Synchronizing Automata

a problem everyone can understand but nobody can solve (so far)

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Can Modern Math be Understood?

"*Most current mathematical research, since the 60s, is devoted to fancy situations: it brings solutions which nobody understands to questions nobody asked*" (quoted from Bernard Beauzamy, "Real Life Mathematics", Irish Math. Soc. Bull. 48 (2002), 43-46).

This provocative claim is certainly exaggerated but it does reflect a really serious problem: a communication barrier between math (and exact science in general) and human common sense.

The barrier results in a paradox: while the achievements of modern mathematics are widely used in many crucial aspects of everyday life, people tend to believe that today mathematicians do "abstract nonsense" of no use at all.

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Well, you have proved Fermat's Last Theorem, congratulations!

Will my cows give more milk now?

Finite Automata

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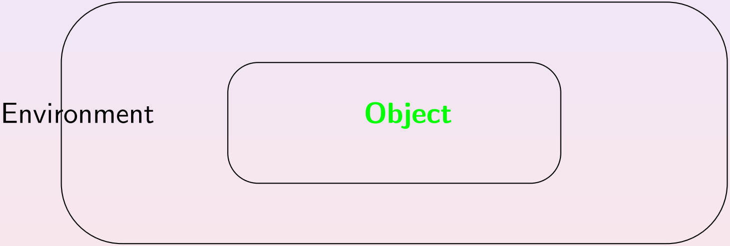

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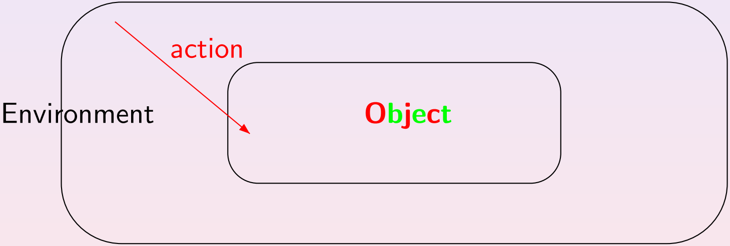

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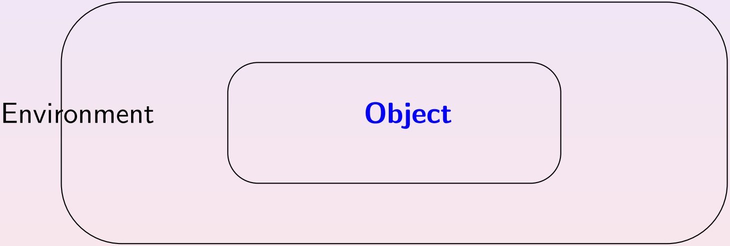

This notion originates in the seminal work by Alan Turing ("On Computable Numbers, With an Application to the Entscheidungsproblem", Proc. London Math. Soc., Ser. 2, 42 (1936), 230–265).

"The behavior of the computer at any moment is determined by the symbols which he is observing, and his state of mind at that moment".

Another important source is the work by neurobiologists Warren McCulloch and Walter Pitts ("A Logical Calculus of the Ideas Immanent in Nervous Activity", Bull. Math. Biophys. 5 (1943), 115–133).

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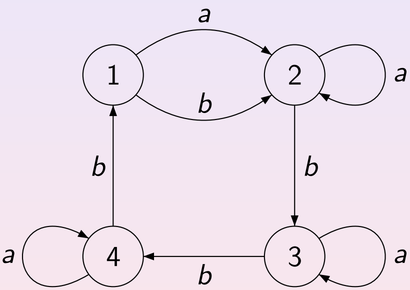

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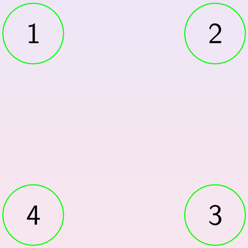

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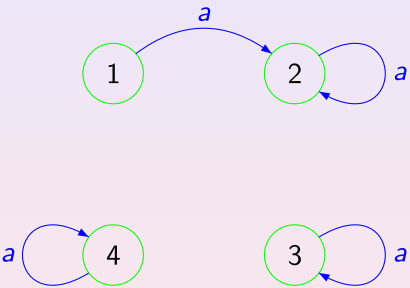

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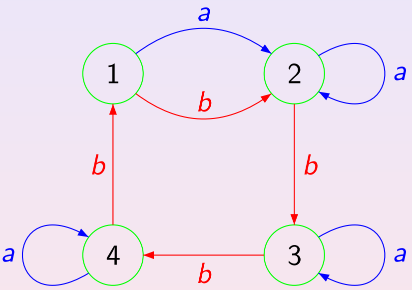

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Definitions and Terminology

We consider complete deterministic finite automata:

$$\mathscr{A} = \langle Q, \Sigma, \delta \rangle.$$

Here

- Q is the state set;
- Σ is the input alphabet;
- $\delta : Q \times \Sigma \to Q$ is the transition function.

We need neither initial nor final states.

Σ^* stands for the set of all words over Σ including the empty word.

The function δ uniquely extends to a function $Q \times \Sigma^* \to Q$ still denoted by δ.

To simplify notation we often write $q \,.\, w$ for $\delta(q, w)$ and $P \,.\, w$ for $\{\delta(q, w) \mid q \in P\}$.

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An automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

We can also write this as $|Q \,.\, w| = 1$.

Any word w with this property is a reset word for $\mathscr{A}$.

Other names:

- for automata: directable, cofinal, collapsible, etc;
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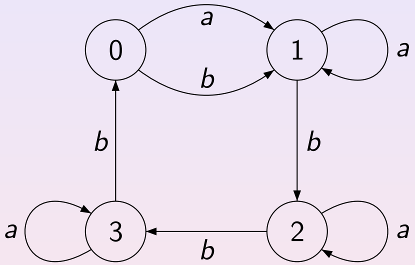

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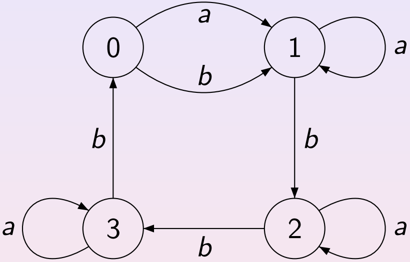

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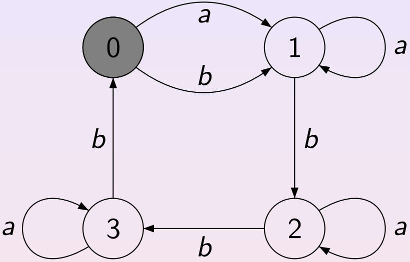

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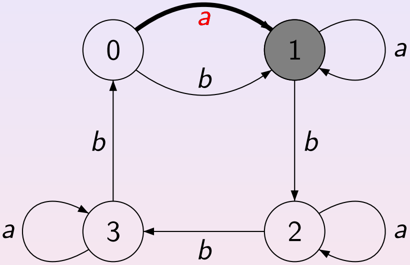

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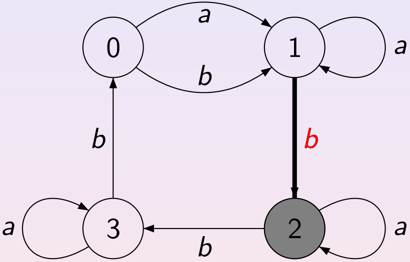

