Developments in Language Theory

Palermo, Italy, July 8th, 2005

Collapsing Words: A Progress Report

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Thus, infinitely many $\boldsymbol{n}$-compressible automata should all be 'served' by the same word! Say, a 3-collapsing word: $\mathbf{4} \longrightarrow \mathbf{1}$; at the same time, $\mathbf{1000000} \longrightarrow \mathbf{999997}$, etc.

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- The goal — to free at least n states (n is a given number).

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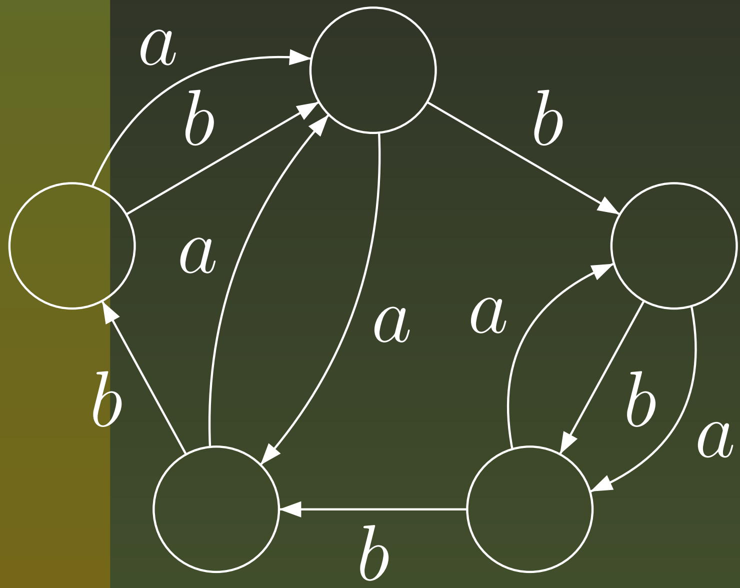

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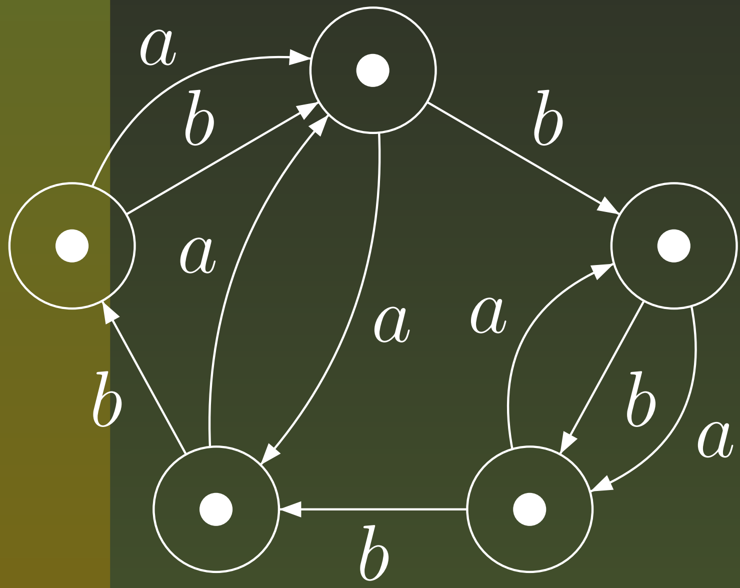

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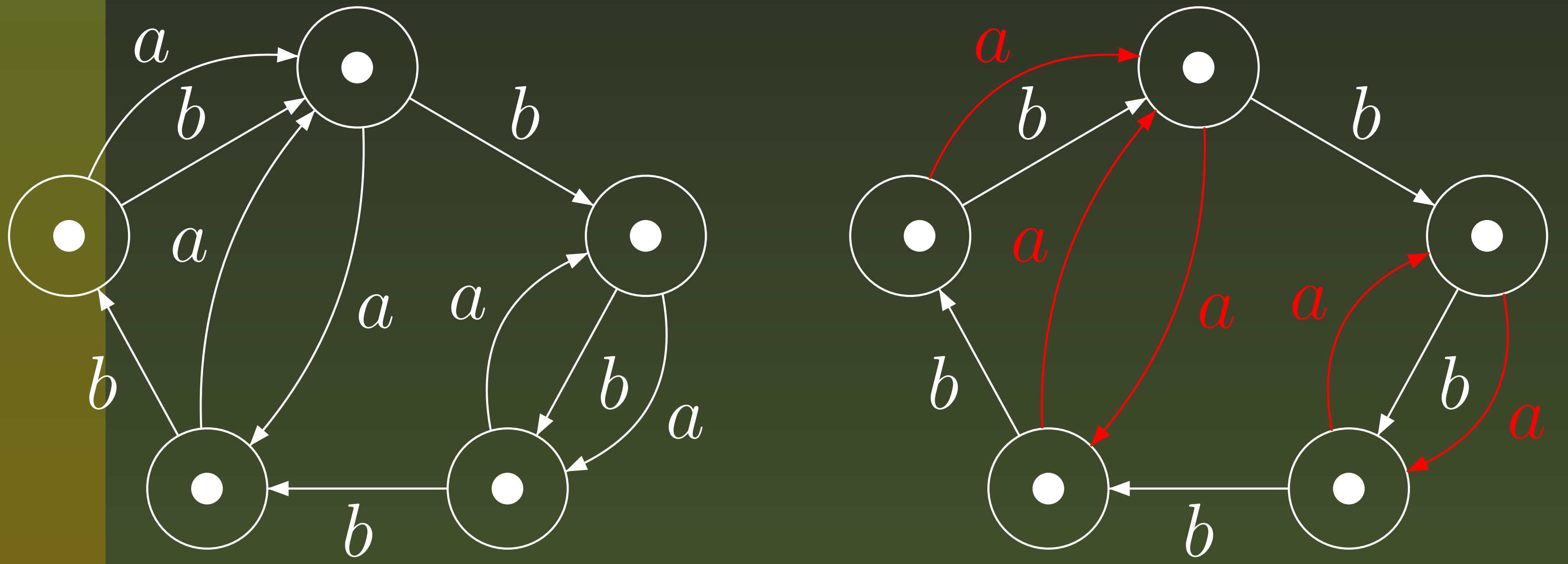

