The Road Coloring Theorem

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Synchronizing Automata

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

$|Q \,.\, w| = 1$. Here $Q \,.\, v = \{\delta(q, v) \mid q \in Q\}$.

Any w with this property is a reset word for $\mathscr{A}$.

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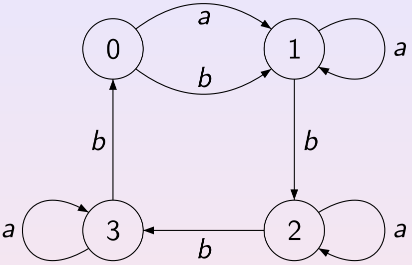

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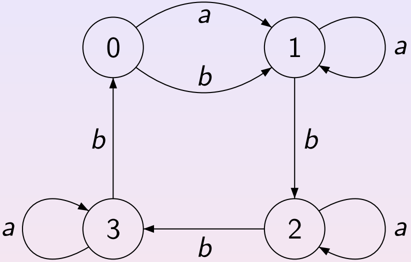

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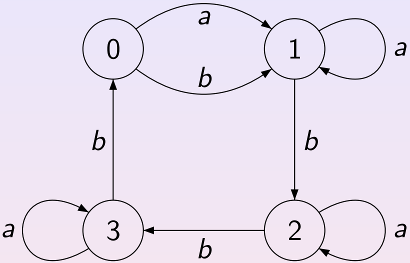

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Strongly Connected Digraphs

Studying synchronizing automata, it is natural to restrict to the strongly connected case. For instance, it suffices to prove the Černý conjecture for this case; the general case follows easily. Thus, we assume that our synchronizing automata are strongly connected as digraphs. Observe that such an automaton can be reset to any state. That is, to every state q of the automaton one can assign an instruction (a reset word) w_q such that following w_q one will surely arrive at q from any initial state.

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$$
\begin{tikzcd}
0 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 1 \arrow[loop right, "a"] \arrow[d, "b"] \\
3 \arrow[u, "b"] \arrow[loop left, "a"] & 2 \arrow[l, "b"] \arrow[loop right, "a"]
\end{tikzcd}
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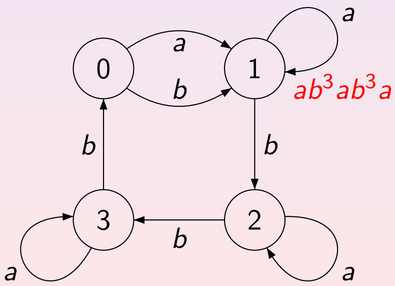

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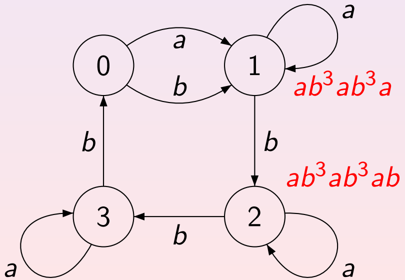

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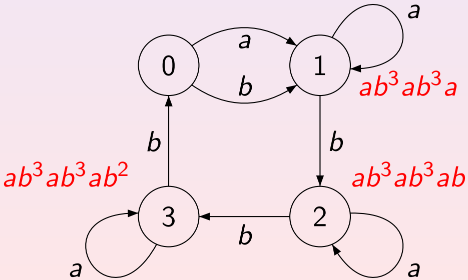

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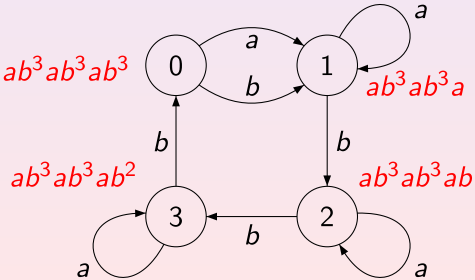
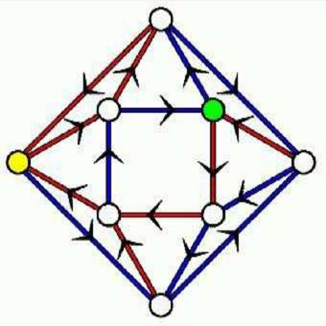


Example

Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.



