Local finiteness for Green's relations in semigroup varieties

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(with Pedro Silva and Filipa Soares)

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AAA94+5th NSAC

My Coauthors

Pedro Ventura Alves da Silva   Filipa Soares de Almeida

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Three More Reasons to Celebrate

This conference is a tribute to our dear colleagues Siniša Crvenković, Reinhard Pöschel, and Branimir Šešelja.

Best wishes to them all from my coauthors and myself as well as Lev Shevrin and all Ekaterinburg algebraists!

However, I think semigroupists have three more anniversaries to celebrate this year.

- 30 years of Szeged 1987 semigroup conference;
- 60th birthday of Mark Sapir;
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Result from Sapir-1987

This talk—is in fact a work-in-progress report—presents results grown from the main problem solved in Sapir's 1987 paper

A semigroup variety is the class of all semigroups satisfying some collection of identities (like $xy = yx$ or $x^2 = x$).

If this collection can be chosen such that all its identities involve in total only finitely many variables, the variety is of finite axiomatic rank. Each variety defined by finitely many identities is such.

A variety is locally finite if all its finitely generated members are finite. For instance, the variety defined by $x^2 = x$ is locally finite while the variety defined by $xy = yx$ is not.

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Result from Sapir-1987, contd

The question addressed in Sapir's 1987 paper is extremely natural:

This a very typical Burnside type problem and it includes the classic Burnside problem as a special case since the class $\mathbf{G}_n$ of all groups of exponent dividing n is the semigroup variety defined by the identities $x^n y = yx^n = y$.

Sapir gave an easy-to-state (but very difficult-to-prove!) solution to this challenging problem. His solution is in fact algorithmic "modulo groups": given a semigroup variety $\mathbf{V}$, Mark's algorithm either returns "No, $\mathbf{V}$ is not locally finite" or reduces the problem to the same question about the class of all groups in $\mathbf{V}$ (which is a subvariety of $\mathbf{G}_n$ for some n computed by the algorithm).

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Result from Sapir-1987, completed

In this talk, we depart from a non-algorithmic version of Mark's characterization of locally finite semigroup varieties (it forms a part of Theorem P in the 1987 paper).

A semigroup S is a nilsemigroup if S has a zero and some power of each element in S is equal to zero.

A semigroup variety is periodic if all its one-generated members are finite. Clearly, a locally finite variety must be periodic.

Clearly, a semigroup S is a periodic group iff S has an identity element 1 and some power of each element in S is equal to 1.

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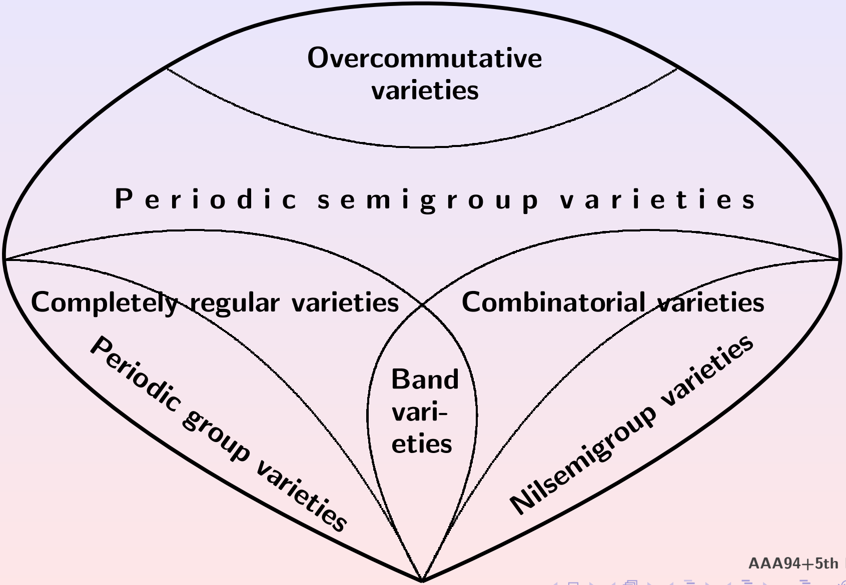
Theorem (Sapir, 1987)

Let $\mathbf{V}$ be a periodic variety of finite axiomatic rank.
If all nilsemigroups and all groups in $\mathbf{V}$ are locally finite, then $\mathbf{V}$ is locally finite.

An Illustration



Green's Relations

For a semigroup S, let S^1 stand for S if S has an identity element and for $S \cup \{1\}$ if S has no identity element.