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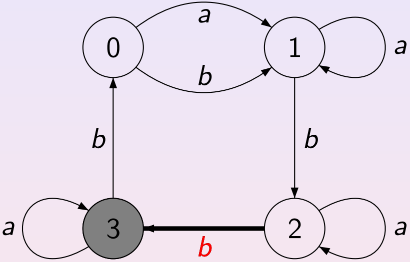

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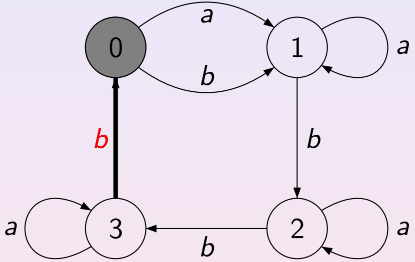

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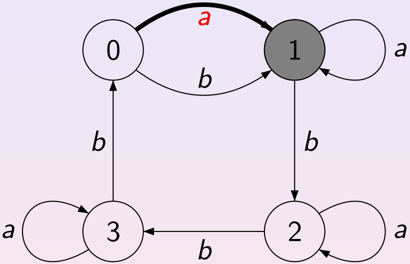

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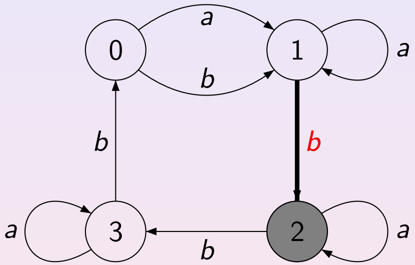

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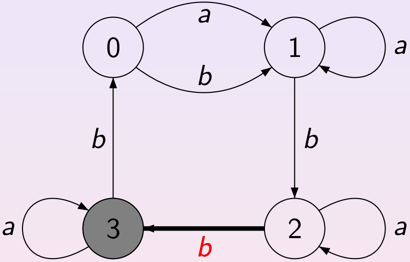

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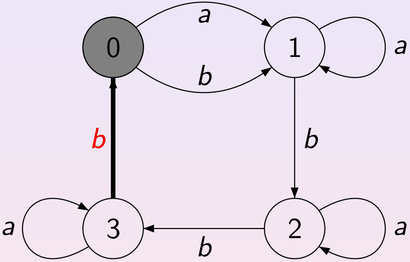

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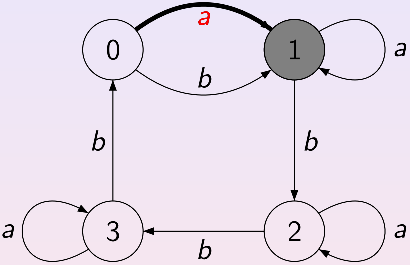

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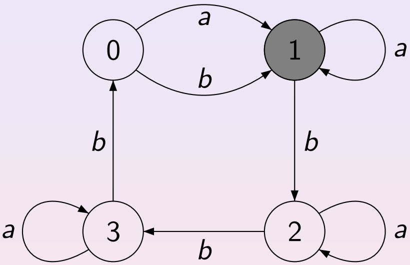

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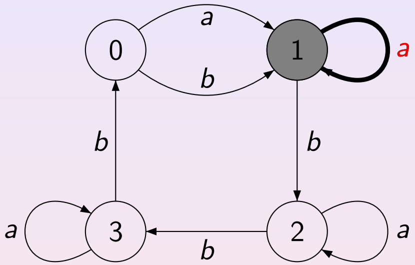

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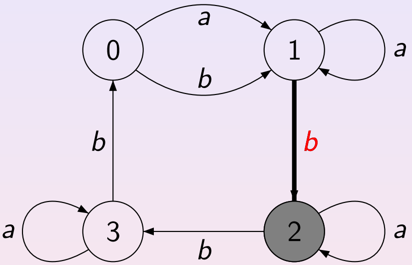

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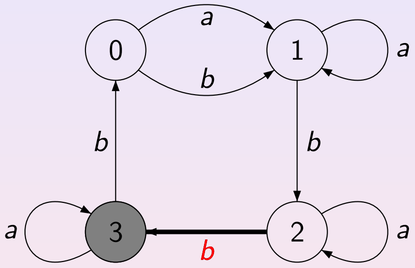

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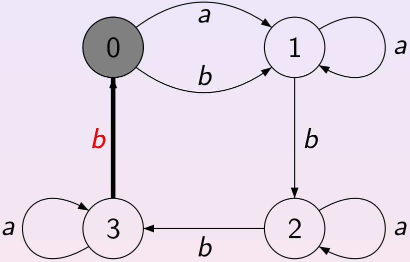

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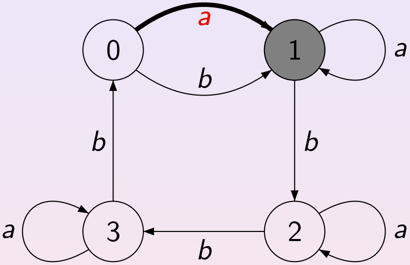

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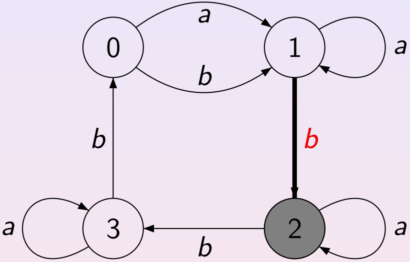

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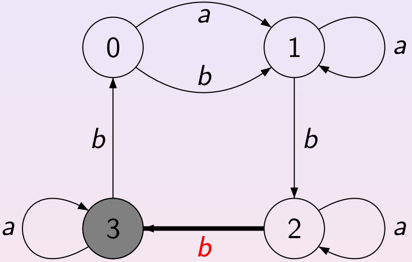

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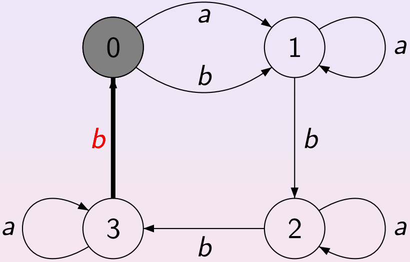

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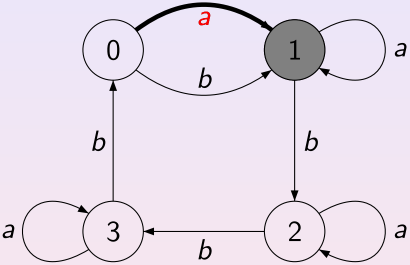

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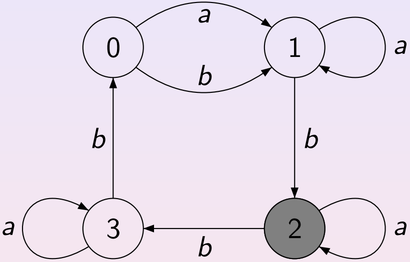

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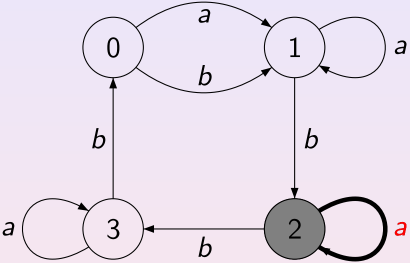

