Synchronizing Finite Automata

Lecture VIII. The Road Coloring Theorem

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1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.
$|Q \,.\, w| = 1$. Here $Q \,.\, v = \{\delta(q, v) \mid q \in Q\}$.

Any w with this property is a reset word for $\mathscr{A}$.

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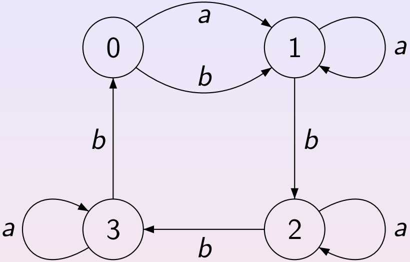

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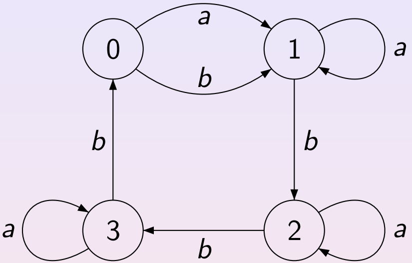
The Černý Conjecture: each synchronizing automaton with n states has a reset word of length $(n-1)^2$.



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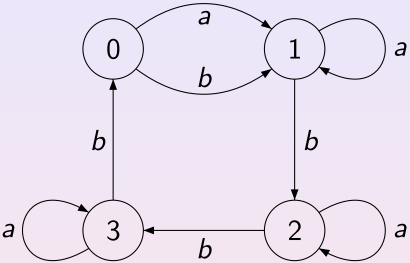

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3. Strongly Connected Digraphs

Studying synchronizing automata, it is natural to restrict to the strongly connected case. (For instance, it suffices to prove the Černý conjecture for this case, the general case would be an easy consequence.) Thus, we assume that our synchronizing automata are strongly connected as digraphs.

Observe that such an automaton can be reset to **any state**. That is, to every state q of the automaton one can assign an instruction (a reset word) w_q such that following w_q one will surely arrive at q from any initial state.

$ab^3ab^3ab^3$ ab^3ab^3a

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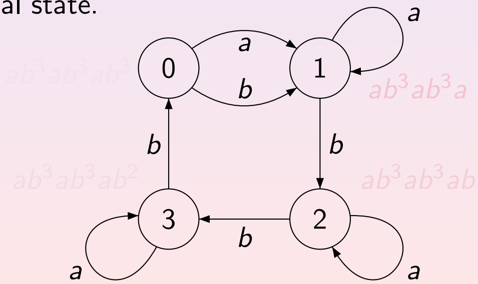

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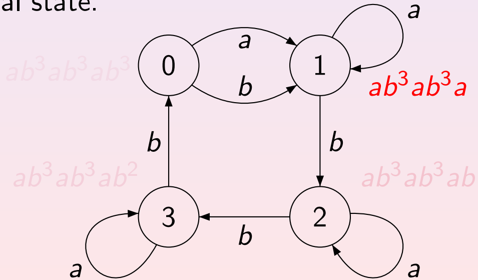

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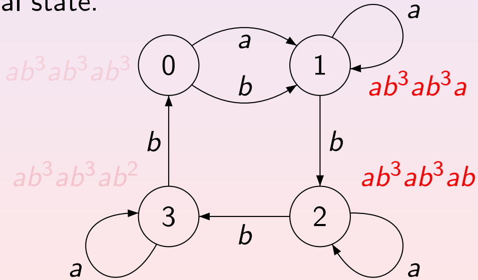

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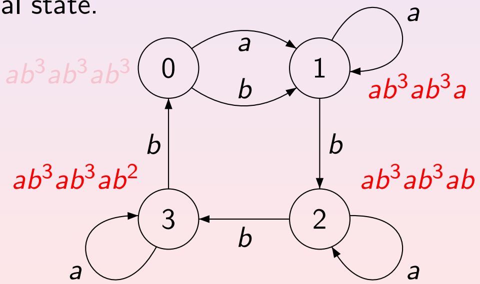

