Algebraic Constructions for Expanders

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AAA88, June 20th, 2014

A Motivating Example I

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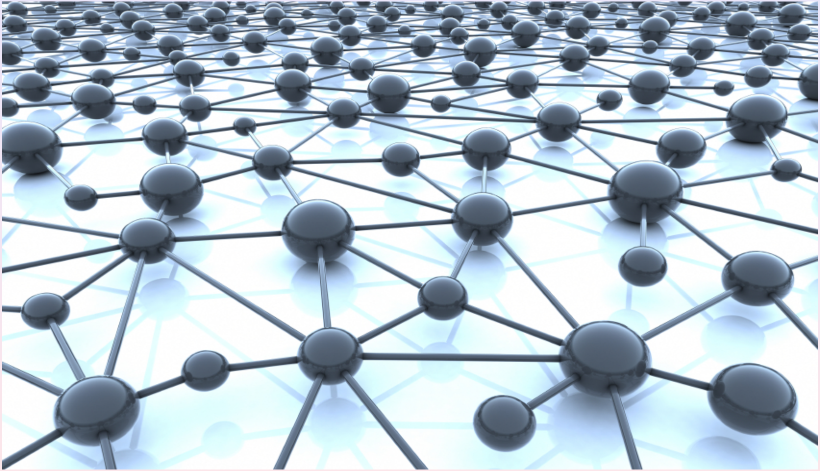

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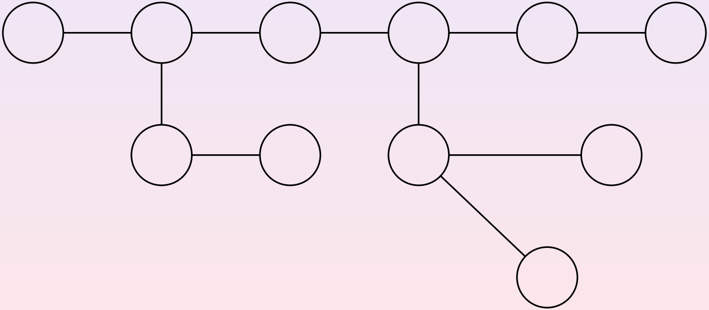

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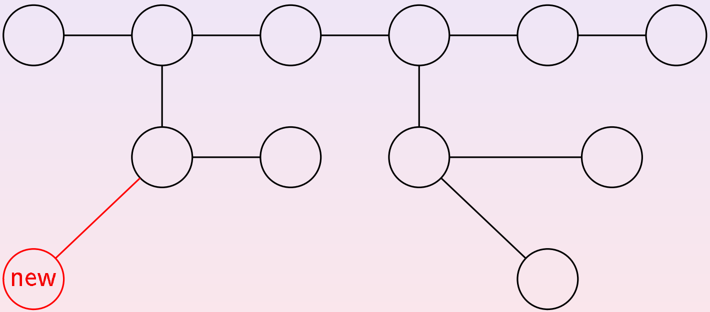
Pro: low costs of expanding — one extra line for each new node:



Thus, expanding a network with n nodes to one with twice as many nodes requires only n extra lines.

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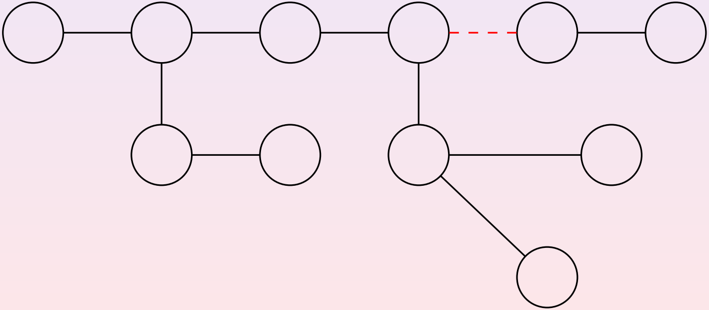
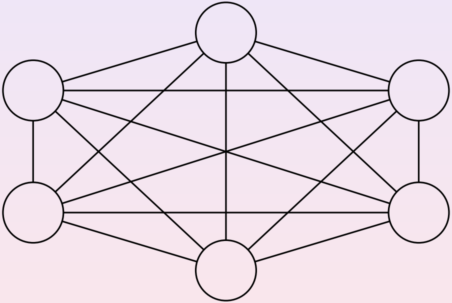
Contra: bad connectivity — removing any edge disconnects the whole network:



