Interpreting graphs in 0-simple semigroups with involution

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(joint work with Marcel Jackson, La Trobe University, Australia)

AAA77, March 20, 2009

Outline

1. Graphs and Algebras
   - Graphs
   - Adjacency Semigroups
   - Universal Horn Classes
2. Main Results and Applications
   - Embedding
   - Applications to Complexity Issues
   - Isomorphism

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Definitions and terminology

We consider a graph as a structure $G := \langle V; \sim\rangle$, where V is a set and $\sim \subseteq V \times V$ is a binary relation.

In other words, we consider directed graphs and do not allow multiple edges.

The adjacency matrix P_G of a graph $G = \langle V; \sim\rangle$ is a $V \times V$-matrix given by $P_G(x, y) = 1$ if $x \sim y$ and 0 otherwise.

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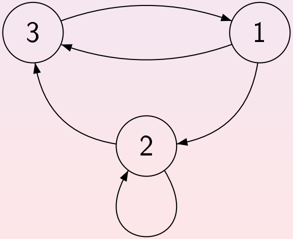

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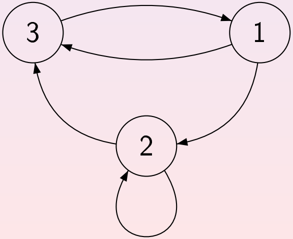

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

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Graph Algebras

Since graphs form a universal language of discrete mathematics, the idea to relate graphs and algebras appear to be natural.

A well known approach: graph algebras

Given a graph $G = \langle V; \sim \rangle$, its graph algebra is a groupoid on the carrier set $V \cup \{\infty\}$ with the multiplication defined by

$$xy = \begin{cases} x & \text{if } x, y \in V \text{ and } x \sim y, \\ \infty & \text{otherwise.} \end{cases}$$

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Adjacency Semigroup of a Graph

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$$(x, y)(z, t) = \begin{cases} (x, t) & \text{if } y \sim z, \\ \mathbf{0} & \text{if } y \nsim z; \end{cases}$$

$$a\mathbf{0} = \mathbf{0}a = \mathbf{0} \quad \text{for all } a \in A(G).$$

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Reversion

Yes, to some extent.

Example: $A(G)$ is 0-simple iff every vertex of G has nonzero indegree and nonzero outdegree.

In general, however, connections between $A(G)$ and G seem to be far too weak in order to be useful.

New idea: to equip $A(G)$ with an additional unary operation (reversion):

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Interpreting Graphs

- $x \sim x$ (reflexivity of G) is equivalent to $A(G) \models XX'X = X$;

Natural graph properties correspond to natural semigroup properties. We encounter identities on the semigroup side. What is the syntactic nature of conditions appearing at the graph side?

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- $x \sim x$ (reflexivity of G) is equivalent to $A(G) \models XX'X = X$; Indeed, $\forall x, y \in V$ $(x, y)(y, x)(x, y) = (x, y)(y, y) = (x, y)$ because $x \sim x$ and $y \sim y$.

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Universal Horn Classes

A universal Horn sentence is a sentence of one of the two forms:

$$(\forall x_1 \forall x_2 \ldots)\left(\left(\mathop{\&}_{1\le i\le n} \Phi_i\right) \to \Phi_0\right),$$

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- Preorders (reflexivity + transitivity)

More about the last example. The following graph C_3 is 3-colorable but not 2-colorable.

C_3 is a generator of minimum size for the uH-class of all 3-colorable graphs (Nešetřil and Pultr, 1978).

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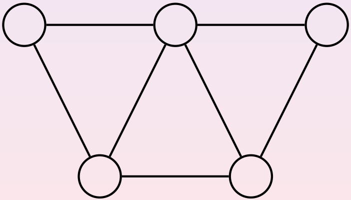

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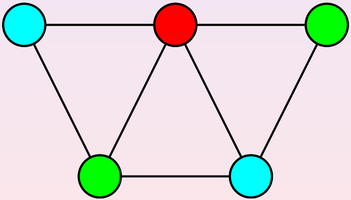

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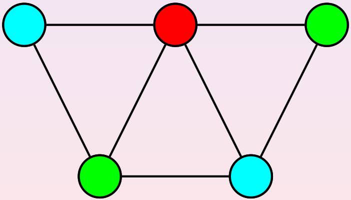

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Theorem

The assignment $G \mapsto A(G)$ induces an injective order-preserving map from the lattice of all uH-classes of graphs to the subvariety lattice of the variety generated by all adjacency semigroups.

Thus, the equational logic of rather a transparent class of unary semigroups interprets the (universal Horn) theory of graphs.

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Let K be any nonempty class of graphs and let G be a graph. G belongs to the uH-class generated by K iff $A(G)$ belongs to the variety generated by the semigroups $A(H)$ with $H \in K$.

The Variety Membership Problem

The theorem in the above strong form has interesting application to the complexity of the variety membership problem for unary semigroups.

Let A be a finite algebra. $\text{VAR-MEMB}(A)$ is the decision problem whose instance is a finite algebra B of the same similarity type and whose question is: does B belong to the variety generated by A?

$\text{VAR-MEMB}(A)$ is decidable. An easy consequence of the HSP-theorem: $B \in \text{var}\, A$ iff B is a homomorphic image of the free $|B|$-generated algebra of var A and the free algebra has at most $|A|^{(|A|^{|B|})}$ elements. But the resulting algorithm requires doubly exponential time (as a function of $|B|$). Can we do better?

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Is there a finite semigroup (a finite semigroup with involution) A such that testing membership in var *A is NP-complete? NP-hard? ... PSPACE-complete? 2-EXPTIME-complete?*

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Recall that the problem 3-COLOR is NP-complete (Levin, 1973) and that the 5-element graph C_3 generates the uH-class of all 3-colorable graphs.

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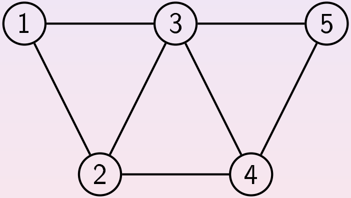

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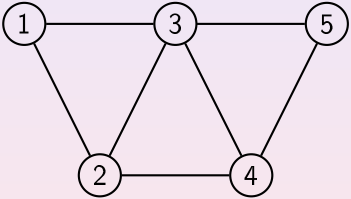

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Some other natural problems about the adjacency semigroup $A(C_3)$ also have high complexity. For instance, checking identities in $A(C_3)$ is co-NP-complete.

Reflexive Case

Recall that a graph G is reflexive (has a loop at each vertex) iff the adjacency semigroup $A(G)$ satisfies $XX'X = X$. When restricted to reflexive graphs, our map becomes a "nearly" isomorphism.

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Applications to Finite Basis Problem

In particular, a subvariety of the variety generated by adjacency semigroups satisfying $XX'X = X$ is finitely based (finitely generated as a variety) iff it corresponds to the inserted element or to a finitely axiomatized (finitely generated, respectively) uH-class of reflexive graphs.

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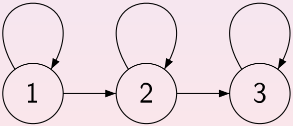

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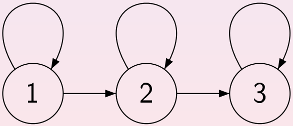

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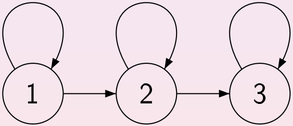

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Then var $A(P)$ is a limit (minimal non-finitely based) variety of unary semigroups and has only 5 subvarieties.

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