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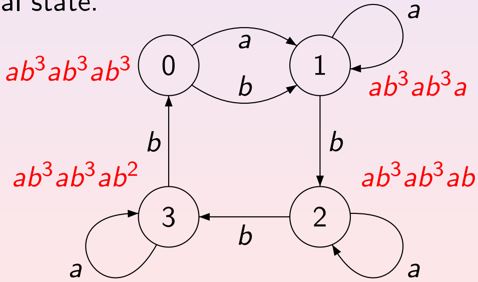
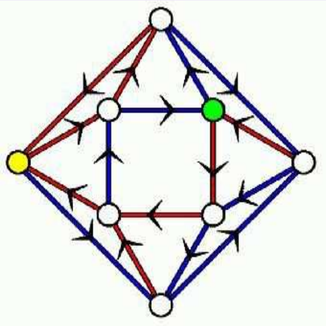


4. Example

Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.



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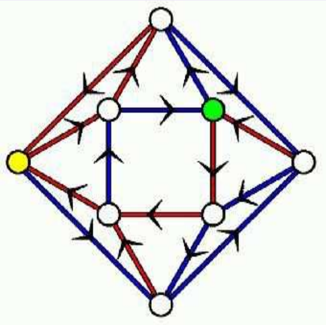

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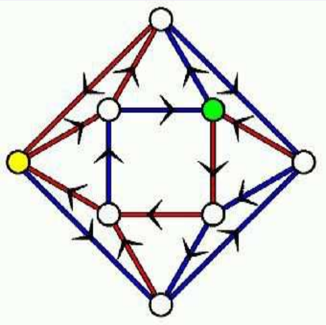

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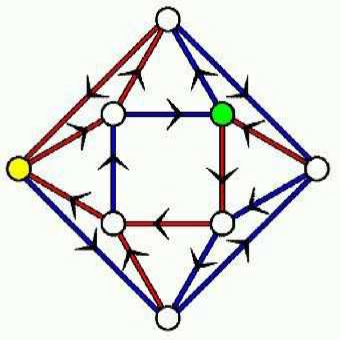
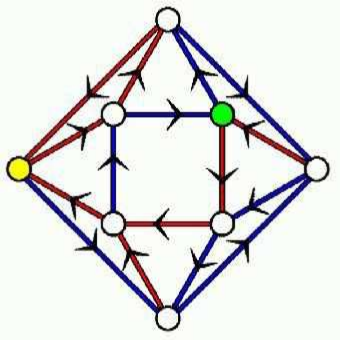


For the green node: blue-blue-red-blue-blue-red-blue-blue-red.

For the yellow node: blue-red-red-blue-red-red-blue-red-red.



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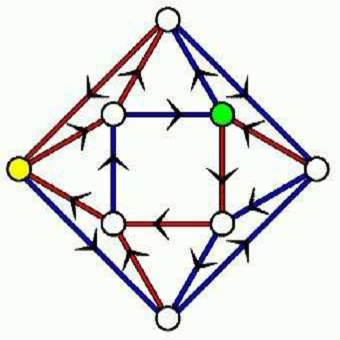

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Now suppose that we have a transport network, that is, a strongly connected digraph.

We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton.
When is such a coloring possible?
In other words: which strongly connected digraphs may appear as underlying digraphs of synchronizing automata?

An obvious necessary condition:
all vertices should have the same out-degree.
In what follows we refer to this as to the constant out-degree condition.

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A less obvious necessary condition is called aperiodicity (not to be confused with aperiodicity from Lecture VII!) or primitivity: *the g.c.d. of lengths of all cycles should be equal to 1*.

To see why primitivity is necessary, suppose that $\Gamma = (V, E)$ is a strongly connected digraph and $k > 1$ is a common divisor of lengths of its cycles. Take a vertex $v_0 \in V$ and, for $i = 0, 1, \dots, k-1$, let

$$V_i := \{v \in V \mid \exists \text{ path from } v_0 \text{ to } v \text{ of length } i \pmod{k}\}.$$

Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

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There is also a path from v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

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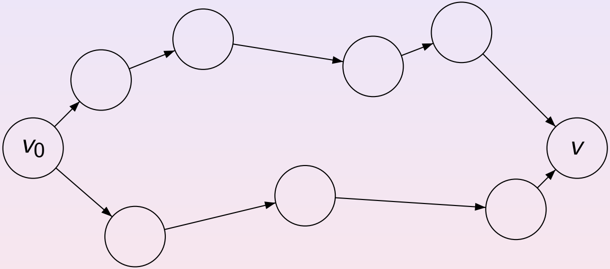