The following five equivalence relations can be defined on every semigroup S:

$x \mathscr{R} y \Leftrightarrow xS^1 = yS^1$, i.e., x and y are prefixes of each other;

$x \mathscr{L} y \Leftrightarrow S^1x = S^1y$, i.e., x and y are suffixes of each other;

$x \mathscr{H} y \Leftrightarrow xS^1 = yS^1 \wedge S^1x = S^1y$, i.e., $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$;

$x \mathscr{D} y \Leftrightarrow (\exists z \in S)\ x \mathscr{R} z \wedge z \mathscr{L} y$, i.e., $\mathscr{D} = \mathscr{R}\mathscr{L}$

$x \mathscr{J} y \Leftrightarrow S^1xS^1 = S^1yS^1$, i.e., x and y are factors of each other.

They were introduced by James Alexander (Sandy) Green in 1951 ("On the structure of semigroups", Ann. Math. (2), 54:163–172) and are collectively referred to as Green's Relations.

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They were introduced by James Alexander (Sandy) Green in 1951 ("On the structure of semigroups", Ann. Math. (2), 54:163–172) and are collectively referred to as Green's Relations.

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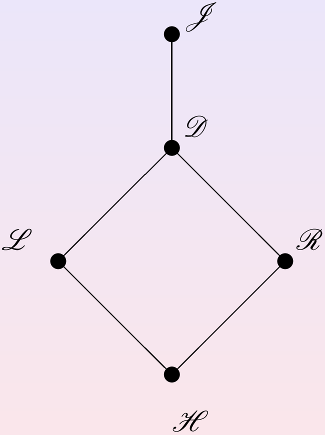
Inclusions between Green's Relations



The diagram shows inclusions between Green's relations in "general position".

In any periodic semigroup, $\mathscr{J} = \mathscr{D}$.

In every group, all Green's relations coincide with the universal relation.

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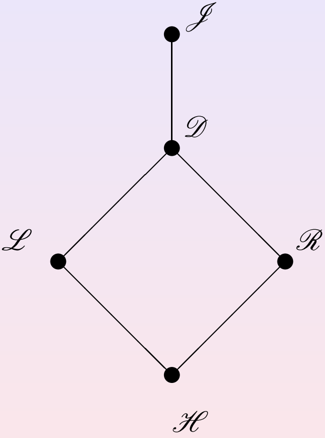

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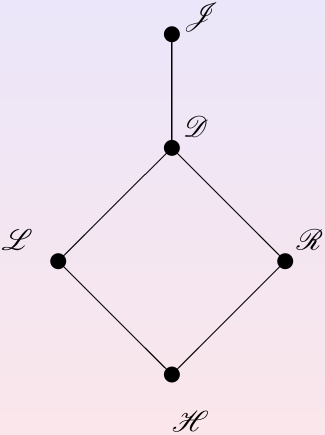

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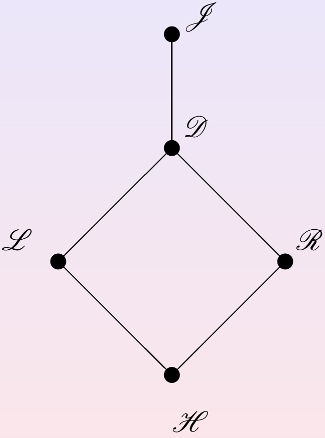

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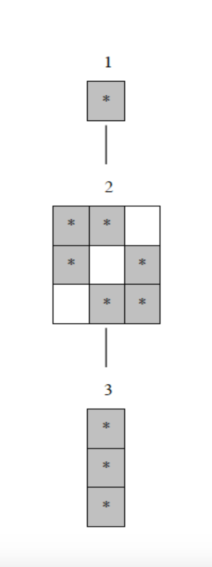

"Egg-box picture" of a semigroup (the 27-element semigroup of all selfmaps of a 3-element set)

Separate rectangles: $\mathscr{D}$-classes

Rows: $\mathscr{R}$-classes; all rows within one rectangle are of the same cardinality

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Shadowed cells: maximal subgroups; all maximal subgroups within one rectangle are isomorphic

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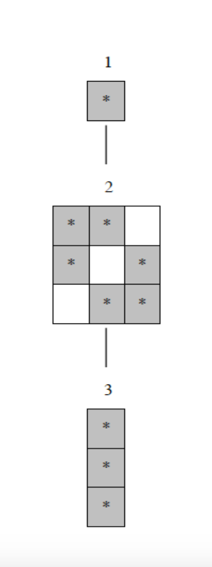