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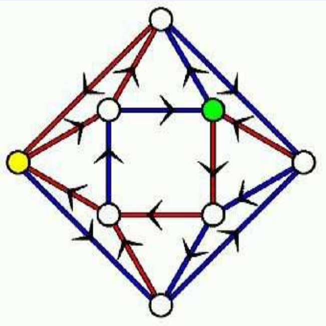

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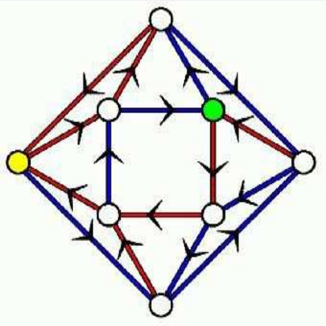

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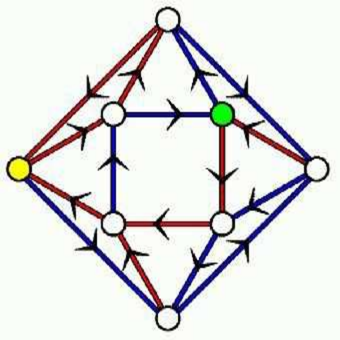

For the green node: blue-blue-red-blue-blue-red-blue-blue-red.

For the yellow node: blue-red-red-blue-red-red-blue-red-red.

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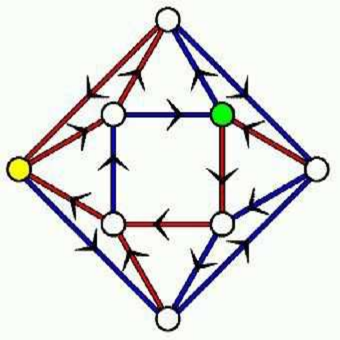

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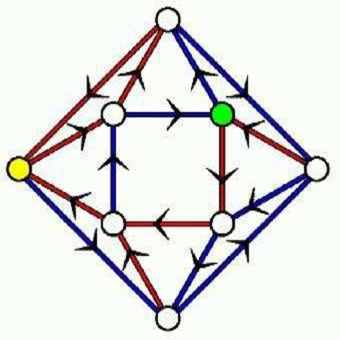

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Now suppose that we have a transport network, that is, a strongly connected digraph.

We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton. When is such a coloring possible?
In other words: which strongly connected digraphs may appear as underlying digraphs of synchronizing automata?

An obvious necessary condition:
all vertices should have the same out-degree.
In what follows we refer to this as to the constant out-degree condition.

Road Coloring

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Necessity of Primitivity

A less obvious necessary condition is called aperiodicity or primitivity:
the g.c.d. of lengths of all cycles should be equal to 1.

To see why primitivity is necessary, suppose that $\Gamma = (V, E)$ is a strongly connected digraph and $k > 1$ is a common divisor of lengths of its cycles. Take a vertex $v_0 \in V$ and, for $i = 0, 1, \dots, k-1$, let

$$V_i = \{v \in V \mid \exists \text{ path from } v_0 \text{ to } v \text{ of length } i \pmod{k}\}.$$

Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

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Let $v \in V_i \cap V_j$ where $i \neq j$. This means that in Γ there are two paths from v_0 to v: of length $\ell \equiv i \pmod{k}$ and of length $m \equiv j \pmod{k}$.

There is also a path from v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

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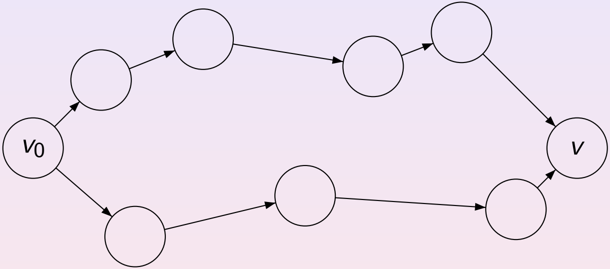

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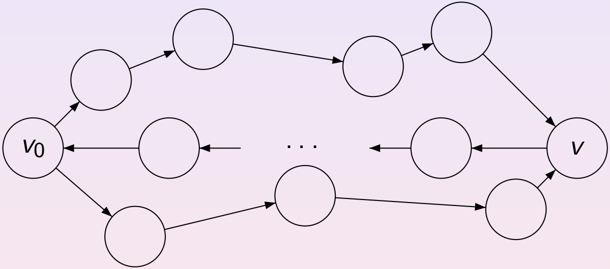

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Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate in the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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Road Coloring Conjecture

The Road Coloring Conjecture claims that the two necessary conditions (constant out-degree and primitivity) are in fact sufficient. In other words: *every strongly connected primitive digraph with constant out-degree admits a synchronizing coloring.*

The Road Coloring Conjecture was explicitly stated by Adler, Goodwyn and Weiss in 1977 (Equivalence of topological Markov shifts, Israel J. Math., 27, 49–63). In an implicit form it was present already in an earlier memoir by Adler and Weiss (Similarity of automorphisms of the torus, Memoirs Amer. Math. Soc., 98 (1970)).