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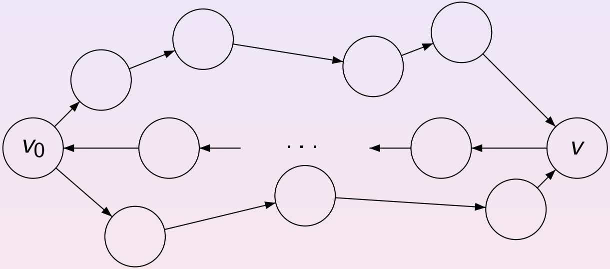

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Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate at the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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The Road Coloring Conjecture claims that the two necessary conditions (constant out-degree and primitivity) are in fact sufficient. In other words: *every strongly connected primitive digraph with constant out-degree admits a synchronizing coloring.*

The Road Coloring Conjecture was explicitly stated by Adler, Goodwyn and Weiss in 1977 (Equivalence of topological Markov shifts, Israel J. Math., 27, 49–63). In an implicit form it was present already in an earlier memoir by Adler and Weiss (Similarity of automorphisms of the torus, Memoirs Amer. Math. Soc., 98 (1970)).

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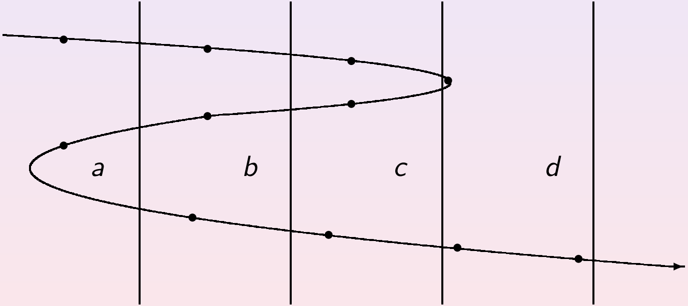

A symbolic trajectory $\ldots abcdcbabcdd \ldots$

12. Road Coloring Conjecture

An archetypical object of symbolic dynamics is the collection of all labels of the bi-infinite walks on a finite automaton (a sofic subshift).

However the conjecture is natural also from the viewpoint of the "reverse engineering" of synchronizing automata as presented here.

The Road Coloring Conjecture has attracted much attention. There were several interesting partial results, and finally the problem was solved (in the affirmative) in August 2007 by Avraham Trahtman. The solution is published in: The Road Coloring Problem, Israel J. Math. 172 (2009) 51–60. Trahtman's solution got much publicity.

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Trahtman's proof heavily depends on a neat idea of stability which is due to Karel Culik II, Juhani Karhumäki and Jarkko Kari (A note on synchronized automata and Road Coloring Problem, Int. J. Found. Comput. Sci., 13 (2002) 459–471). Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. We define the relation $\sim$ on Q as follows:

$$q \sim q' \iff \forall u \in \Sigma^* \ \exists v \in \Sigma^* \ q \, . \, uv = q'.uv.$$

$\sim$ is called the *stability relation* and any pair (q, q') such that $q \sim q'$ is called *stable*. It is immediate that $\sim$ is a congruence of the automaton $\mathscr{A}$. Also observe that

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Proposition CKK. *Suppose every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Then the Road Coloring Conjecture holds true.*

The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/{\sim}$ admits a synchronizing recoloring by the induction assumption. Then one lifts the correct coloring of $\mathscr{A}/\sim$ to a coloring of Γ in the most natural way: every edge gets the color of its image. Any reset word for the correct coloring of $\mathscr{A}/\sim$ brings all states of the new coloring of Γ into a single stability class.