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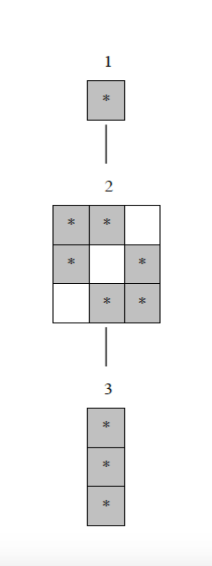

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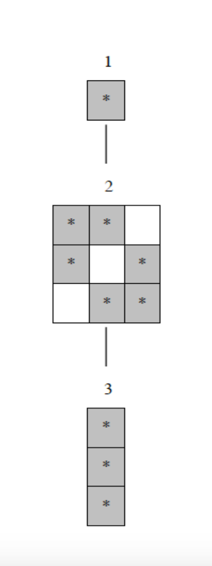

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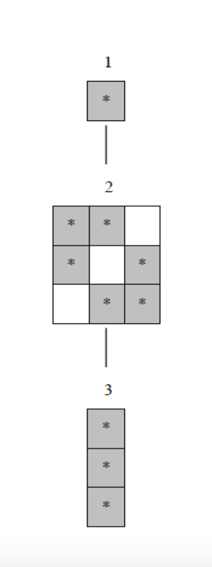

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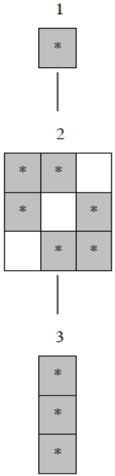

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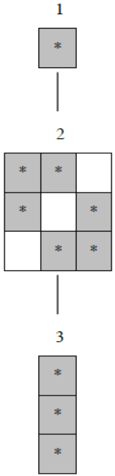

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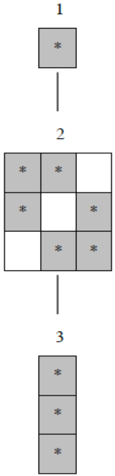

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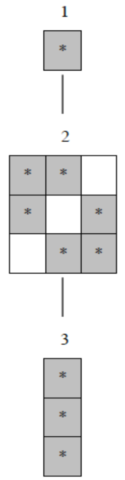

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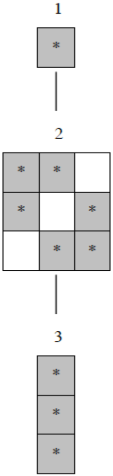

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Let $\mathscr{K}$ be one of the five Green relations. A variety $\mathbf{V}$ is said to be locally $\mathscr{K}$-finite if each finitely generated semigroup in $\mathbf{V}$ has only finitely many $\mathscr{K}$-classes.

Our goal: we aim to classify locally $\mathscr{K}$-finite semigroup varieties.

Motivation: a natural generalization of local finiteness that "bypasses" the classical Burnside problem (as all semigroup varieties consisting of groups are locally $\mathscr{K}$-finite for any $\mathscr{K}$).

Of course, every locally finite variety is locally $\mathscr{K}$-finite for any $\mathscr{K}$; hence, only locally $\mathscr{K}$-finite varieties which are not locally finite are of interest.

We refer to infinite finitely generated groups of finite exponent as Novikov–Adian groups (NAGs). NAGs are definitely amongst the most complicated and mysterious objects of algebra.

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Recall:

Theorem (Sapir, 1987)

Let **V** be a periodic variety of finite axiomatic rank. If all nilsemigroups in **V** are locally finite and **V** contains no NAGs, then **V** is locally finite.

The infinite one-generated semigroup has infinitely many $\mathscr{J}$-classes $\Longrightarrow$ for every $\mathscr{K} \in \{\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}\}$, locally $\mathscr{K}$-finite varieties are periodic.

Every nilsemigroup is $\mathscr{J}$-trivial $\Longrightarrow$ if $\mathscr{K} \in \{\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}\}$ and **V** is a locally $\mathscr{K}$-finite variety, then nilsemigroups in **V** are locally finite.

Thus, Sapir's result tells us that every locally $\mathscr{K}$-finite variety of finite axiomatic rank which is not locally finite contains a NAG.

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A semigroup is **completely regular** if it is a union of groups.

A non-empty subset R of a semigroup S is called a **right ideal** of S if $rs \in R$ for every $r \in R$ and every $s \in S$. Left ideals are defined dually, and non-empty subset I of S is said to be an **ideal** of S if I is both left and right ideal of S.