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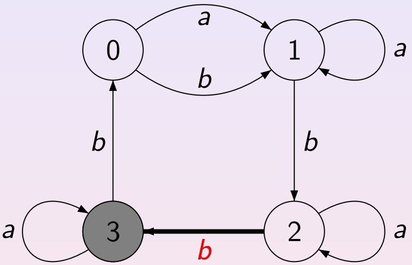

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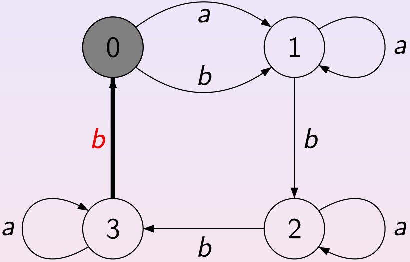

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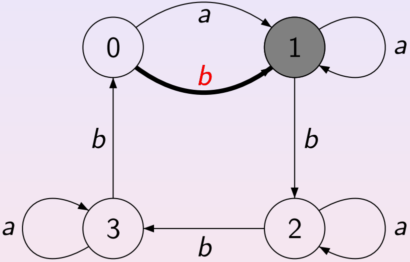

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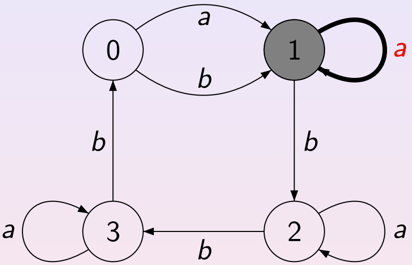

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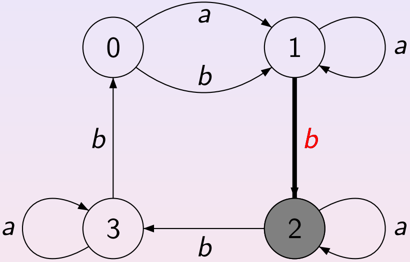

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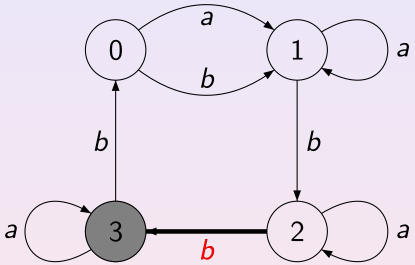

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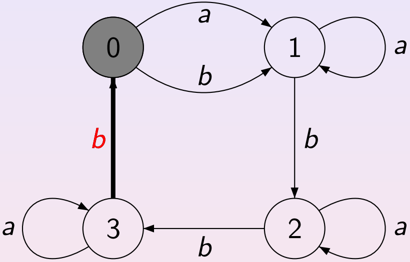

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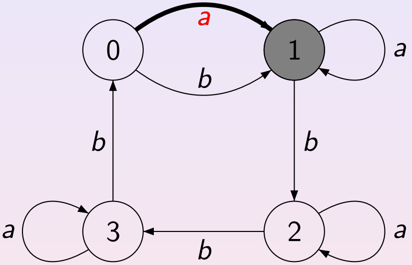

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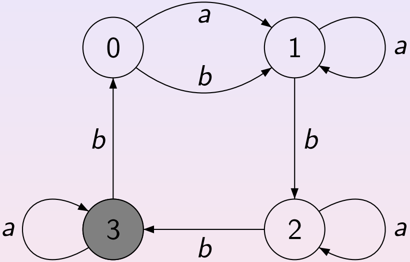

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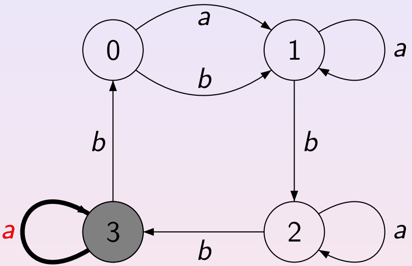

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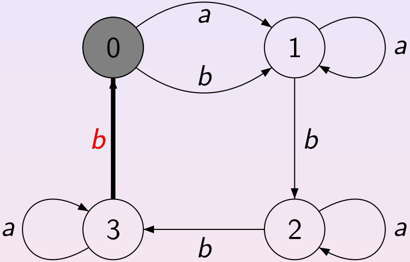

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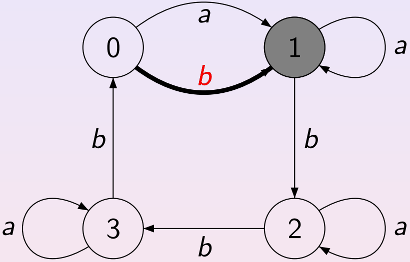

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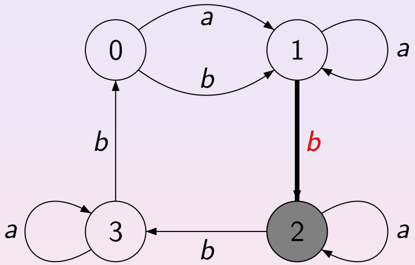

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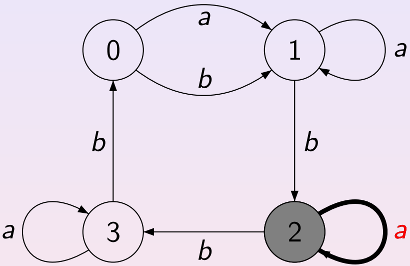

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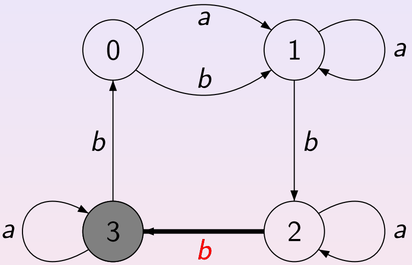

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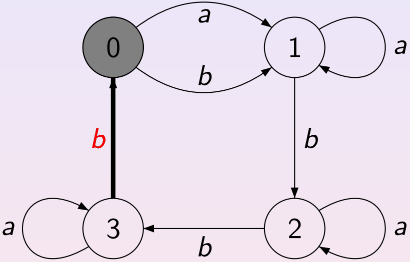

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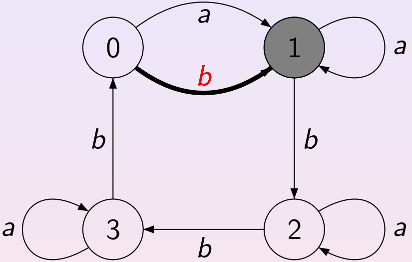

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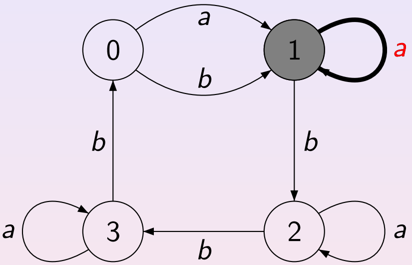