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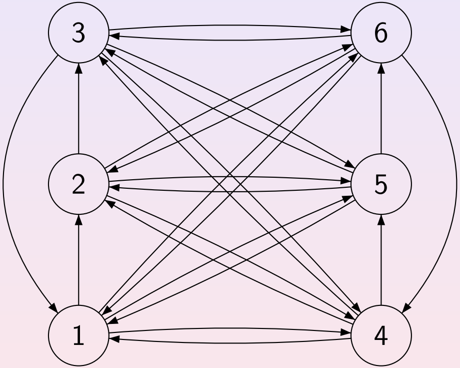

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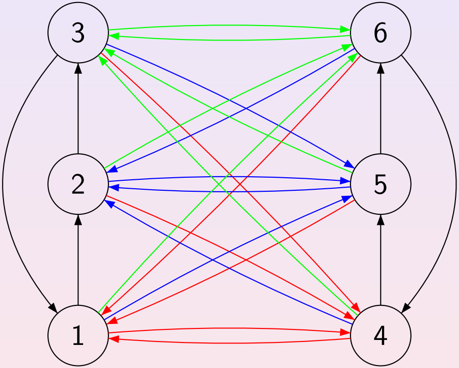

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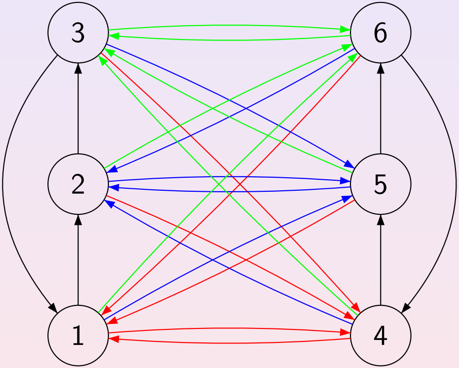

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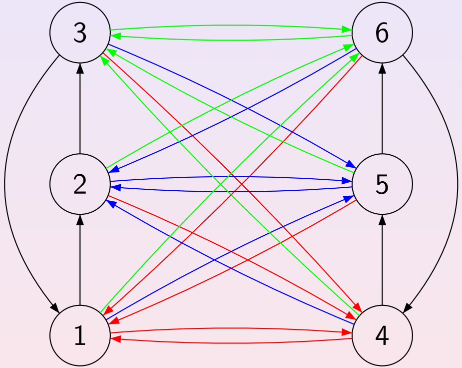

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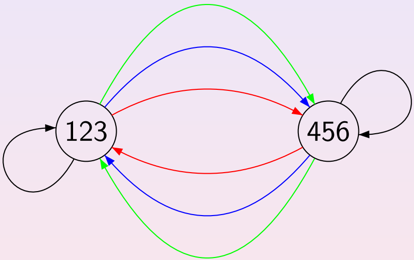
This is the quotient automaton of the above coloring. It is easy to recolor this quotient to get a synchronizing automaton.



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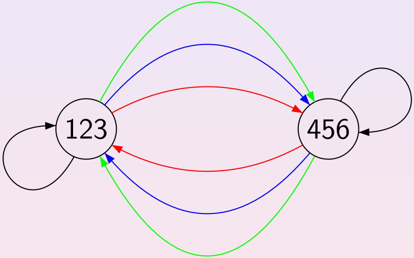

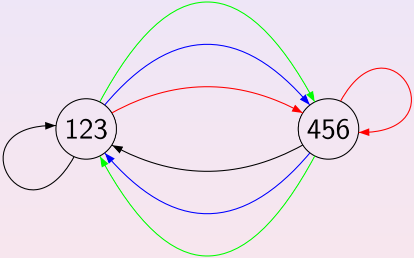

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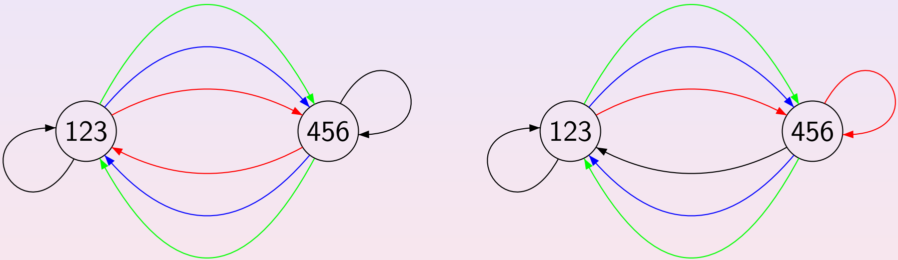