If I is an ideal of S, one can define a multiplication $\star$ on the set $S/I := (S \setminus I) \cup \{0\}$, where 0 is a fresh symbol, letting for all $s, t \in S/I$,

$$s \star t := \begin{cases} st & \text{if } st \in S \setminus I, \\ 0 & \text{otherwise.} \end{cases}$$

$(S/I, \star)$ becomes a semigroup, and we say that S is an extension of the ideal I by the quotient S/I. If S/I happens to be locally finite, we say that S is a locally finite extension of I.

Some Definitions

A semigroup is completely regular if it is a union of groups.

A non-empty subset R of a semigroup S is called a right ideal of S if $rs \in R$ for every $r \in R$ and every $s \in S$. Left ideals are defined dually, and non-empty subset I of S is said to be an ideal of S if I is both left and right ideal of S.

If I is an ideal of S, one can define a multiplication $\star$ on the set $S/I := (S \setminus I) \cup \{0\}$, where 0 is a fresh symbol, letting for all $s, t \in S/I$,

$$s \star t := \begin{cases} st & \text{if } st \in S \setminus I, \\ 0 & \text{otherwise.} \end{cases}$$

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A Construction

Let G be a group and let H be a proper subgroup in G.
Denote by $L_H(G)$ the union of G with the set $G_H := \{gH \mid g \in G\}$ of the left cosets of H in G and define a multiplication on $L_H(G)$ by keeping products in G and letting for all $g_1, g_2 \in G$,

$$g_1(g_2H) := g_1g_2H \text{ and } (g_1H)g_2 = (g_1H)(g_2H) := g_1H.$$

Note that we view the coset gH as different from g even if H is the trivial subgroup!

$L_H(G)$ becomes a semigroup in which G is the group of units and G_H is an ideal.

In the dual way, for every group G and its proper subgroup H, we define the semigroup $R_H(G)$ which is the union of G, being the group of units, and the set ${}_HG := \{Hg \mid g \in G\}$ of the right cosets of H in G, being an ideal.

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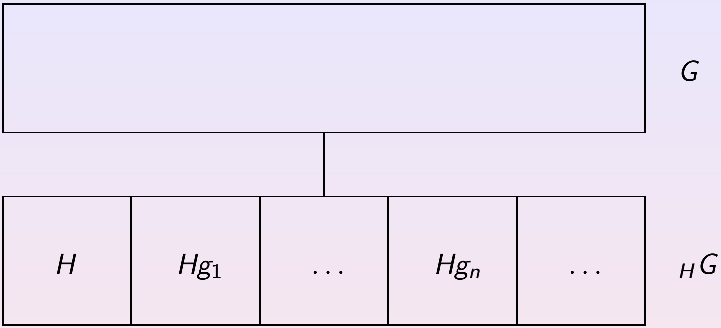
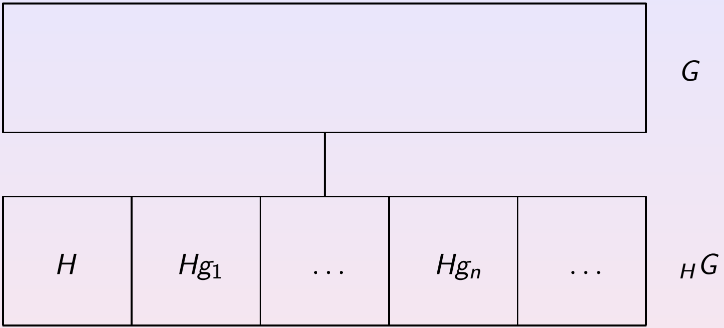


This is the egg-box picture of $R_H(G)$. It is a completely regular semigroup (the union of G and the one-element groups $\{Hg\}$) with two $\mathscr{R}$-classes but infinitely many $\mathscr{L}$-classes (and $\mathscr{H}$-classes) provided that H is of infinite index in G.

If G is finitely generated, so is $R_H(G)$.