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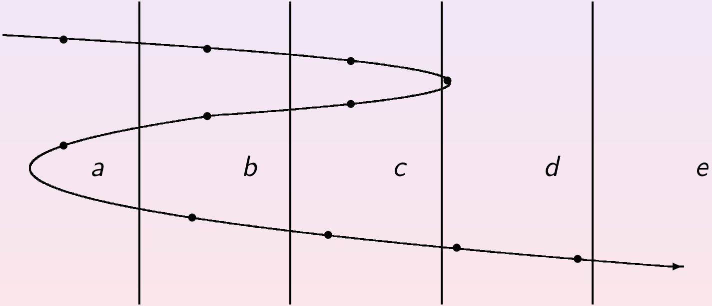

A symbolic trajectory $\ldots abcdcbabcdd \ldots$

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Stability

Trahtman's proof heavily depends on a neat idea of stability which is due to Karel Culik II, Juhani Karhumäki and Jarkko Kari (A note on synchronized automata and Road Coloring Problem, Int. J. Found. Comput. Sci., 13 (2002) 459–471). Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. We define the relation $\sim$ on Q as follows:

$$q \sim q' \iff \forall u \in \Sigma^* \ \exists v \in \Sigma^* \ q\,.\,uv = q'.uv\,.$$

$\sim$ is called the *stability relation* and any pair (q, q') such that $q \sim q'$ is called *stable*. It is immediate that $\sim$ is a congruence of the automaton $\mathscr{A}$. Also observe that $\mathscr{A}$ is synchronizing iff all pairs are stable.

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Proposition CKK. *Suppose every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Then the Road Coloring Conjecture holds true.*

The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/{\sim}$ admits a synchronizing recoloring by the induction assumption.
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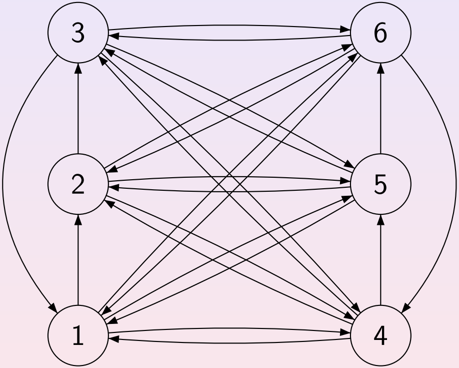

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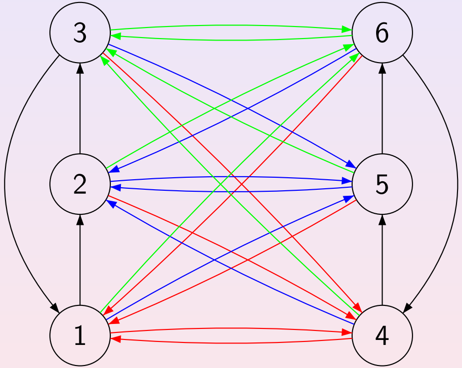

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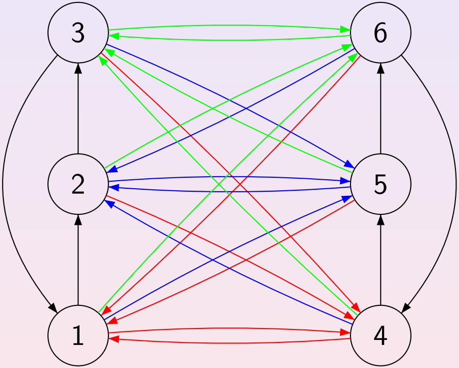

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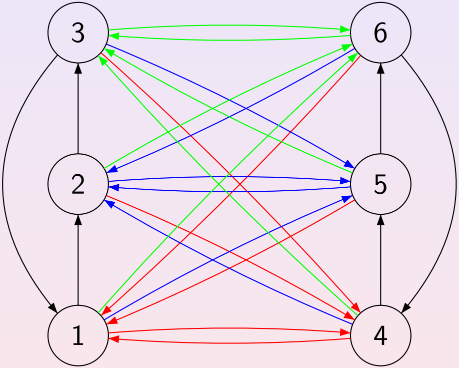

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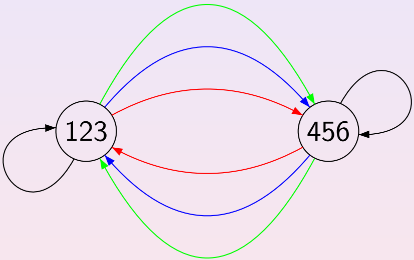

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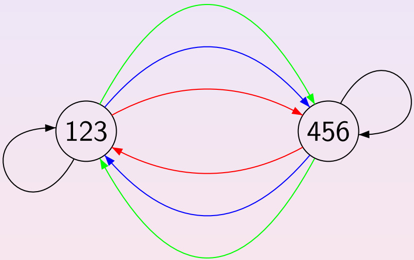