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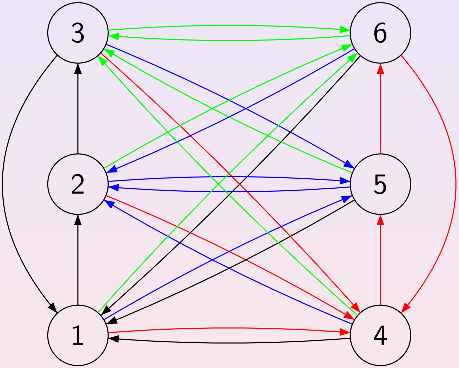

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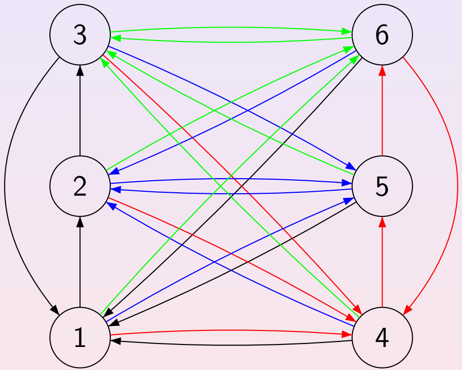

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18. Trahtman's Proof

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First, we need a couple of notions.

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. A pair (p, q) of distinct states is a deadlock if $\forall w \in \Sigma^* \ p \,.\, w \neq q \,.\, w$. If an automaton is not synchronizing, it must have deadlocks!

Moreover, if a pair (p, q) is not stable, then for some word $u \in \Sigma^*$ the pair $(p \,.\, u, q \,.\, u)$ is a deadlock.

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Lemma 1. *Let $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ be an automaton. If $F, G \subseteq Q$ are two cliques in $\mathscr{A}$ such that*

$$|F| - |F \cap G| = |G| - |F \cap G| = 1,$$

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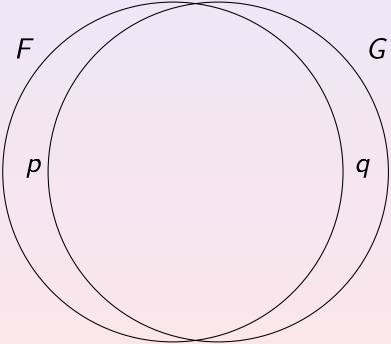

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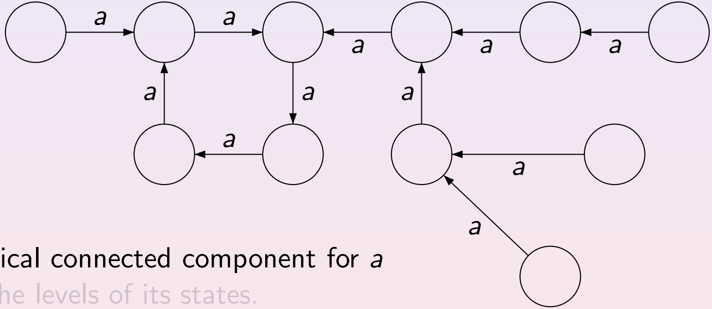

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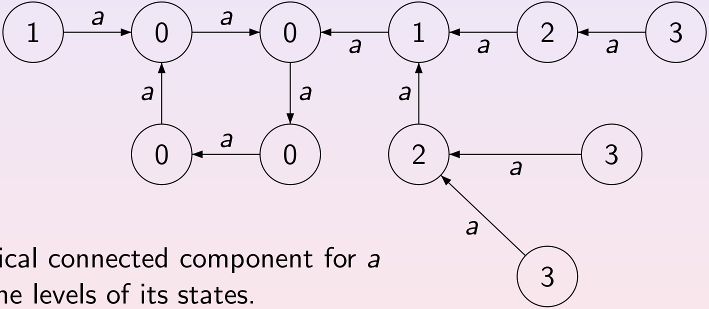