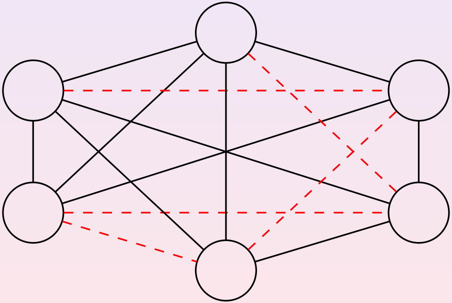
An "opposite" solution: a clique (complete graph).



A Motivating Example II

An "opposite" solution: a clique (complete graph).

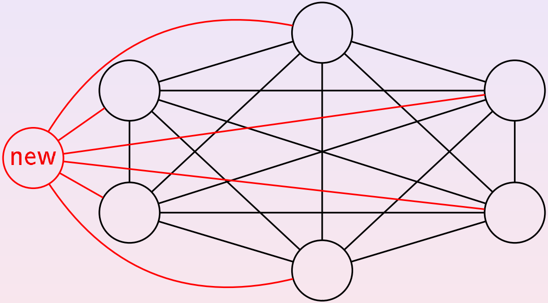
Pro: good connectivity — the network stays connected even after removing several edges:



A Motivating Example II

An “opposite” solution: a clique (complete graph).

Contra: high costs of expanding — n extra lines for each new node:



Thus, expanding a network with n nodes to one with twice as many nodes requires around $\frac{3}{2}n^2$ extra lines.

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How do we formalize good connectivity?

Towards A Formal Definition

Let $G = (V, E)$ be an undirected and d-regular graph; $|V| = n$; loops and multiple edges are allowed. For $S, T \subset V$, let $E(S, T)$ be the set of edges from S to T:

$$E(S, T) = \{(u, v) \mid u \in S,\ v \in T,\ (u, v) \in E\}$$

The boundary of $S \subset V$ is the set $\partial S = E(S, V \setminus S)$; the set of edges emanating from S to its complement.

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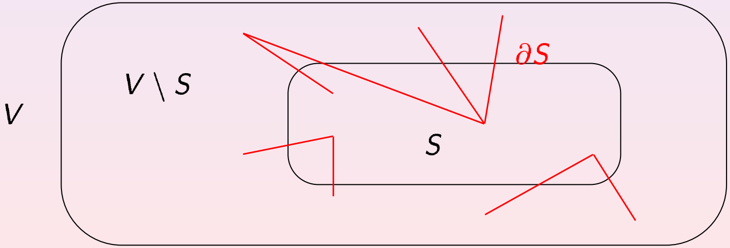

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Good connectivity $\Rightarrow$ every "small" set of vertices has a relatively big boundary $\Rightarrow$ the expansion ratio is not too small.

A Formal Definition

Definition

A sequence of d-regular graphs $\{G_i\}_{i\in\mathbb{N}}$ of size increasing with i is a Family of Expander Graphs if there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i.

Families of expander graphs have found extensive and surprising applications in designing algorithms and error correcting codes; they have also been used in proofs of many important results in computational complexity theory, in cryptography, and also in several areas of pure mathematics and statistical physics.
An excellent source for basic material, a wide range of applications as well as research up to 2005 is the monograph by Shlomo Hoory, Nati Linial, and Avi Wigderson: "Expander graphs and their applications", Bull. Amer. Math. Soc., 43(4):439–561, 2006.
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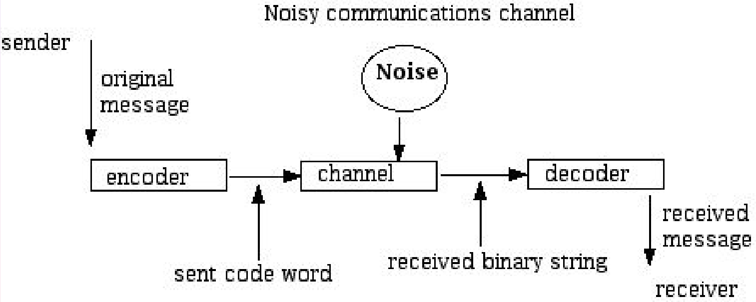