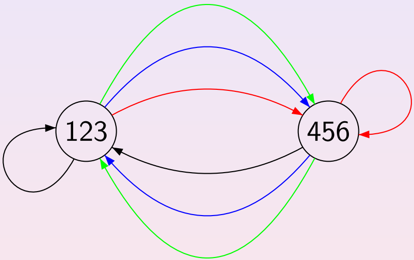

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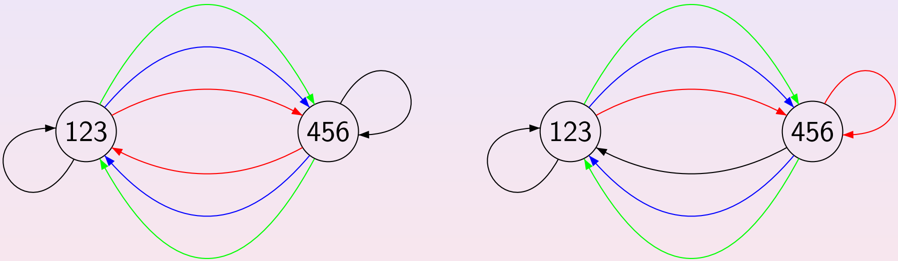

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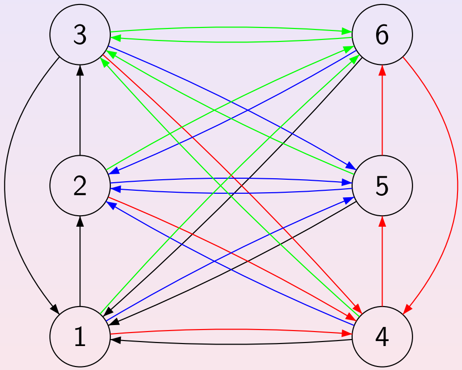

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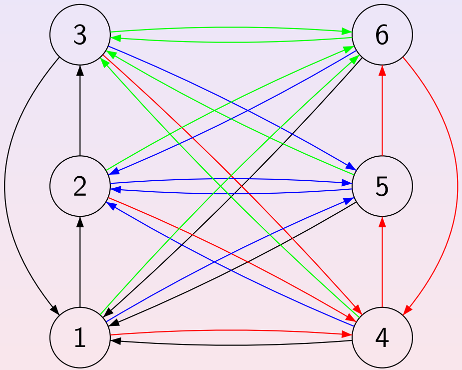

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Trahtman has managed to prove exactly what was needed to use Proposition CKK: every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Thus, Road Coloring Conjecture holds true.

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Moreover, if a pair (p, q) is not stable, then for some word $u \in \Sigma^*$ the pair $(p \,.\, u, q \,.\, u)$ is a deadlock.

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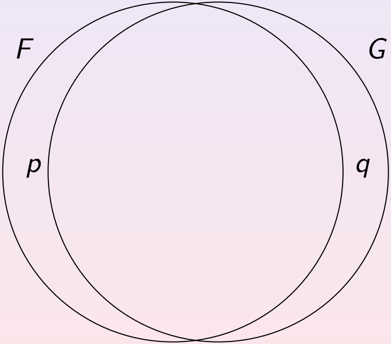

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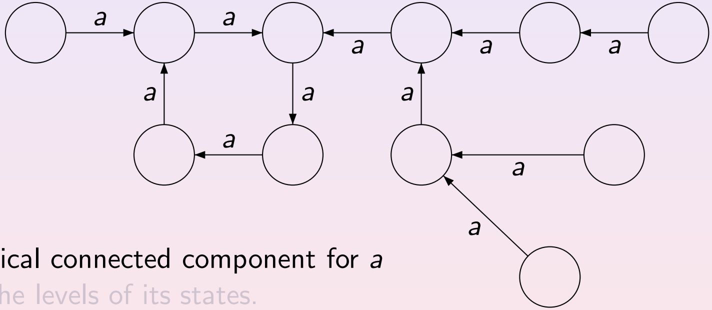

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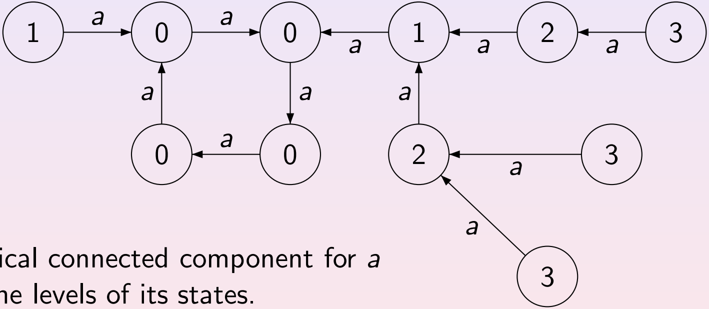