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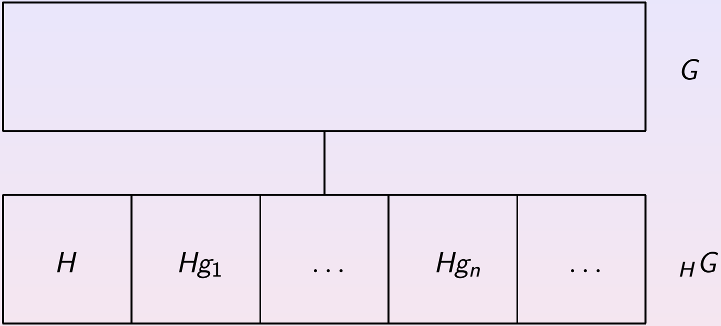

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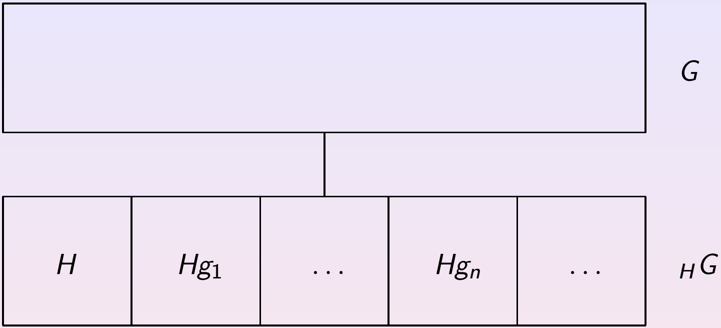

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Theorem 1

A semigroup variety **V** of finite axiomatic rank is locally $\mathscr{H}$-finite if and only if either **V** is locally finite or every semigroup in **V** is a locally finite extension of a periodic completely regular ideal and **V** contains none of the semigroups $L_H(G)$, $R_H(G)$ where G is a NAG and H is its subgroup of infinite index.

The proof proceeds in two steps. First, one reduces the problem to the case when **V** consists of completely regular semigroups. The reduction is relatively easy and invokes the three 3-element semigroups P, P^*, and N_2^1 that often pop up in studying semigroup varieties. They share the base set $\{e, a, 0\}$.

Locally $\mathscr{H}$-Finite Varieties

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In P the Cayley table is

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In N_2^1 the Cayley table is

	e	a	0
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Reduction: an Example

Here is a sample argument used in the reduction step.

Let G be a NAG with generators $g_1, \ldots, g_n$. Assume that a semigroup variety $\mathbf{V}$ contains G and the semigroup P.

Recall that the Cayley table of P is

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The direct product $S := P \times G \in \mathbf{V}$ is easily seen to be generated by the pairs (e, g_i), $i = 1, \ldots, n$, and $(a, 1)$. On the other hand, since $(a, g)x \in \{0\} \times G$ for every $x \in S$, no pairs (a, g) and (a, h) with $g \neq h$ can be $\mathscr{R}$-related in S. Therefore, S has at least $|G|$ different $\mathscr{R}$-classes whence $\mathbf{V}$ is not locally $\mathscr{R}$-finite.

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A divisor of S is a homomorphic image of a subsemigroup of S.

Locally $\mathscr{R}$-Finite and Locally $\mathscr{L}$-Finite Varieties

Theorem 2

A semigroup variety $\mathbf{V}$ of finite axiomatic rank is locally $\mathscr{R}$-finite if and only if either $\mathbf{V}$ is locally finite or every semigroup $S \in \mathbf{V}$ is a locally finite extension of an ideal of the form SR, where R is a periodic completely regular right ideal in S, and $\mathbf{V}$ contains none of the semigroups $L_H(G)$, where G is a NAG and H is its subgroup of infinite index.

Of course, for locally $\mathscr{L}$-finite varieties we get a dual statement.

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Theorem 2'

A semigroup variety $\mathbf{V}$ of finite axiomatic rank is locally $\mathscr{L}$-finite if and only if either $\mathbf{V}$ is locally finite or every semigroup $S \in \mathbf{V}$ is a locally finite extension of an ideal of the form LS, where L is a periodic completely regular left ideal in S, and $\mathbf{V}$ contains none of the semigroups $R_H(G)$, where G is a NAG and H is its subgroup of infinite index.

Recall that in any periodic semigroup, $\mathscr{J} = \mathscr{D}$. Hence, local $\mathscr{J}$-finiteness is the same as local $\mathscr{D}$-finiteness.

A semigroup has central idempotents if it satisfies the quasi-identity $e^2 = e \to es = se$.

Conversely, if every $S \in \mathbf{V}$ is a locally finite extension of an ideal of the form SR or LS, where R (resp. L) is a periodic completely regular right (resp. left) ideal in S, then $\mathbf{V}$ is locally $\mathscr{J}$-finite.

However, there exist finitely generated semigroups with just two idempotents—0 and 1—and infinitely many $\mathscr{J}$-classes.

A classification in this case still has to be completed.

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Conclusion

Thank you for your attention, and once again, all the best for Siniša, Reinhard, and Branimir!!