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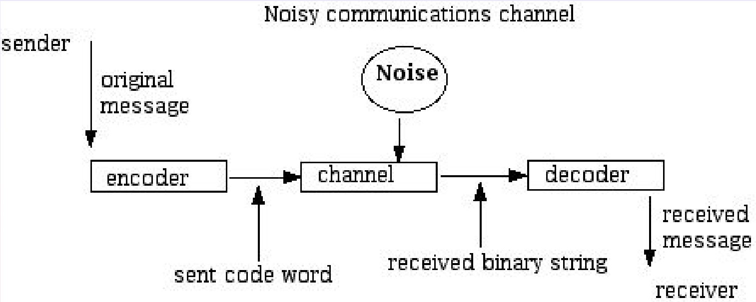

Alice and Bob communicate over a noisy channel. A fraction p of the bits sent may be altered. What is the smallest number of bits that Alice can send, assuming she wants to communicate an arbitrary k-bit message, so that Bob should be able to unambiguously recover the original message?

Error Correcting Codes I

The basic idea (due to Claude Shannon) is to create a code $C \subset \{0,1\}^n$ of size $|C| = 2^k$ such that the Hamming distance between any two strings in C is greater than $2pn$. (The Hamming distance $d_H(u,v)$ is the number of coordinates i such that $u_i \neq v_i$.) Alice and Bob agree about the encoding: a bijection $\varphi : \{0,1\}^k \to C$. If Alice needs to send a message $x \in \{0,1\}^k$, she transmits $\varphi(x) \in C$. Bob receives $y \in \{0,1\}^n$ which is a corrupted version of $\varphi(x)$. Since $d_H(y, \varphi(x)) \leq pn$, Bob can recover $\varphi(x)$ as the string $z \in C$ that is closest to y and then find $x = \varphi^{-1}(z)$.

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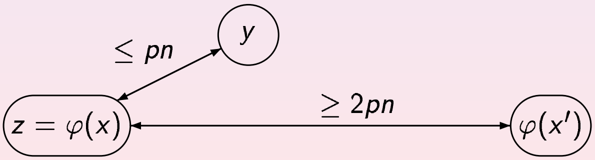

Error Correcting Codes II

For a code $C \subset \{0,1\}^n$, its rate is

$$r = \frac{\log |C|}{n}$$

and its (normalized) distance is

$$\delta = \frac{\min_{c_1 \neq c_2 \in C} d_H(c_1, c_2)}{n}.$$

The distance of a code controls its power to overcome noise while its rate measures its efficiency in channel utilization.

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Problem

Is it possible to explicitly design arbitrarily large codes $\{C_k\}$ of size $|C_k| = 2^k$, with $r(C_k) \geq r_0$ and $\delta_(C_k) \geq \delta_0$ for some absolute constants $r_0, \delta_0 > 0$?

Magic Bipartite Graphs

We utilize a bipartite version of expanders.

A bipartite graph $G = (L \cup R, E)$ is said to be (n, d)-magic if $|L| = n$, $|R| = 3n/4$, every left vertex has degree d, and the following two properties hold:

(1) for every $S \subset L$ with $|S| \leq \frac{n}{10d}$, the set $\Gamma(S)$ of all neighbors of S in R is of size at least $\frac{5d}{8}|S|$;

(2) for every $S \subset L$ with $\frac{n}{10d} < |S| \leq \frac{n}{2}$, the set $\Gamma(S)$ is of size at least $|S|$.

Observe that for every nonempty $S \subset L$ with $s = |S| \leq \frac{n}{10d}$, there is a vertex in R with exactly one neighbor in S. Indeed, there are exactly ds edges between S and $\Gamma(S)$. Since $|\Gamma(S)| \geq \frac{5ds}{8}$, the average number of neighbors in S that a vertex in $\Gamma(S)$ has is at most $ds : \frac{5ds}{8} = \frac{8}{5} < 2$. Since each vertex in $\Gamma(S)$ has at least one neighbor in S, some vertices must have exactly one neighbor in S.