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Proof. Let M be the set of all states of level L w.r.t a. Then $p\,.\,a^L = q\,.\,a^L$ for all $p, q \in M$ whence no pair of states from M forms a deadlock. Thus, if $C \subseteq Q$ is a clique then $|C \cap M| \leq 1$. Take a clique C such that $|C \cap M| = 1$ (it exists since $\mathscr{A}$ is strongly connected). Then $F = C.a^{L-1}$ is a clique that has all its states except one in the a-cycles. If m is the l.c.m. of the lengths of all a-cycles, $r\,.\,a^m = r$ for any r in any a-cycle. Hence $G = F\,.\,a^m$ is a clique such that

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Proof. Let M be the set of all states of level L w.r.t a. Then $p \,.\, a^L = q \,.\, a^L$ for all $p, q \in M$ whence no pair of states from M forms a deadlock. Thus, if $C \subseteq Q$ is a clique then $|C \cap M| \leq 1$. Take a clique C such that $|C \cap M| = 1$ (it exists since $\mathscr{A}$ is strongly connected). Then $F = C.a^{L-1}$ is a clique that has all its states except one in the a-cycles. If m is the l.c.m. of the lengths of all a-cycles, $r \,.\, a^m = r$ for any r in any a-cycle. Hence $G = F \,.\, a^m$ is a clique such that

$$|F| - |F \cap G| = |G| - |F \cap G| = 1.$$

By Lemma 1 $\mathscr{A}$ has a stable pair.

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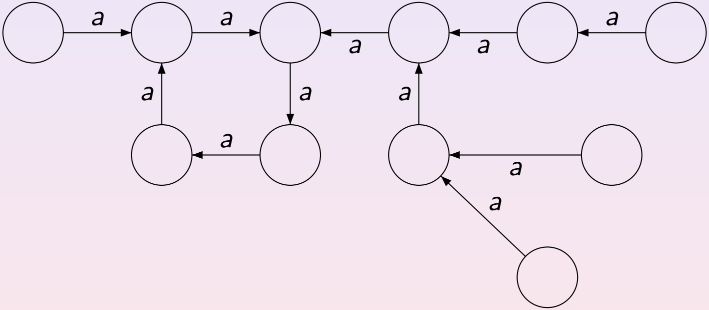

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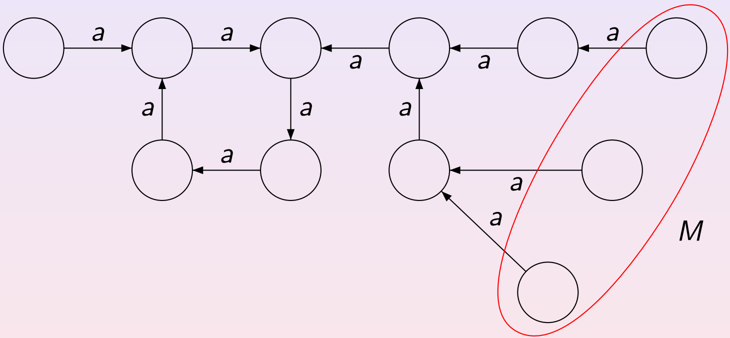

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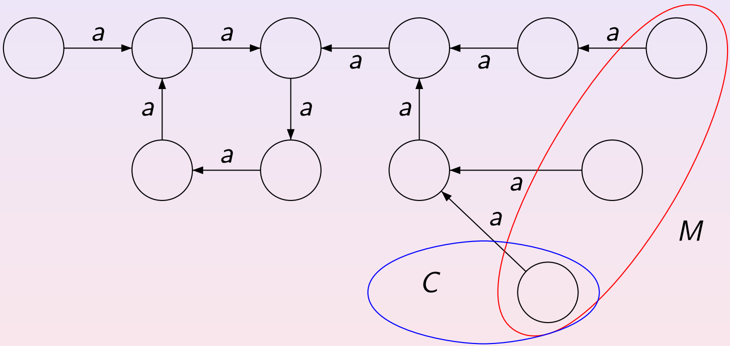