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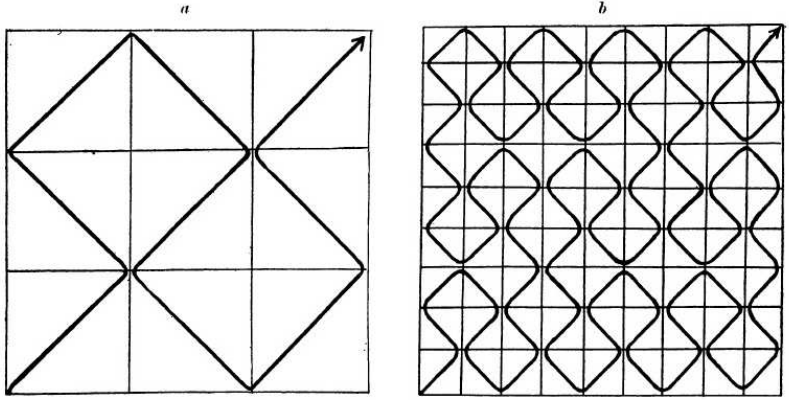
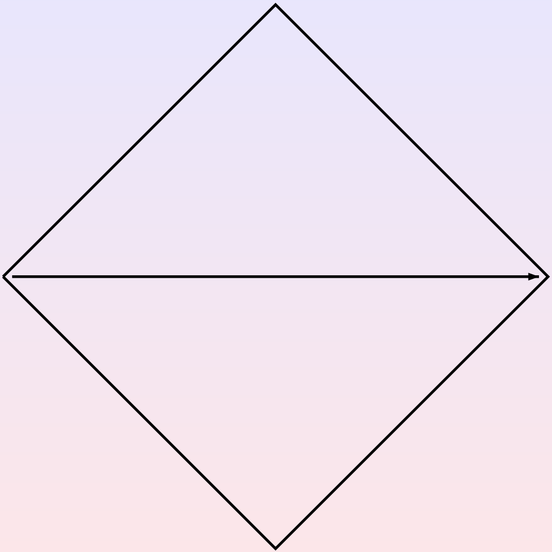


Line $C_1^{(3)}$ Line $C_2^{(3)}$

Moore's Curve as Substitution



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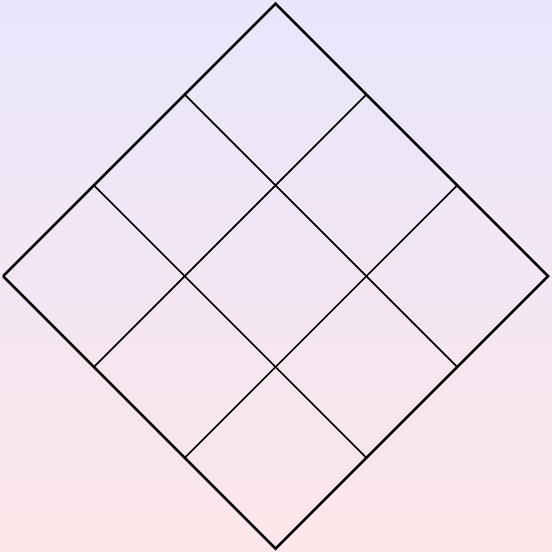

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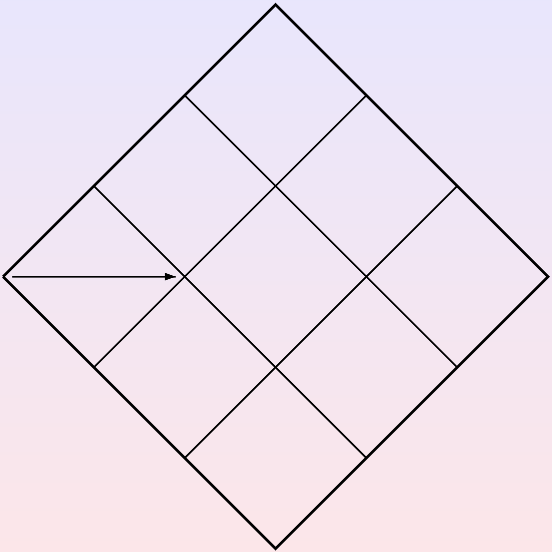

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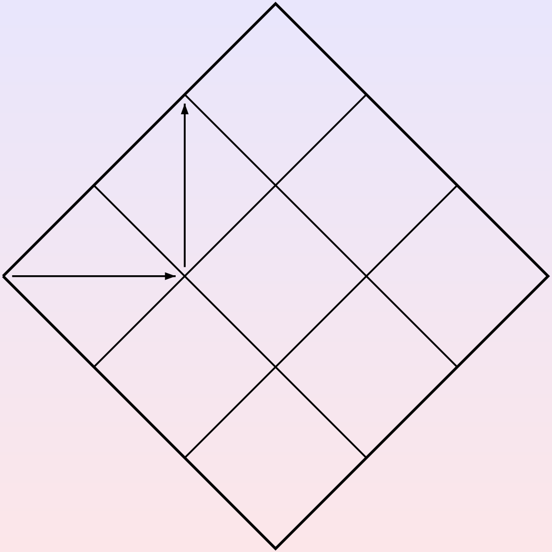

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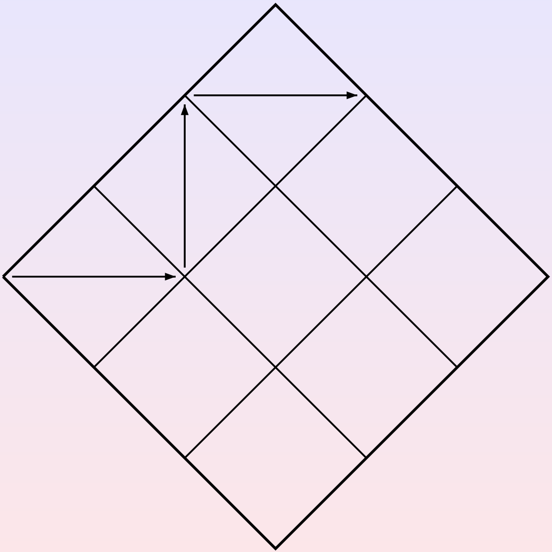

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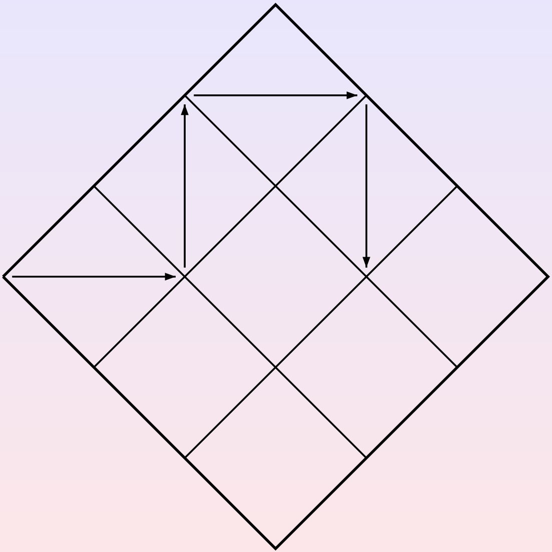