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(1) for every $S \subset L$ with $|S| \leq \frac{n}{10d}$, the set $\Gamma(S)$ of all neighbors of S in R is of size at least $\frac{5d}{8}|S|$;
(2) for every $S \subset L$ with $\frac{n}{10d} < |S| \leq \frac{n}{2}$, the set $\Gamma(S)$ is of size at least $|S|$.

Observe that for every nonempty $S \subset L$ with $s = |S| \leq \frac{n}{10d}$, there is a vertex in R with exactly one neighbor in S. Indeed, there are exactly ds edges between S and $\Gamma(S)$. Since $|\Gamma(S)| \geq \frac{5ds}{8}$, the average number of neighbors in S that a vertex in $\Gamma(S)$ has is at most $ds : \frac{5ds}{8} = \frac{8}{5} < 2$. Since each vertex in $\Gamma(S)$ has at least one neighbor in S, some vertices must have exactly one neighbor in S.

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Thus, $d_H(c, 0) > \frac{n}{10d}$ for every $c \in C \setminus \{0\}$ whence $d_H(c_1, c_2) = d_H(c_1 - c_2, 0) > \frac{n}{10d}$ for all distinct $c_1, c_2 \in C$.

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One gets $\delta(C) = \frac{\min_{c_1 \neq c_2 \in C} d_H(c_1, c_2)}{n} > 1/10d$.

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The corresponding LDPC codes give simultaneously the best coding parameters as well as best algorithmic performance.

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Recall the formal definition:

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A sequence of d-regular graphs $\{G_i\}_{i\in\mathbb{N}}$ of size n_i increasing with i is a Family of Expander Graphs if there exists $\varepsilon > 0$ such that $h(G_i) = \min_{|S|\leq\frac{n_i}{2}} \frac{|\partial S|}{|S|} \geq \varepsilon$ for all i.

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However, it is not what we really need: the applications require explicit and efficient constructions for expander families.

The first explicitly constructed family of expander graphs is due to Grigory Margulis, "Explicit Construction of Concentrators", Prob. Per. Infor. 9(4):325–332, 1975 (in Russian). This is a family of 8-regular graphs M_n where n runs over the set of positive integers. The vertex set of M_n is $\mathbb{Z}_n \times \mathbb{Z}_n$; the neighbors of the vertex (x, y) are $(x \pm y, y)$, $(x \pm (y + 1), y)$, $(x, y \pm x)$, and $(x, y \pm (x + 1))$, where the arithmetics is modulo m. This is an example of a very explicit expander family.

Margulis had not provided any specific bound on $h(M_n)$; later it was shown that $h(M_n) \geq \frac{8-5\sqrt{2}}{2}$ (Gabber and Galil, 1981). Both Margulis's and Gabber–Galil's results are rather hard to prove and use some heavy mathematical machinery.

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Another family consists of 3-regular graphs I_p indexed by primes.
The vertex set of I_p is $\mathbb{Z}_p$; the neighbors of the vertex x are $x \pm 1$ and x^{-1}, where the arithmetic is modulo p and $0^{-1} := 0$.
The proof relies on a deep result in number theory (Selberg's 3/16 theorem). Observe that the family I_p is in a sense less explicit than Margulis's family since no deterministic polynomial algorithm for generating large primes is known.

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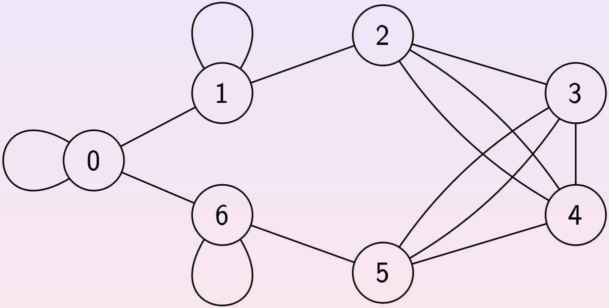

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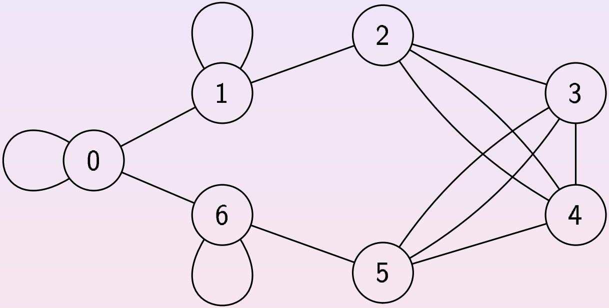