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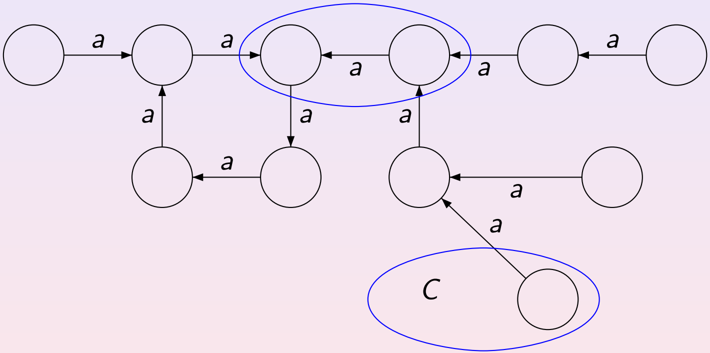

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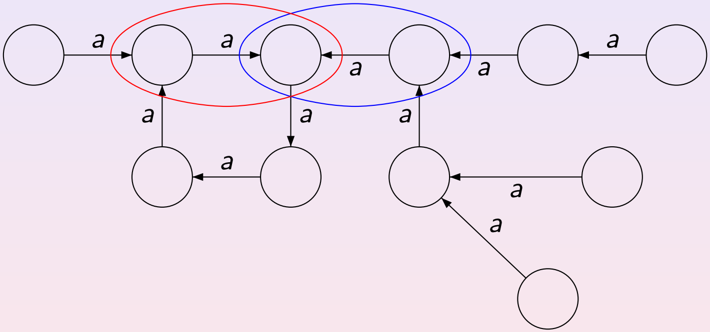

Recall, that we try to prove that every admissible digraph Γ has a stable coloring. By Lemma 2 for this it suffices to show that every such Γ may be colored into an automaton satisfying the premise of the lemma. This is, of course, much easier task because basically we only need to deal with one color, that is, with the action of one letter.

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Suppose that $N = 0$. This means that all states lie on a-cycles.

We say that a vertex p of Γ is a bunch if all edges that begin at p lead to the same vertex q.

If all vertices in Γ are bunches, then there is just one a-cycle (since Γ is strongly connected) and all cycles in Γ have the same length. This contradicts the assumption that Γ is primitive.

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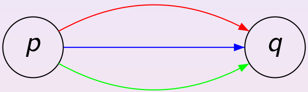

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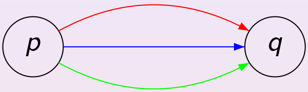

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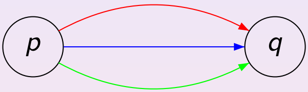

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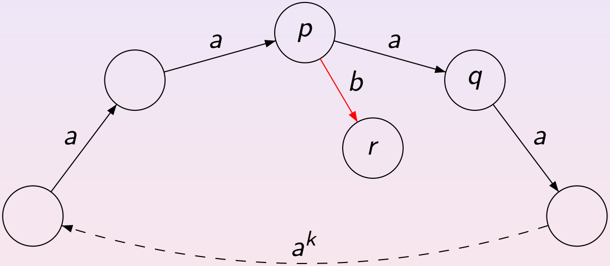
Thus, let p be a state which is not a bunch, let $q = p \,.\, a$ and let $b \neq a$ be such that $r = p \,.\, b \neq q$. We exchange the labels of the edges $p \xrightarrow{a} q$ and $p \xrightarrow{b} r$.



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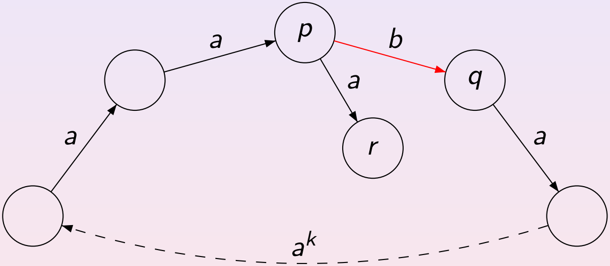

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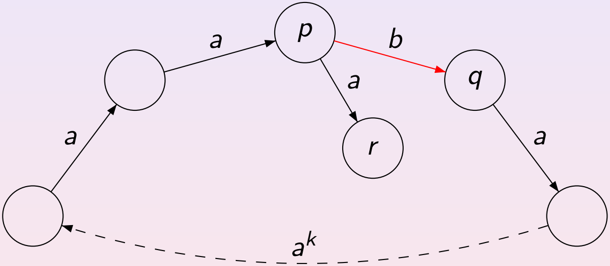

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The following considerations split in several cases. In each case except one we can recolor Γ by swapping the labels of two edges such the new coloring either satisfies the premise of Lemma 2 (all states of maximal level w.r.t. a belong to the same tree) or has more states on the a-cycles (and the induction assumption applies). The remaining case turns out to be easy.

