $\mathscr{A}$ is strongly connected, there is a word u_1 of length $< n$ such that $p\,.\,u_1 = q$. Then for each $r \in T$ such that $p \succeq_K r$ we have $q = p\,.\,u_1 \succeq_K r\,.\,u_1$ whence $q = r\,.\,u_1$ as q is minimal. If still $|T\,.\,u_1| > 1$, we can take a maximum state $p' \in T\,.\,u_1$ and repeat the process by taking a word u_2 of length $< n$ such that $p'\,.\,u_2 = q$, and so on.

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How long can be a reset word constructed this way?

From the minimum-maximum symmetry it follows that the number of steps is at most $\frac{|T|}{2}$. In the case when $T = Q$ (actually, this is the worst case) we get at most $\frac{n}{2}$ steps and a word of length at most $n-1$ is added at each step. The resulting reset word is of length at most $\frac{n(n-1)}{2}$. If $|T| = m < n$, then the quotient automaton $\mathscr{A}/\pi_K$ has at most $n-m+1$ states and we first need a word v of length at most $\frac{(n-m+1)(n-m)}{2}$ to send Q to T and then a word of length at most $\frac{m(n-1)}{2}$ to synchronize T. It remains to calculate that

$$\frac{(n-m+1)(n-m)}{2} + \frac{m(n-1)}{2} \leq \frac{n(n-1)}{2}$$

for all $m = 2, \dots, n-1$. If $m = 1$, then v itself is a reset word for $\mathscr{A}$.

27. Estimations

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28. Weakly Monotonic Automata

In my paper (Synchronizing automata preserving a chain of partial orders, Theor. Comput. Sci. 410 (2009) 3513–3519) Trahtman's theorem has been extended to a larger class automata.

A DFA $\mathscr{A}$ is weakly monotonic of level ℓ if it has a strictly increasing chain of stable binary relations $\rho_0 \subset \rho_1 \subset \cdots \subset \rho_\ell$ satisfying the following conditions:

- ρ_0 is the equality relation;
- for each $i = 1, \ldots, \ell$, the congruence π_{i-1} generated by ρ_{i-1} is contained in ρ_i and the relation ρ_i/π_{i-1} is a partial order on Q/π_{i-1};
- π_ℓ is the universal relation.

This differs from the notion of a generalized monotonic automaton by just dropping the restriction that the order ρ_i/π_{i-1} is linear on each π_i/π_{i-1}-class.

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29. Examples

- every aperiodic automaton is weakly monotonic;
- every automaton with 0 is weakly monotonic (of level 1).

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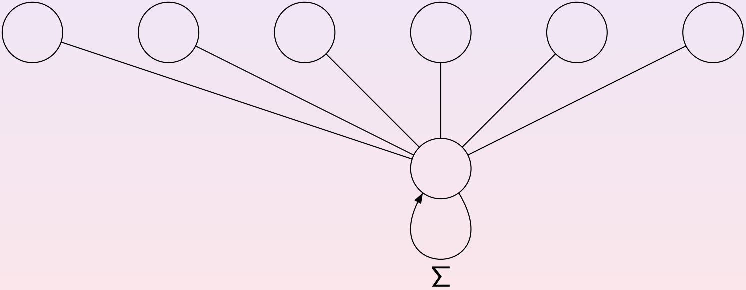

30. WM + SC Automata

- Every weakly monotonic automaton with a strongly connected underlying digraph is synchronizing. (A non-trivial generalization of the corresponding result for aperiodic automata.)

- Every weakly monotonic automaton with a strongly connected underlying digraph and n states has a reset word of length $\leq \left\lfloor \frac{n(n+1)}{6} \right\rfloor$. (This upper bound is new even for the aperiodic case – recall that Trahtman's bound was 3 times higher, namely, $\frac{n(n-1)}{2}$.)

- Every weakly monotonic synchronizing automaton with n states has a reset word of length $\frac{n(n-1)}{2}$.

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A bad news is that there are no matching lower bounds for the upper bounds just discussed. No aperiodic (even no weakly monotonic) strongly connected automaton is known for which the minimum length of reset words would be greater than or equal to the number of states.

Let $C_{SCA}(n)$ denote the restriction of the Černý function to the class of all strongly connected aperiodic automata, that is, $C_{SCA}(n)$ is the maximum length of shortest reset words for strongly connected aperiodic automata with n states. Then our current knowledge can be summarized as follows:

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$$n-1 \leq C_{SCA}(n) \leq \left\lfloor \frac{n(n+1)}{6} \right\rfloor \quad \text{(Volkov, 2009).}$$

Similarly, if $C_A(n)$ denotes the restriction of the Černý function to the class of all aperiodic automata, we have:

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$$\text{(Ananichev, 2010)} \quad n + \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq C_A(n)$$

32. Ananichev's Series

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(Ananichev, 2010) $n+\left\lfloor\frac{n}{2}\right\rfloor-2 \le C_A(n) \le \frac{n(n-1)}{2}$ (Trahtman, 2007).

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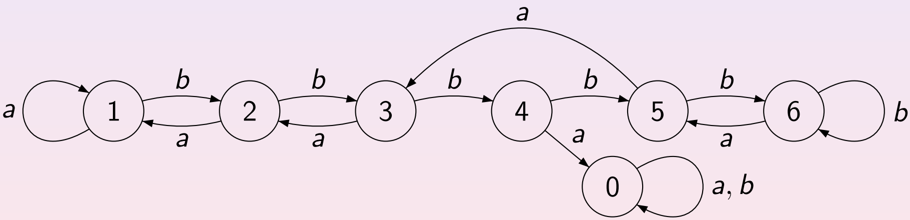

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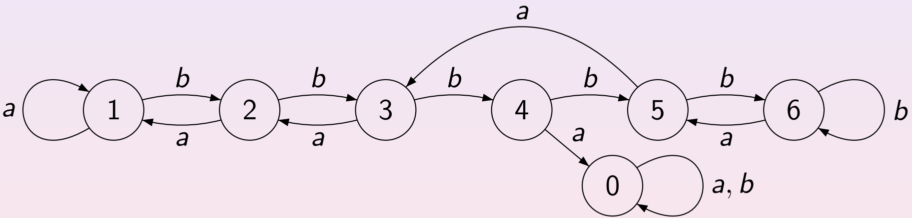

This is the first automaton in Ananichev's series that yields the best up to now lower bound for $C_A(n)$.

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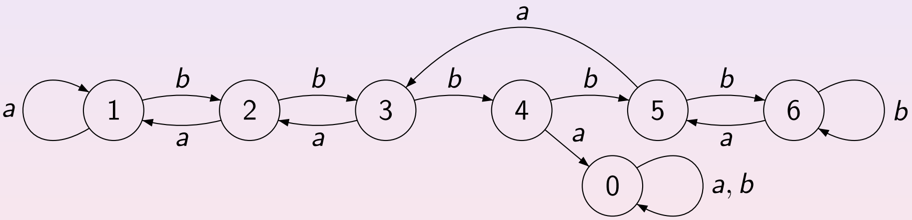

This is the first automaton in Ananichev's series that yields the best up to now lower bound for $C_A(n)$. It has 7 states and its shortest reset word is a^4b^3a of length $7+\left\lfloor\frac{7}{2}\right\rfloor-2=8$.