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Proof. Let M be the set of all states of level L w.r.t a. Then $p \,.\, a^L = q \,.\, a^L$ for all $p, q \in M$ whence no pair of states from M forms a deadlock. Thus, if $C \subseteq Q$ is a clique then $|C \cap M| \leq 1$. Take a clique C such that $|C \cap M| = 1$ (it exists since $\mathscr{A}$ is strongly connected). Then $F = C . a^{L-1}$ is a clique that has all its states except one in the a-cycles. If m is the l.c.m. of the lengths of all a-cycles, $r \,.\, a^m = r$ for any r in any a-cycle. Hence $G = F \,.\, a^m$ is a clique such that

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$$|F| - |F \cap G| = |G| - |F \cap G| = 1.$$

By Lemma 1 $\mathscr{A}$ has a stable pair.

23. Lemma on Level

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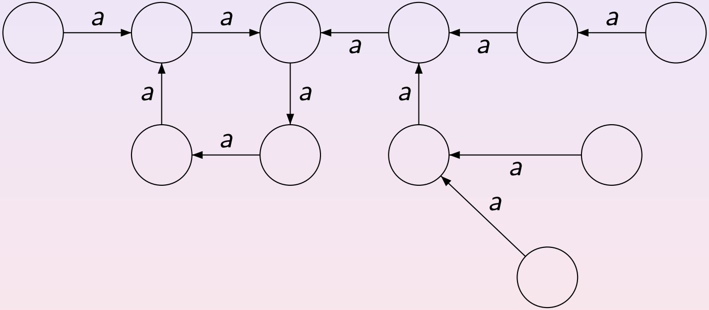
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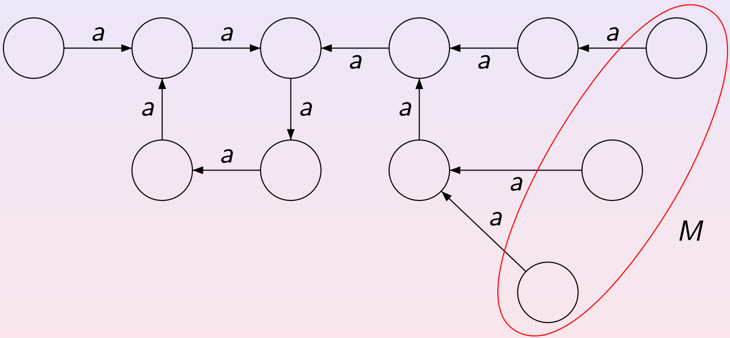
24. Lemma on Level



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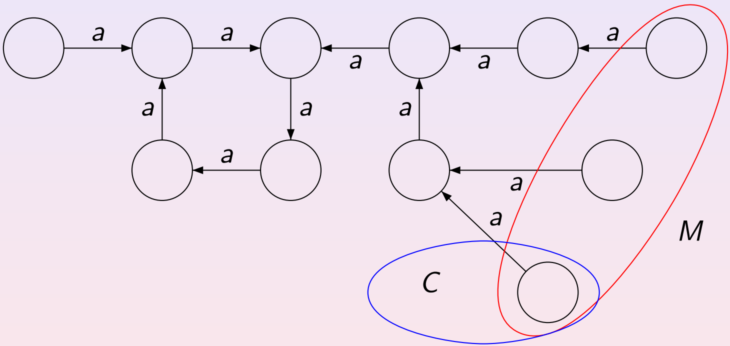

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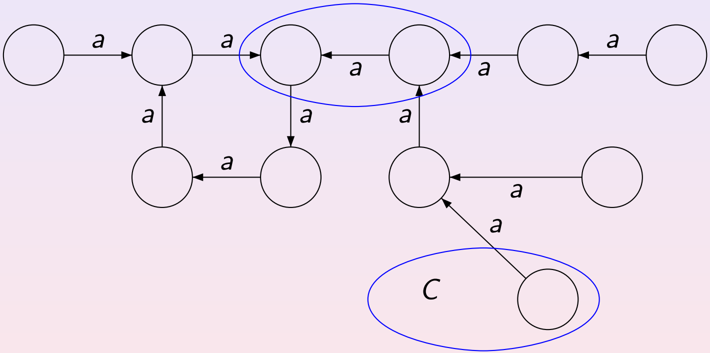
$$F = C \, . \, a^{L-1}$$