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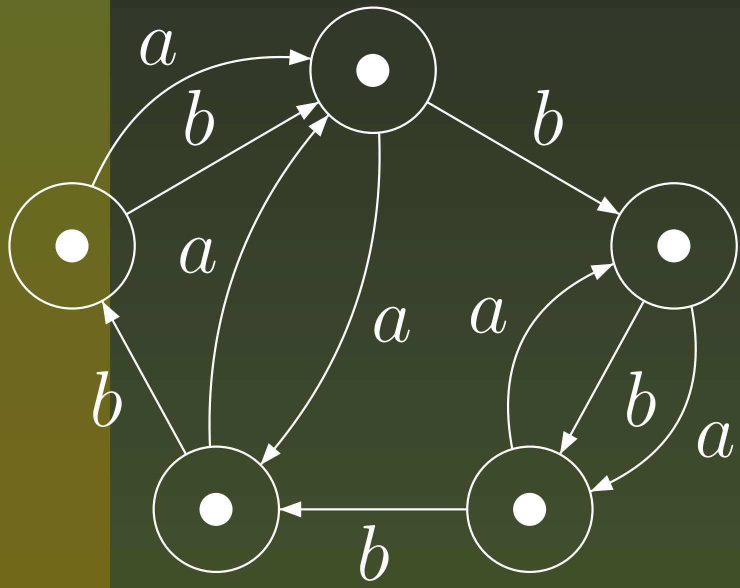