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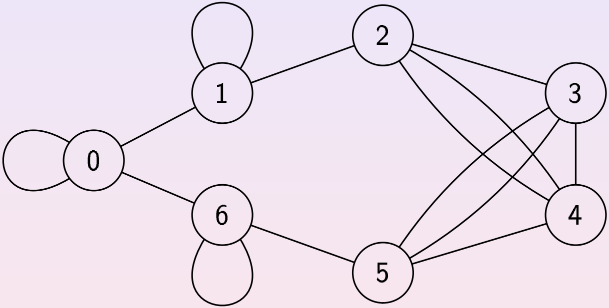

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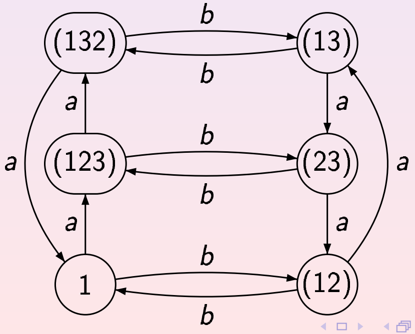
Here is the Cayley graph of the symmetric group $\mathbb{S}_3$ with respect to its generating set $\{a=(123), b=(12)\}$:



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Algebra Comes Into The Play: Cayley Graphs

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Problem

For which finite groups H and their generating set S do the Cayley graphs $\mathrm{Cay}(H, S)$ form a family of expander graphs?

All Groups Have Small Generating Sets

Observation

Every finite group H has a generating set of size $\log|H|$.

Proof: It is a simple greedy algorithm: having picked i elements $g_1, g_2, \ldots, g_i$ from H into the generating set, we list out the elements of the subgroup H_i generated by the set $\{g_1, g_2, \ldots, g_i\}$. If $H_i \neq H$, we pick any $g_{i+1} \in H \setminus H_i$ as the next element in the generating set. The subgroup H_{i+1} generated by $\{g_1, \ldots, g_i, g_{i+1}\}$ contains H_i properly whence $|H_{i+1}| \geq 2|H_i|$ by Lagrange's theorem. The process will stop after at most $\log|H|$ steps.

This bound is tight: the group $H = \mathbb{Z}_2^n$ has 2^n elements and dimension n as the vector space over $\mathbb{Z}_2$. Hence every generating set of H contains at least $n = \log|H|$ elements.

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Alon–Roichman Theorem

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Theorem (Alon and Roichman)

For every ε such that $1 > \varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ such that the following holds. Let H be a group of order n, and let S be a random set of $c(\varepsilon) \log n$ elements of H, then the Cayley graph $G = \mathrm{Cay}(H, S)$ is an expander with $h(G) \geq \varepsilon$ almost surely. (The probability that G is such an expander tends to 1 as $n \to \infty$.)

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Recently, it has been shown by Arvind, Mukhopadhyay, and Nimbhorkar ("Erdös-Rényi sequences and deterministic construction of expanding Cayley graphs", LATIN 2012: 37–48) that the Alon–Roichman theorem admits an efficient derandomization.

Theorem (Derandomized version of the Alon–Roichman theorem)

For every ε such that $1 > \varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ such that the following holds. There exists a polynomial in n algorithm that, given a group H of order n, produces a generating set S of H of size $c(\varepsilon)\log n$ elements of H such that the Cayley graph $G = \mathrm{Cay}(H, S)$ is an expander with $h(G) \geq \varepsilon$.

Thus, the algebraic approach can be used to produce explicit families of expanders.

What else should be done? If H is given in a more efficient way, we want the algorithm to work in time polynomial in the size of the description of H rather than the size of H itself. A more efficient way—as a group of permutations or matrices; if, say, H is specified as a subgroup of $\mathbb{S}_m$, the algorithm is to be polynomial in m, etc.

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Conclusion

Modern computer science asks many questions that:

- are of major theoretical and practical importance;
- can be stated in term of very "classical" algebra;
- are somewhat different from the "standard" questions studied by algebraists and thus often require new algebraic tools.

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