24. Lemma on Level

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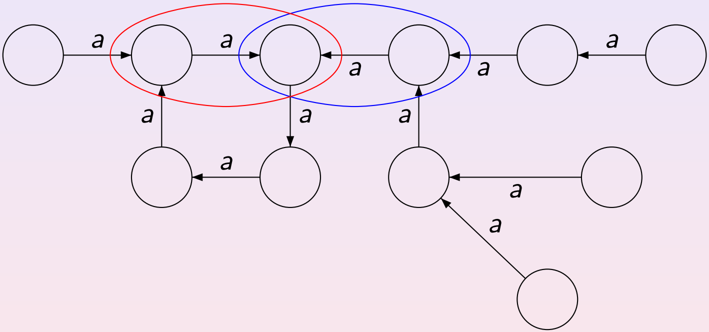

Recall, that we try to prove that every admissible digraph Γ has a stable coloring. By Lemma 2 for this it suffices to show that every such Γ may be colored into an automaton satisfying the premise of the lemma. This is, of course, much easier task because basically we only need to deal with one color, that is, with the action of one letter.

Take an arbitrary admissible digraph Γ. We start with an arbitrary coloring of Γ, take an arbitrary color (=letter) a, and induct on the number N of states that do not lie on any a-cycle in the initial coloring.

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Suppose that $N = 0$. This means that all states lie on a-cycles.

We say that a vertex p of Γ is a bunch if all edges that begin at p lead to the same vertex q.

If all vertices in Γ are bunches, then there is just one a-cycle (since Γ is strongly connected) and all cycles in Γ have the same length. This contradicts the assumption that Γ is primitive.

It is quite interesting that this is the only place in the whole proof where the primitivity condition is invoked.

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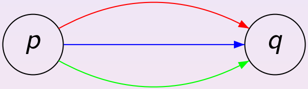

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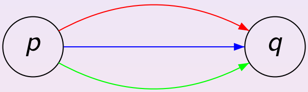

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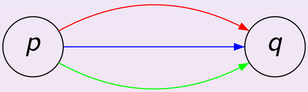

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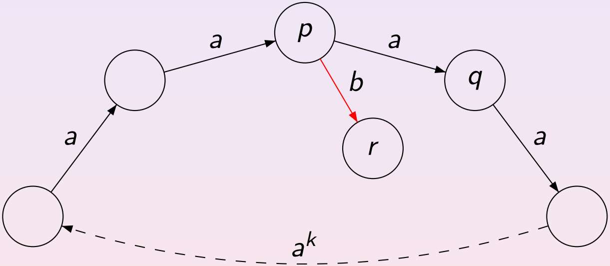
Thus, let p be a state which is not a bunch, let $q = p \,.\, a$ and let $b \neq a$ be such that $r = p \,.\, b \neq q$. We exchange the labels of the edges $p \xrightarrow{a} q$ and $p \xrightarrow{b} r$.



It is clear that in the new coloring there is only one state of maximal level w.r.t. a, namely q. Thus, the induction basis is verified.

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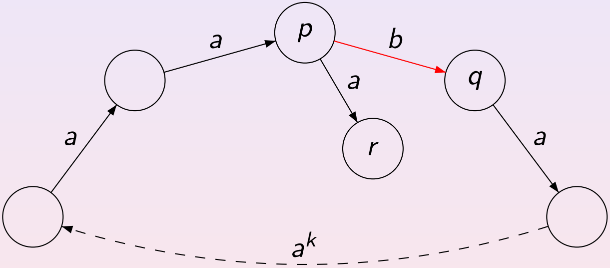

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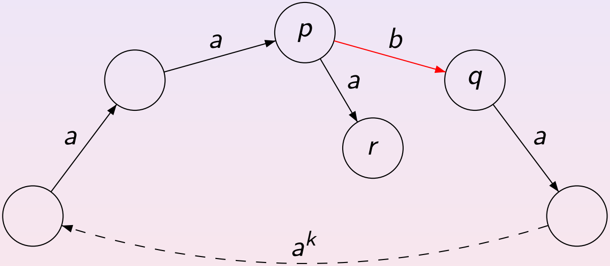

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