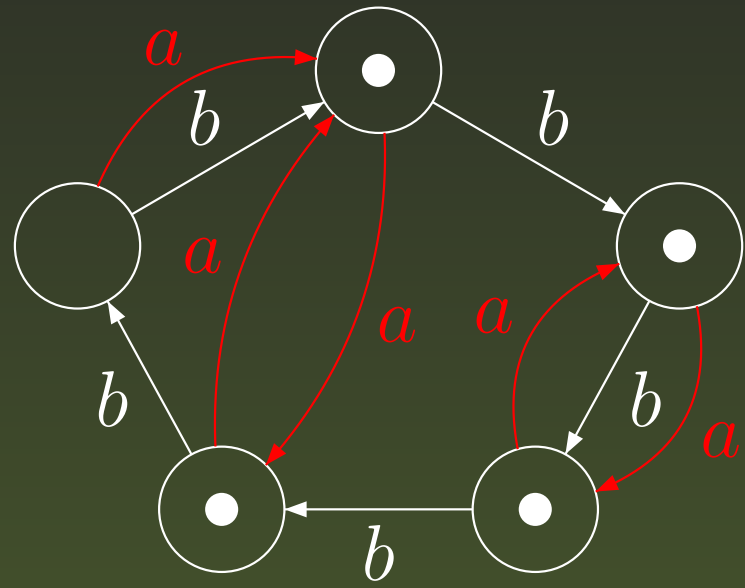

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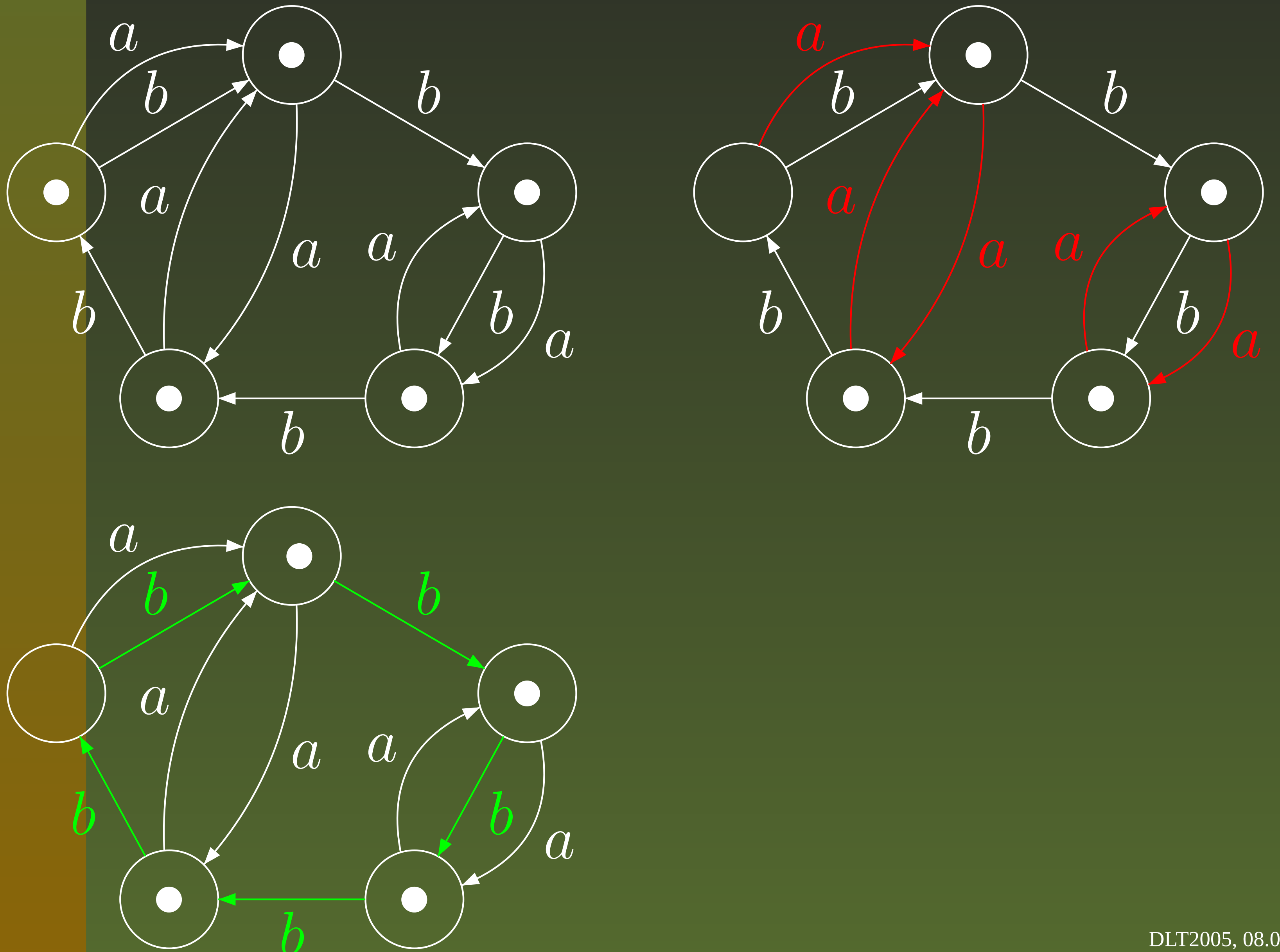

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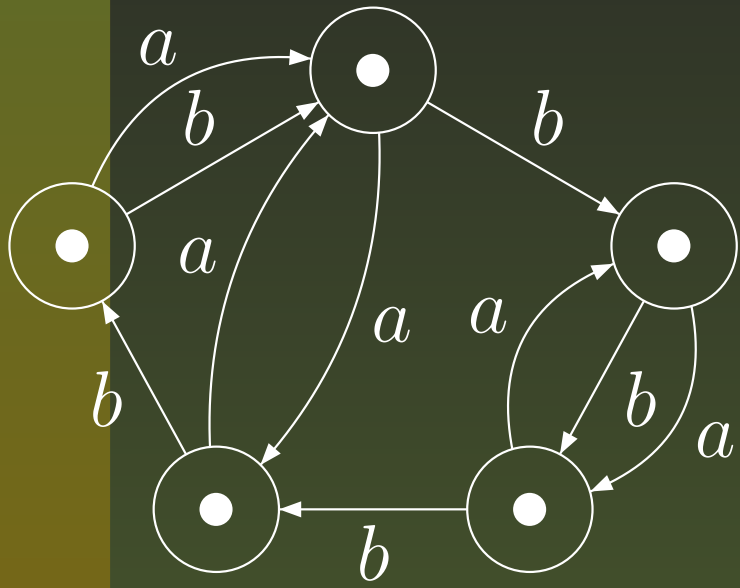