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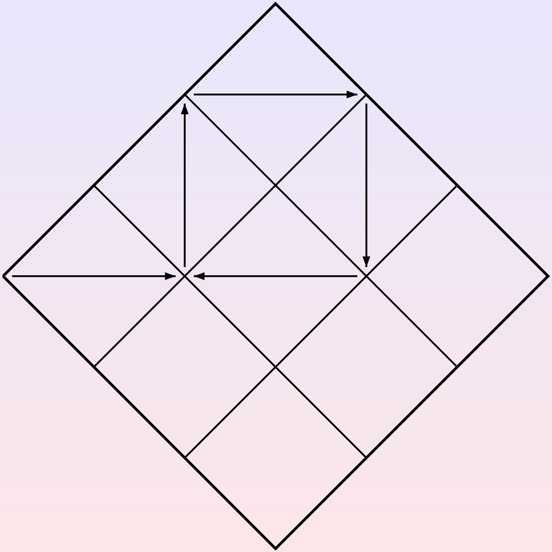

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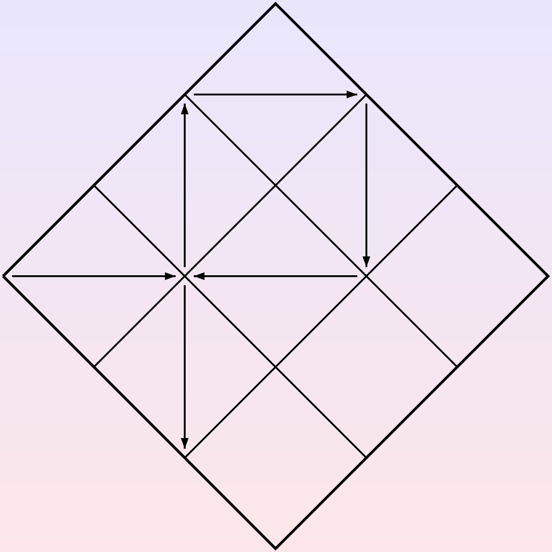

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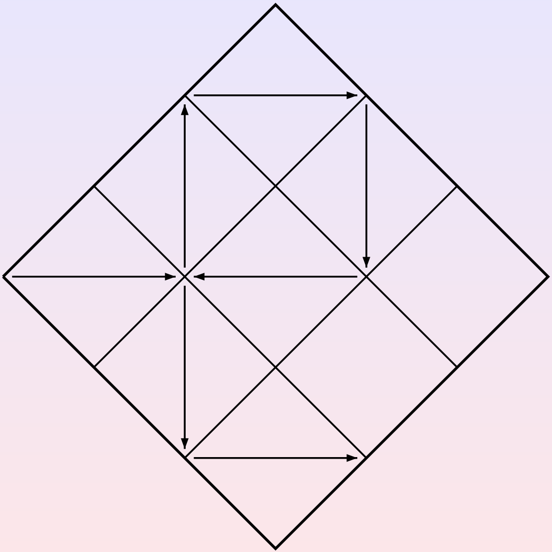

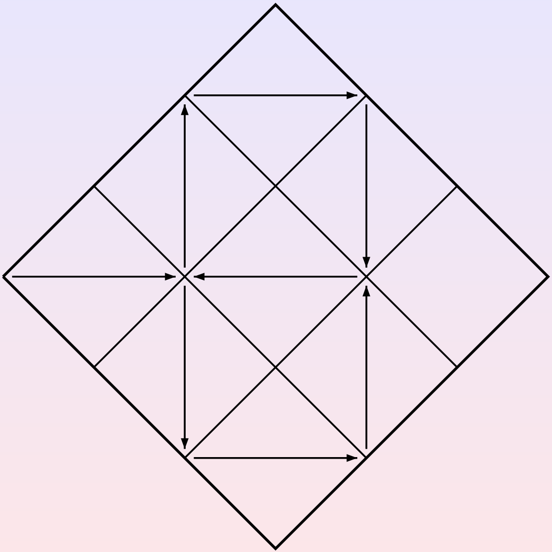

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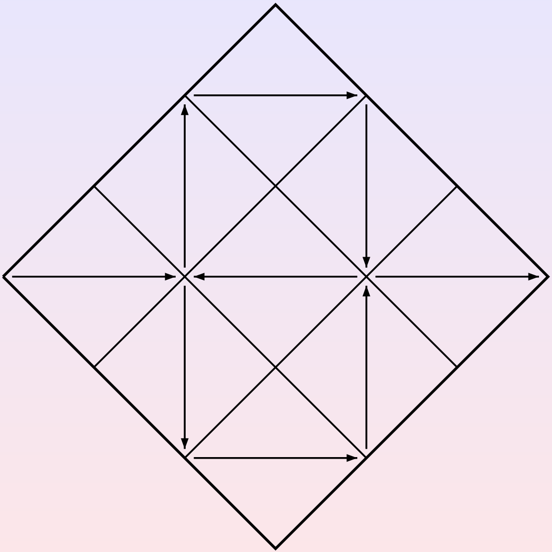

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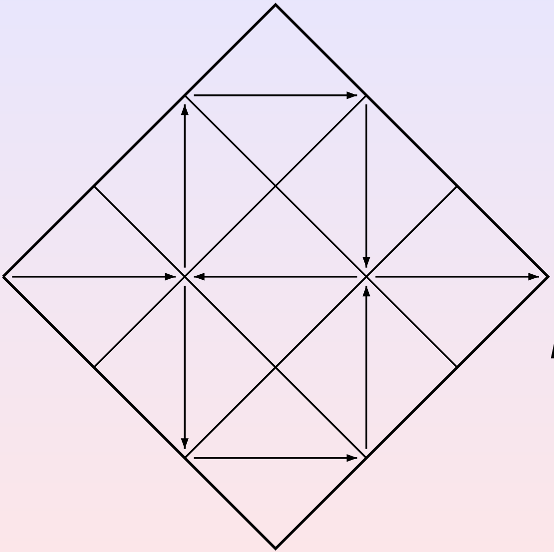

$E \mapsto ENE\ SWS\ ENE$

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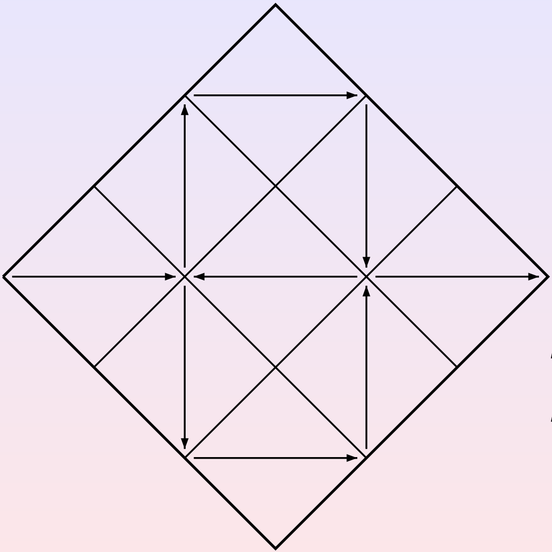