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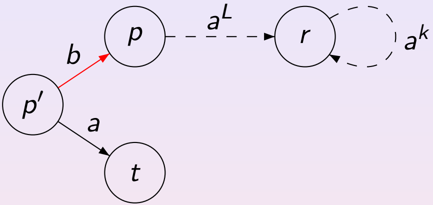

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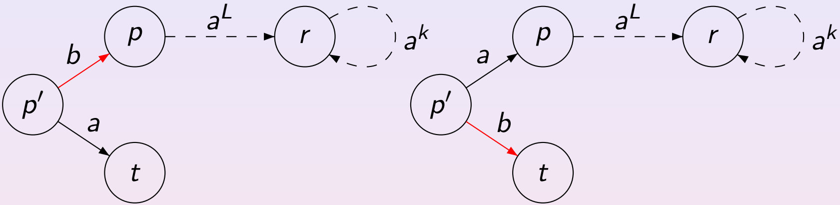

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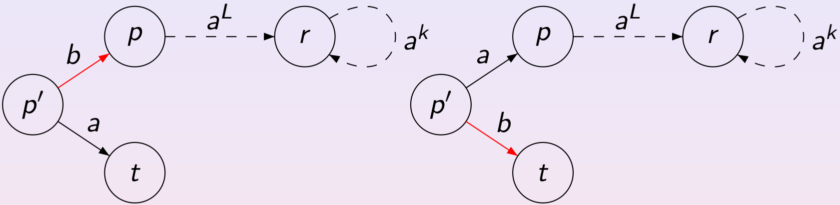

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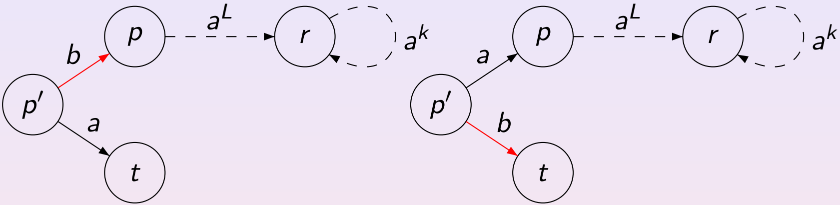

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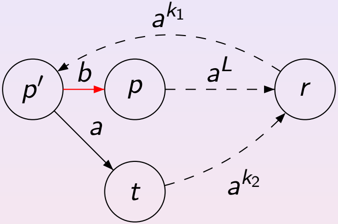

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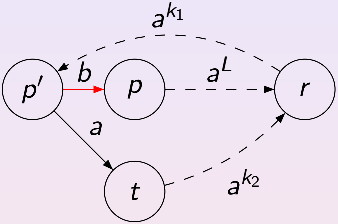

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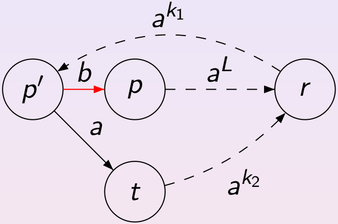

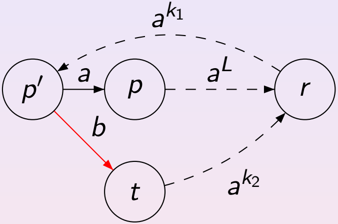

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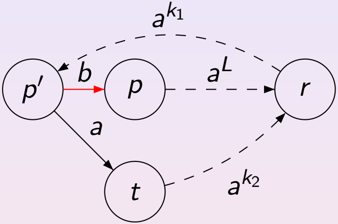

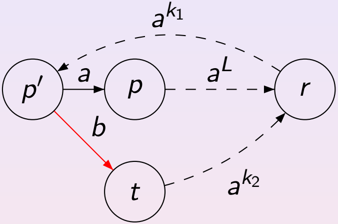

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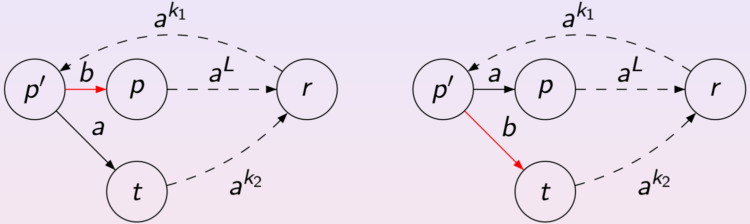

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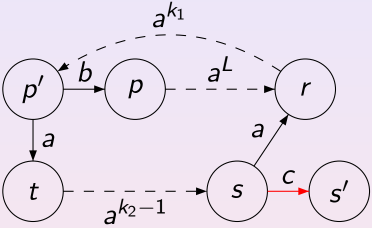

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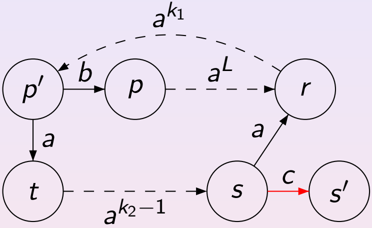