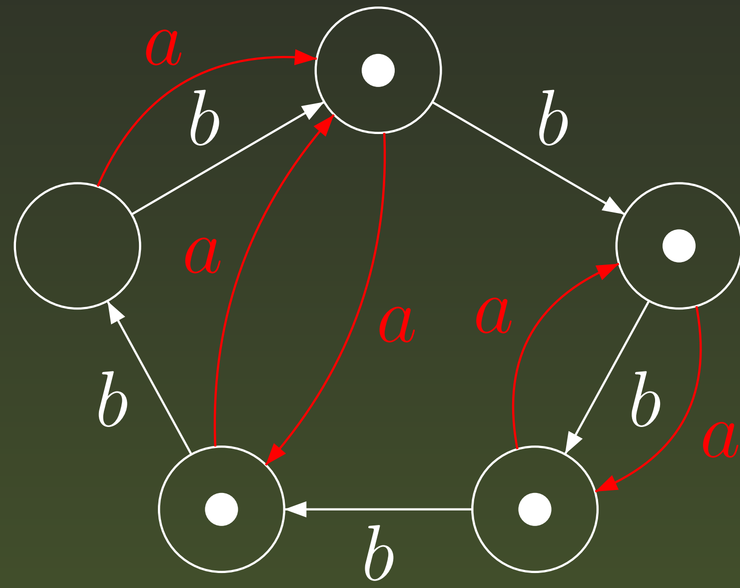

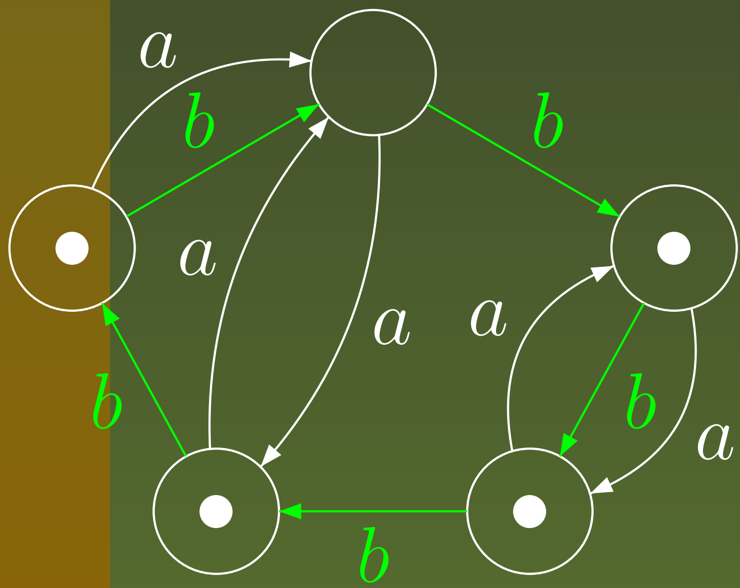

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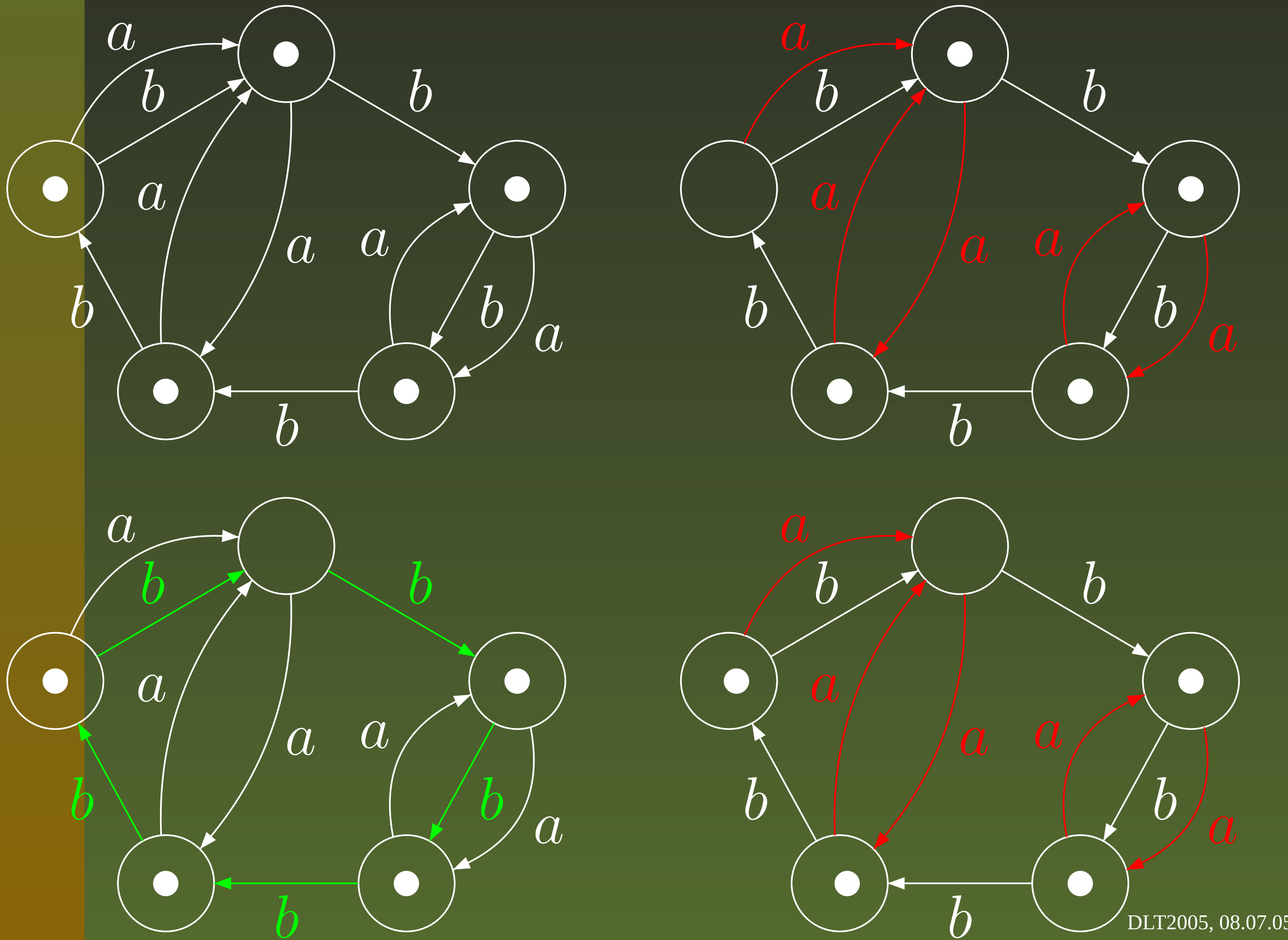