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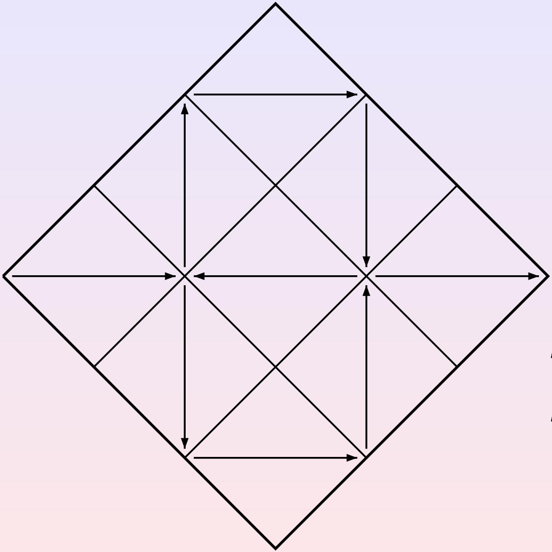

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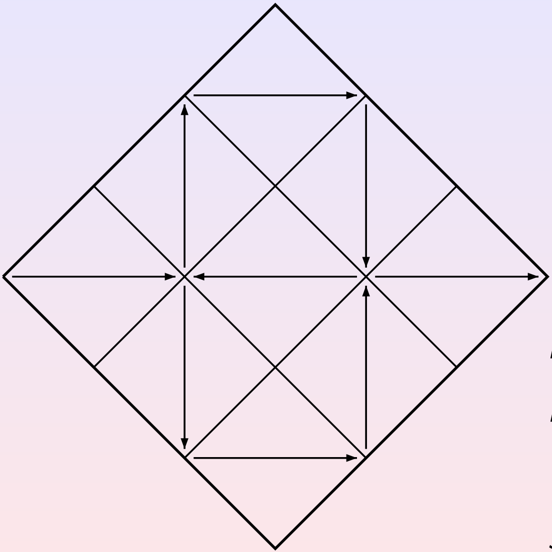

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Thus, τ satisfies the coincidence condition (with $k = 4$, $m = 7$). The coincidence condition completely characterizes the constant length substitutions that give rise to dynamical systems measure-theoretically isomorphic to a translation on a compact Abelian group (Dekking, 1978).

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2	$\mapsto$	20	$\mapsto$	2011	$\mapsto$	20111212	$\mapsto$	201112**1**212201220

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A substitution on a finite alphabet X is a map $\sigma : X \to X^+$; the substitution is said to be of constant length if all words $\sigma(x)$, $x \in X$, have the same length. One says that σ satisfies the coincidence condition if there exist positive integers m and k such that all words $\sigma^k(x)$ have the same letter in the m-th position. For an example, consider the substitution τ on $X = \{0, 1, 2\}$ defined by $0 \mapsto 11,\ 1 \mapsto 12,\ 2 \mapsto 20$. Calculate the iterations of τ up to τ^4:

$$\begin{array}{ccccccccc} 0 & \mapsto & 11 & \mapsto & 1212 & \mapsto & 12201220 & \mapsto & 122020\mathbf{1}112202011 \\ 1 & \mapsto & 12 & \mapsto & 1220 & \mapsto & 12202011 & \mapsto & 122020\mathbf{1}120111212 \\ 2 & \mapsto & 20 & \mapsto & 2011 & \mapsto & 20111212 & \mapsto & 201112\mathbf{1}212201220 \end{array}$$

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Substitution Systems = DFAs

There is a straightforward bijection between DFAs and constant length substitutions. Each DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ with $\Sigma = \{a_1, \ldots, a_\ell\}$ defines a length ℓ substitution on Q that maps every $q \in Q$ to the word $(q \, . \, a_1) \ldots (q \, . \, a_\ell) \in Q^+$.

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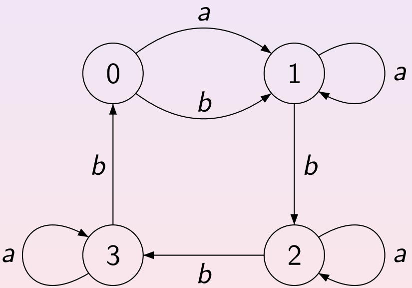

induces the substitution $0 \mapsto 11,\ 1 \mapsto 12,\ 2 \mapsto 23,\ 3 \mapsto 30.$

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Conversely, each substitution $\sigma : X \to X^+$ such that all words $\sigma(x)$, $x \in X$, have the same length ℓ gives rise to a DFA for which X is the state set and which has ℓ input letters $a_1, \dots, a_\ell$ acting on X as follows: $x \,.\, a_i$ is the symbol in the i-th position of the word $\sigma(x)$.

Under this bijection substitutions satisfying the coincidence condition correspond precisely to synchronizing automata, and moreover, given a substitution, the number of iterations at which the coincidence first occurs is equal to the minimum length of reset word for the corresponding automaton.

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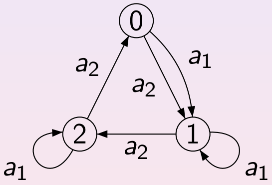

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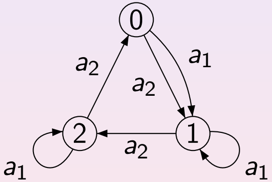

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The Černý Series

Suppose a synchronizing automaton has n states. What is its reset threshold, i.e., the minimum length of its reset words?

In his 1964 paper Jan Černý constructed a series $\mathscr{C}_n$, $n = 2, 3, \ldots$, of synchronizing automata over 2 letters.

The states of $\mathscr{C}_n$ are the residues modulo n, and the input letters a and b act as follows:

$$\delta(0, a) = 1,\ \ \delta(m, a) = m \text{ for } 0 < m < n,$$

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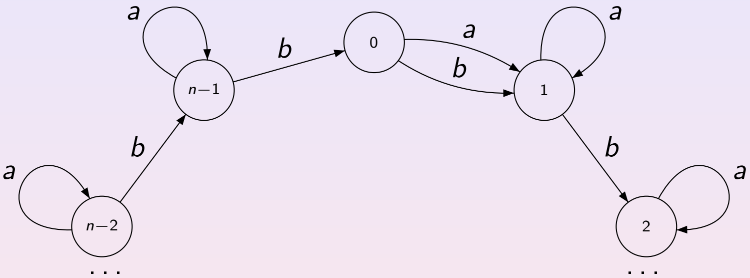

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$. As other results from Černý's paper of 1964, this nice series of automata has been rediscovered many times.

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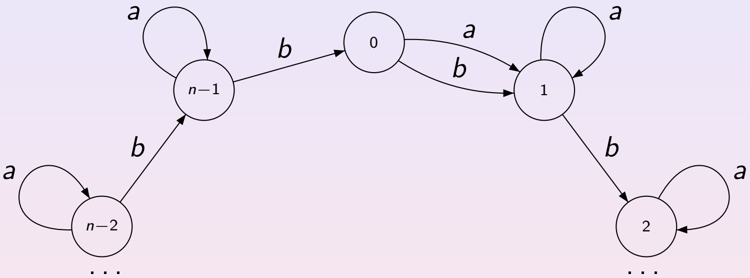

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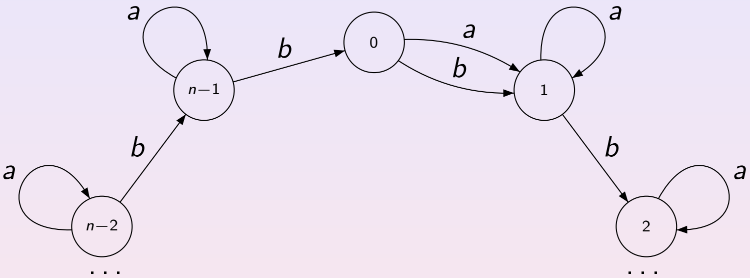

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Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, $n = 2, 3, \ldots$, yields the inequality $C(n) \geq (n-1)^2$.