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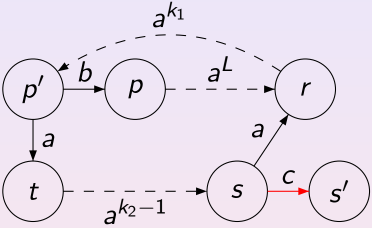

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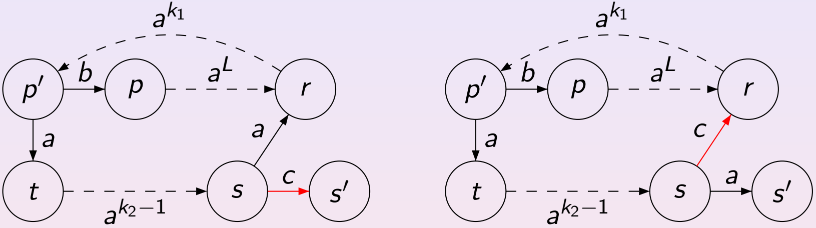

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Induction Step: Subcase 2.3

Let q be the state on the a-path from p to r such that $q\,.\,a = r$.

Subcase 2.3: $k_2 = L$ and q is not a bunch. Since q is not a bunch, there is a letter c such that $q' = q\,.\,c \neq r$.

If we swap the labels of $p' \xrightarrow{b} p$ and $p' \xrightarrow{a} t$, we find ourselves in the conditions of Subcase 2.2 (with q and q' playing the roles of s and s' respectively).

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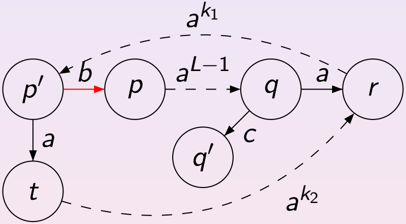

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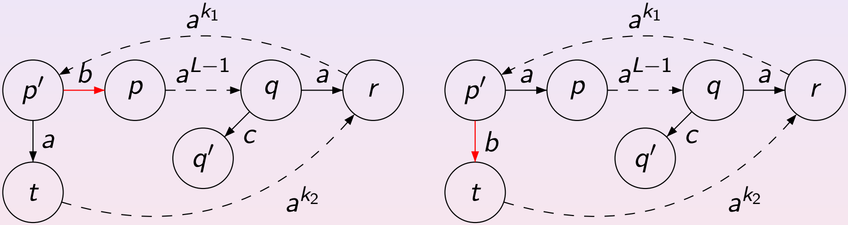

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Induction Step: Subcase 2.4

Subcase 2.4: $k_2 = L$ and both s and q are bunches.

In this case it is clear that q and s form a stable pair. This completes the proof.

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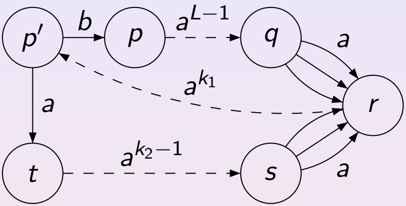

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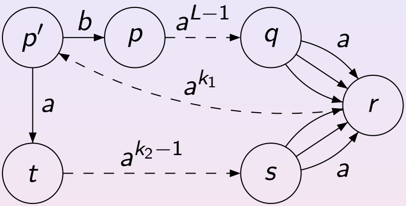

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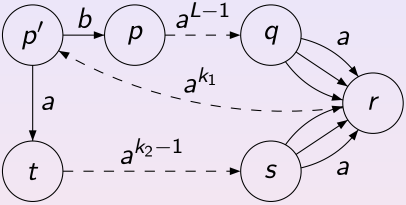

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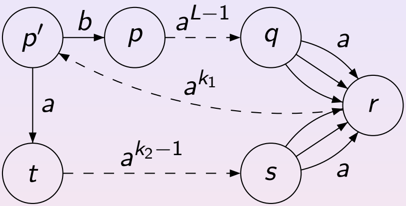

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Open Questions

1. Characterize totally synchronizing graphs, i.e., graphs such that every coloring makes them become synchronizing automata.

2. (Hybrid Road Coloring – Černý problem) What is the maximum value of the minimum length of reset words for synchronizing colorings of admissible graphs with n vertices?
Conjecture: $n^2 - 3n + 3$, achieved by the Wielandt graph.

3. (Quantitative Road Coloring problem) An admissible graph with n vertices and common out-degree k has k^n colorings. How many of them are synchronizing?
Conjecture: at least one half with exactly one exception which is the Cayley graph of the symmetric group in 3 points.

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