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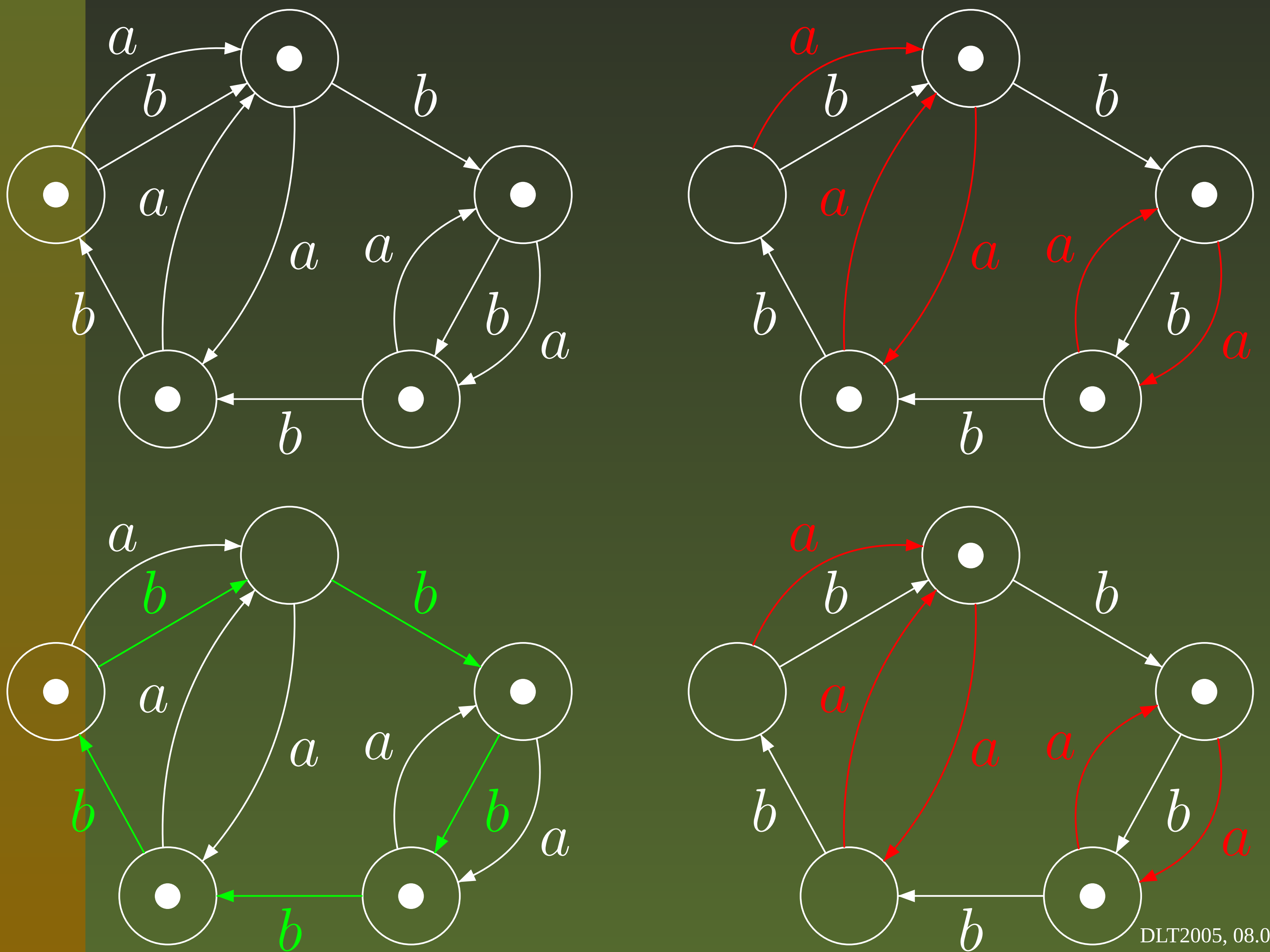

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Therefore:

- the deficiency $\mathrm{df}_{\mathscr{A}}(v)$ of v is exactly the number of freed states;
- $\mathscr{A}$ is n-compressible iff there is a winning sequence of moves for the game on the board $\Gamma(\mathscr{A})$ with the score n to be achieved;
- a word w is n-collapsing iff it represents a universal winning strategy for all games (with the same names of moves and the score n) that can be won.