The Černý conjecture is the claim that in fact the equality $C(n) = (n-1)^2$ holds true. This simply looking conjecture is arguably the most longstanding open problem in the combinatorial theory of finite automata. Everything we know about the conjecture in general can be summarized in one line:

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Thus, the pattern is:

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Exponents of Non-negative Matrices

A non-negative matrix A is said to be primitive if some power A^k is positive. The minimum k with this property is called the exponent of A, denoted $\exp A$.

Helmut Wielandt proved in 1950 that for any primitive $n \times n$-matrix A, one has $\exp A \leq n^2 - 2n + 2 = (n-1)^2 + 1$, and this bound is tight. Possible exponents of $n \times n$-matrices were intensively studied in the 1960s, and it was discovered that two extreme values are each attained by a unique matrix, then there is a gap followed by an island followed by another gap. The sizes of the gaps and of the island perfectly match the sizes of the gaps and of the islands in possible reset thresholds of synchronizing automata with n states — basically one has the same picture shifted by 1. Clearly, this cannot be a mere coincidence.

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Digraphs and Matrices

A directed graph (digraph) is a pair $D = \langle V, E \rangle$.

- V set of vertices
- $E \subseteq V \times V$ set of edges

This definition allows loops but excludes multiple edges.

The matrix of a digraph $D = \langle V, E \rangle$ is just the incidence matrix of the edge relation, that is, a $V \times V$-matrix whose entry in the row v and the column v' is 1 if $(v, v') \in E$ and 0 otherwise.

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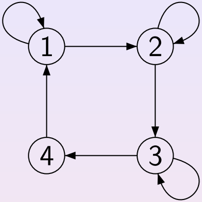

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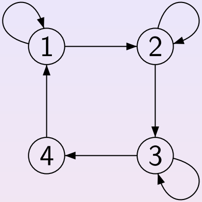

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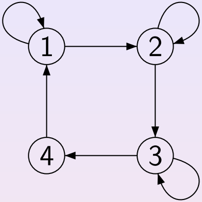

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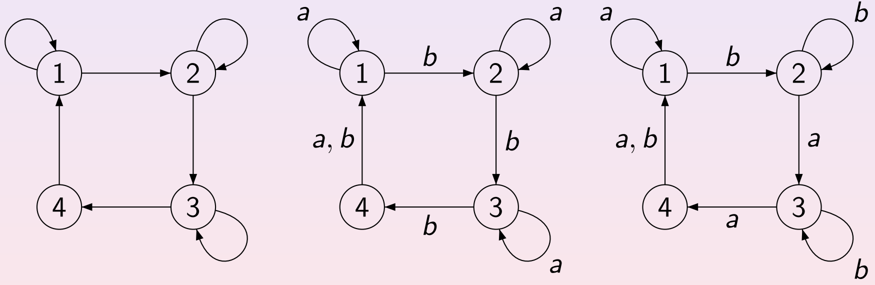

A digraph D is primitive if D is strongly connected and the greatest common divisor of the lengths of all cycles in D is equal to 1.

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Exponents vs Reset Lengths

Exponents of primitive digraphs with 9 vertices vs reset thresholds of 2-letter strongly connected synchronizing automata with 9 states

N	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51
# of primitive digraphs with exponent N	1	1	0	0	0	0	0	1	1	2	0	0	0	0	3
# of 2-letter synchronizing automata with reset threshold N	0	1	0	0	0	0	0	1	2	3	0	0	0	4	4

The Wielandt Automaton

The Wielandt automaton $\mathscr{W}_n$ is a (unique) coloring of the Wielandt digraph W_n with $\gamma(W_n) = (n-1)^2 + 1$. The Wielandt digraph has n vertices $1, 2, \ldots, n$, say, and the following $n+1$ edges: $(i, i+1)$ for $i = 1, \ldots, n-1$, $(n, 1)$, and $(n, 2)$.

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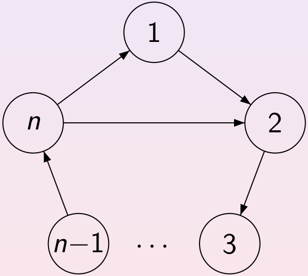

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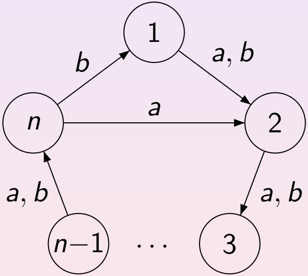

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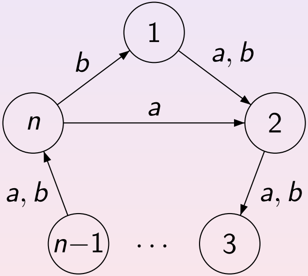

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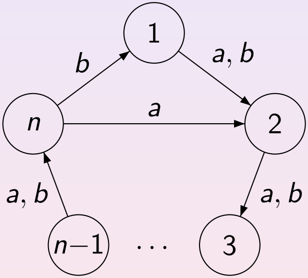

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In a similar way, each digraph with large exponent generates slowly synchronizing automata.

Observation

Let a strongly connected synchronizing automaton with n states and reset threshold t be a coloring of a digraph D. Then

$$\gamma(D) \leq t + n - 1.$$

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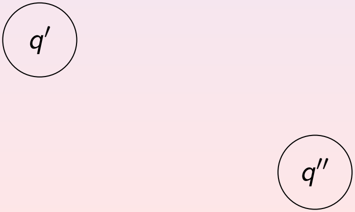

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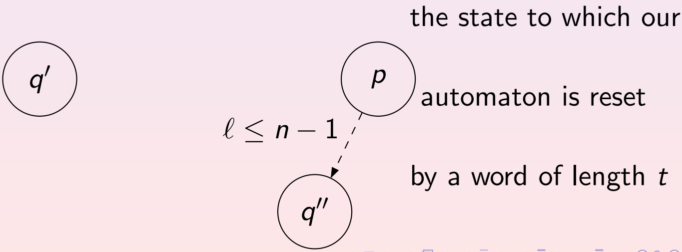

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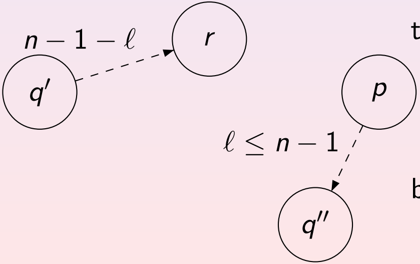

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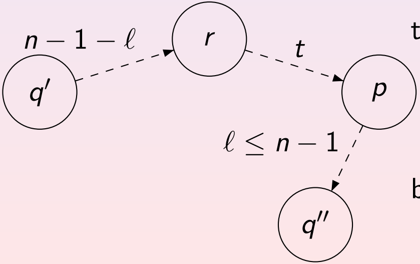

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$$\gamma(D) \leq t + n - 1.$$

For instance, the reset threshold t of the Wielandt automaton $\mathscr{W}_n$ must satisfy

$$t \geq \gamma(W_n) - n + 1 = (n-1)^2 + 1 - n + 1 = n^2 - 3n + 3,$$

and it is easy to find a reset word of length $n^2 - 3n + 3$.