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Roughly speaking, collapsing words allow one to reduce in a uniform way studying arbitrary transformations (semigroup theory) to studying reversible transformations (group theory).

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Here collapsing words (over the DNA alphabet) might play a role of a 'RESET' button.

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Here collapsing words precisely correspond to engineers' dreams about UFO (Universal Feeder-Orienter).

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  It leads to a word of length $|\Sigma|^{\frac{n^2-n}{2}} + \ldots$.
- A construction related to the so-called rank conjecture, a generalization of the well known Černý's conjecture. It leads to a word of length $|\Sigma|^{n^2} + \ldots$ modulo the conjecture and of length $|\Sigma|^{\frac{n(n+1)(n+2)}{6}} + \ldots$ unconditionally.

The Inductive Construction

Observe that if a DFA $\mathscr{A}$ is 1-compressible then there exists a letter $\boldsymbol{a} \in \boldsymbol{\Sigma}$ with $\mathrm{df}_{\mathscr{A}}(\boldsymbol{a}) \geq \mathbf{1}$.

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In Sauer-Stone's version, the product was over all $v \in \Sigma^*$ with $|v| \leq 2^{n-3} + 1$.

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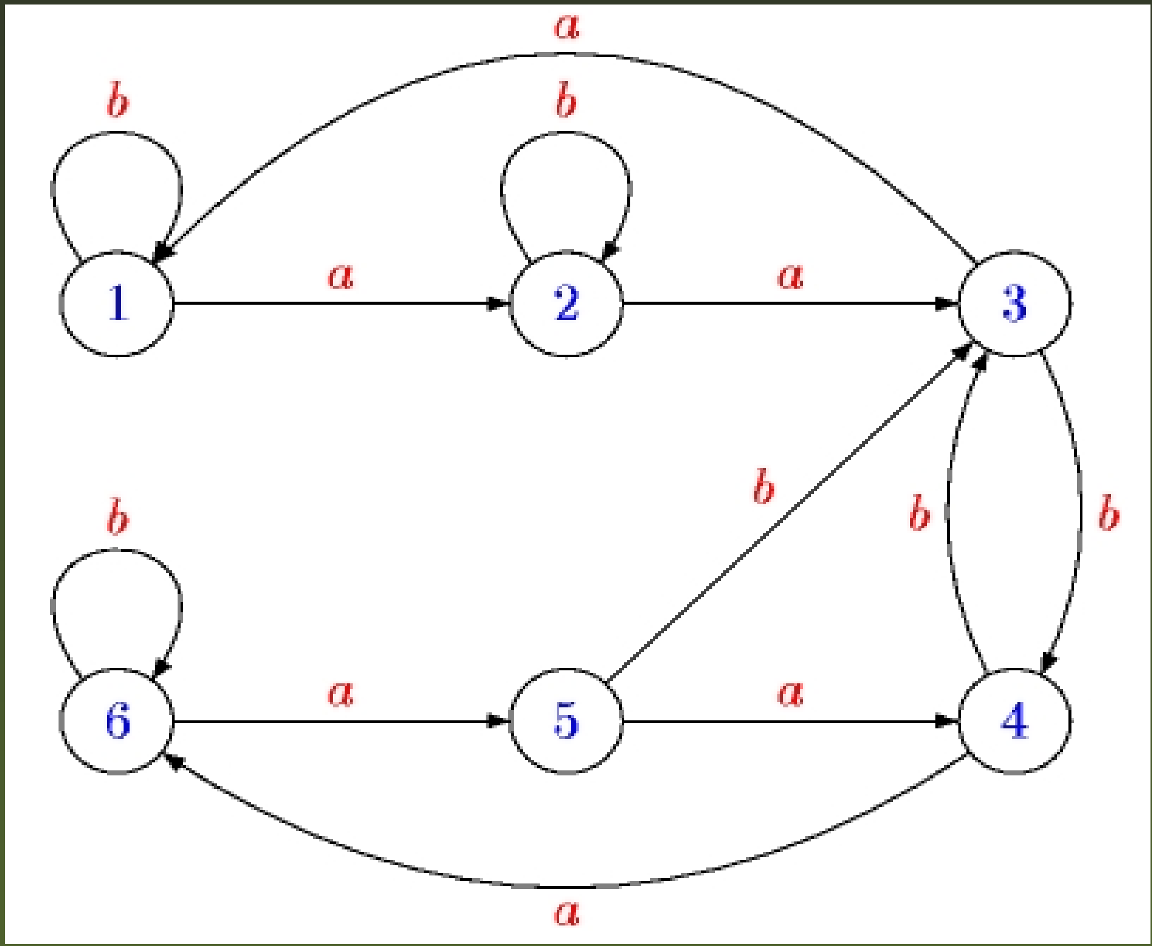
Kari's automaton does **not** refute the rank conjecture!