The Černý Automaton

There are slowly synchronizing automata that cannot be obtained as colorings of a digraph with large exponent. For instance, the Černý automaton $\mathscr{C}_n$ has reset threshold $(n-1)^2$ while its underlying digraph has exponent $n-1$.

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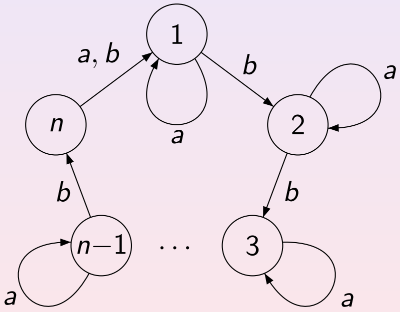

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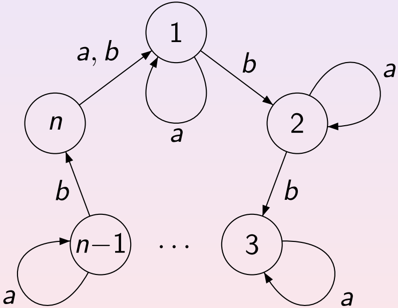

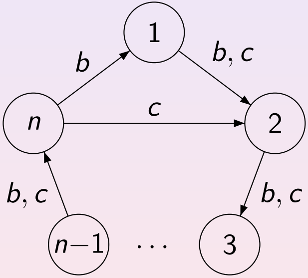

However, $\mathscr{C}_n$ becomes $\mathscr{W}_n$ under the action of b and $c = ab$.

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Let w be a shortest reset word for $\mathscr{C}_n$. It must end with a and every other occurrence of a in w is followed by an occurrence of b. Thus, $w = w'a$ where w' can be rewritten into a word v over the alphabet $\{b, c\}$. Since w' and v act in the same way, the word vc is a reset word for $\mathscr{W}_n$. Hence $|v| \geq n^2 - 3n + 2$.

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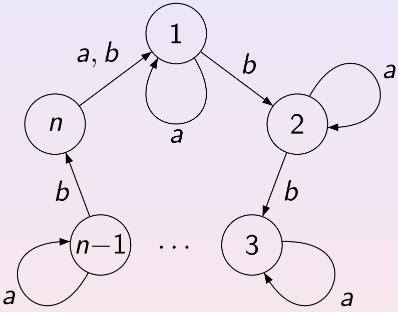

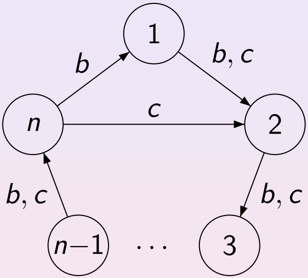

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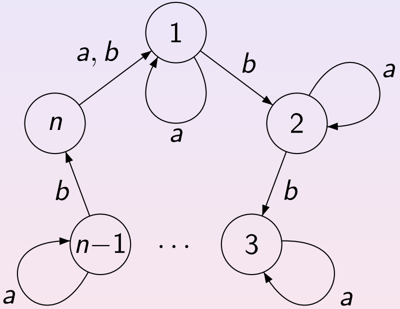

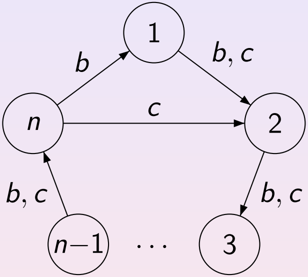

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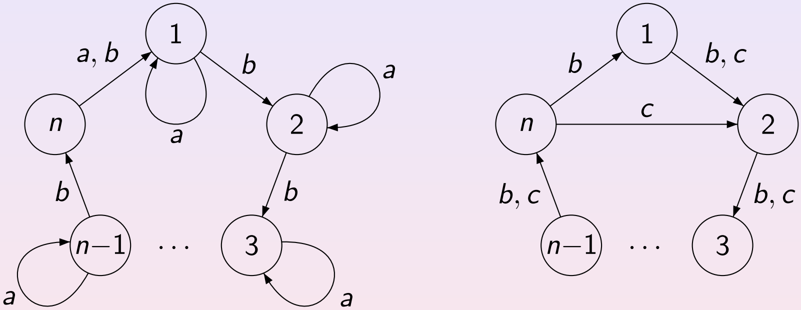

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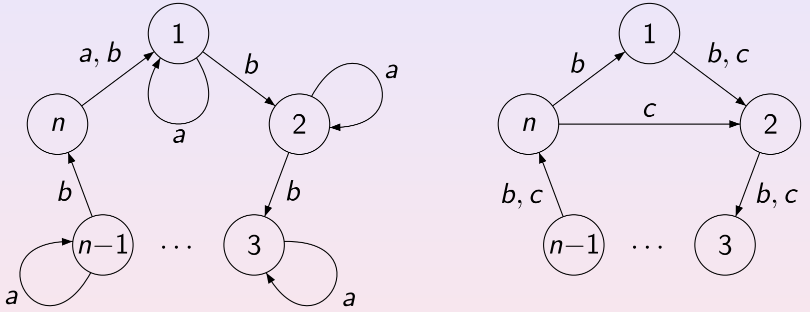

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Thus, it is the Wielandt digraph that stays behind the Černý automaton!

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Digraphs vs Automata

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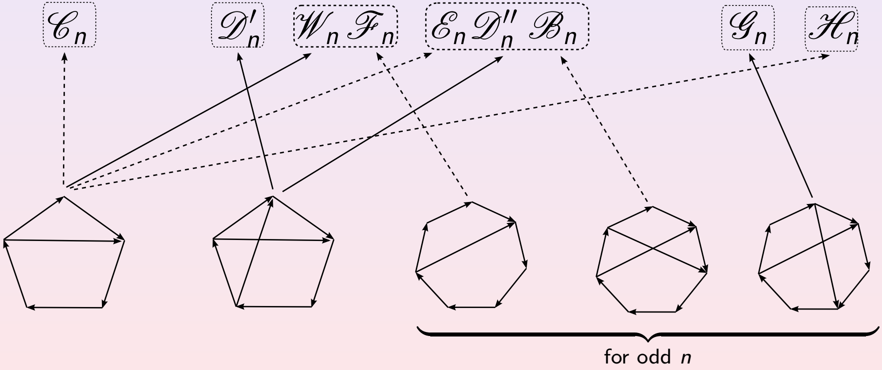

New Conjectures

(a) (The Černý conjecture) The reset threshold of every synchronizing n-automaton does not exceed $(n-1)^2$.

(b) If $n > 6$, then there exists exactly one n-automaton with reset threshold $(n-1)^2$, namely, $\mathscr{C}_n$.

(c) If $n > 6$, then there exists no n-automaton whose reset threshold is greater than $n^2 - 3n + 4$ but less than $(n-1)^2$.

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