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- The upper bounds provided by the constructions are very far from being precise.
- The inductive construction is better even if the rank conjecture is true. The optimal universal winning strategy should not consist of optimal local winning strategies glued together!
- The rank conjecture construction may be better for some restricted classes (for aperiodic automata, say).

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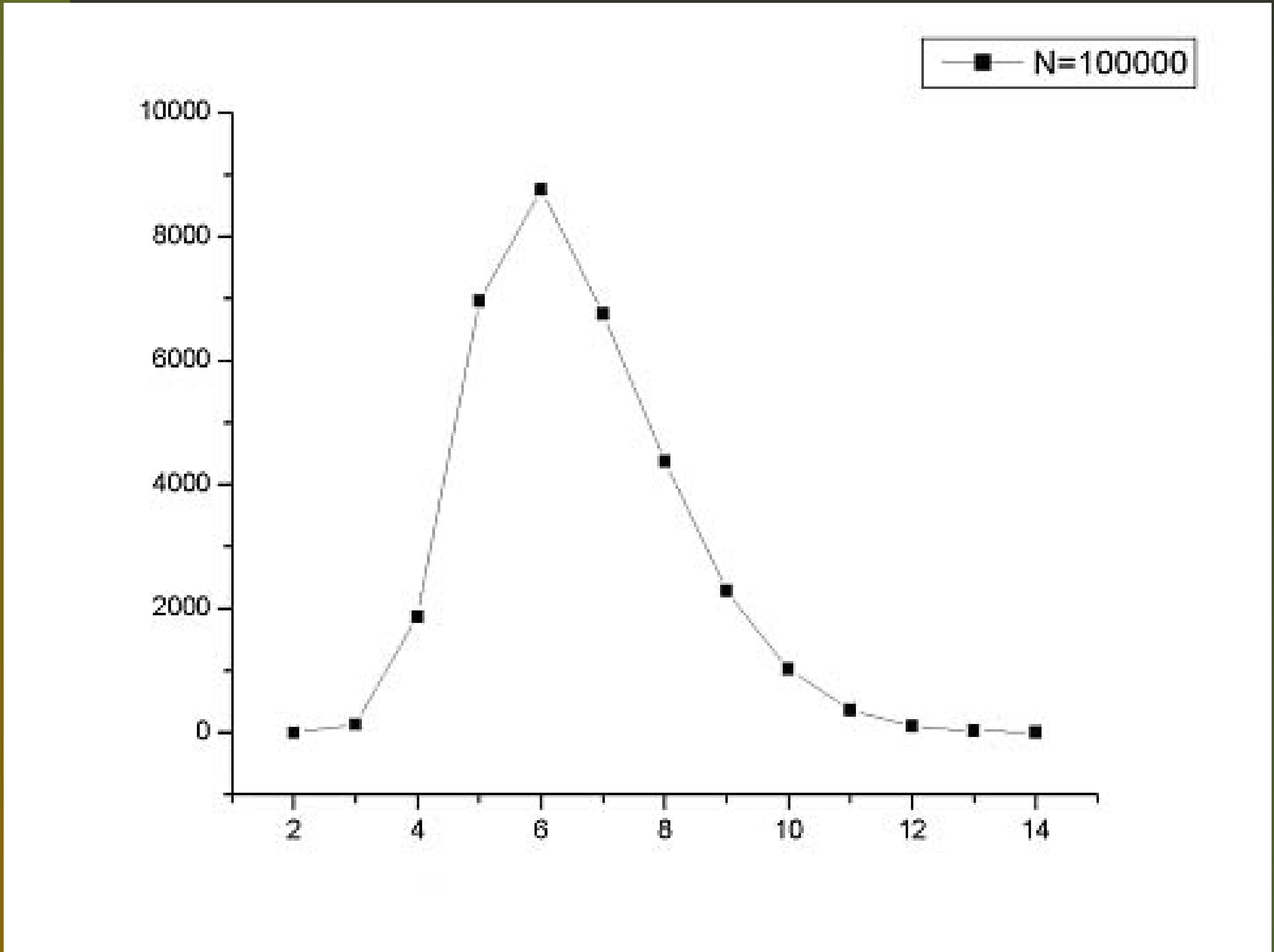
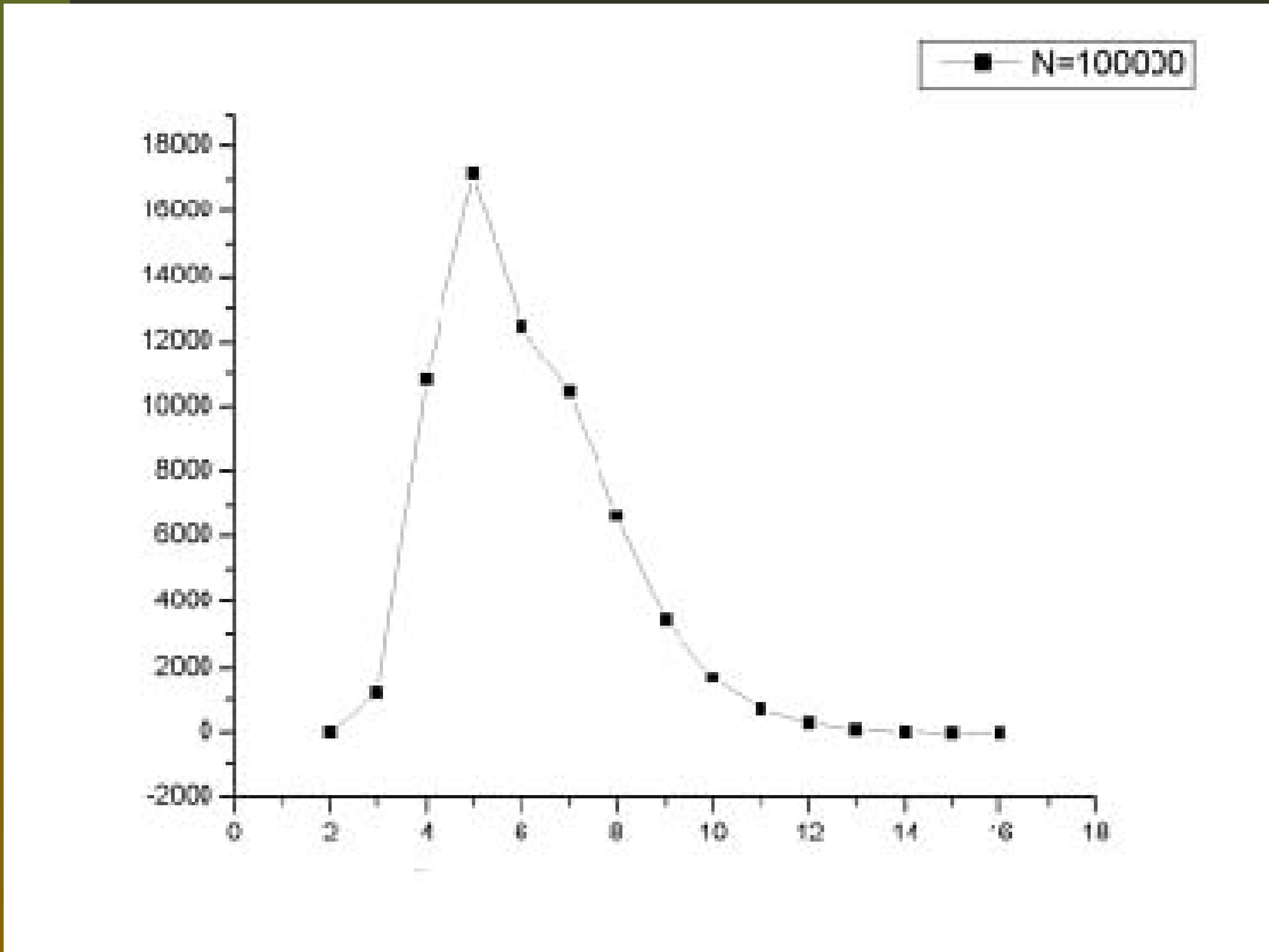
They give some indication that a more economic construction for n-collapsing words may based on the average case approach: include all $v \in \Sigma^*$ with $|v| = n$ as factors (this is certainly necessary) and add just a few longer factors to serve all possible ‘complicated’ game-boards.

20-State Game-Boards



30-State Game-Boards



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The answer is 'yes' for $n = 2$ (Ananichev, Cherubini and $\sim$, 2003).

Recognizing Collapsing Words

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The answer is 'yes' for $n = 2$ (Ananichev, Cherubini and $\sim$, 2003). The algorithm that relies on combinatorial group theory has been implemented by Petrov, and experiments with the implementation have led to many interesting examples and several useful observations.

Recognizing Collapsing Words

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A byproduct of our algorithm for $n = 2$ is the following fact: if w is not 2-collapsing, then $\mathbf{df}_{\mathscr{A}}(w) < 2$ for some 2-compressible DFA $\mathscr{A}$ with $\leq |w|$ states.

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Moreover, it leads to a non-deterministic linear space and polynomial time algorithm recognizing the complement of $\mathcal{C}_2$ whence $\mathcal{C}_2$ is a context-sensitive language.

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Namely, for every word $\boldsymbol{w}$ which is not $\boldsymbol{n}$-collapsing, there exists an $\boldsymbol{n}$-compressible DFA $\mathscr{A}$ with $\leq 3|\boldsymbol{w}|(\boldsymbol{n}-1)+\boldsymbol{n}+1$ states and such that $\mathrm{df}_{\mathscr{A}}(\boldsymbol{w})<\boldsymbol{n}$.

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This implies that the set of all 2-collapsing words over each finite alphabet is a recursive and even context-sensitive language.

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- the set of all current free states;
- the set of all 'post-free' states to which the current letter brings states that were empty before the move;
- the set of all 'next-to-free' states that could be achieved from current free states if we would repeat the same move.

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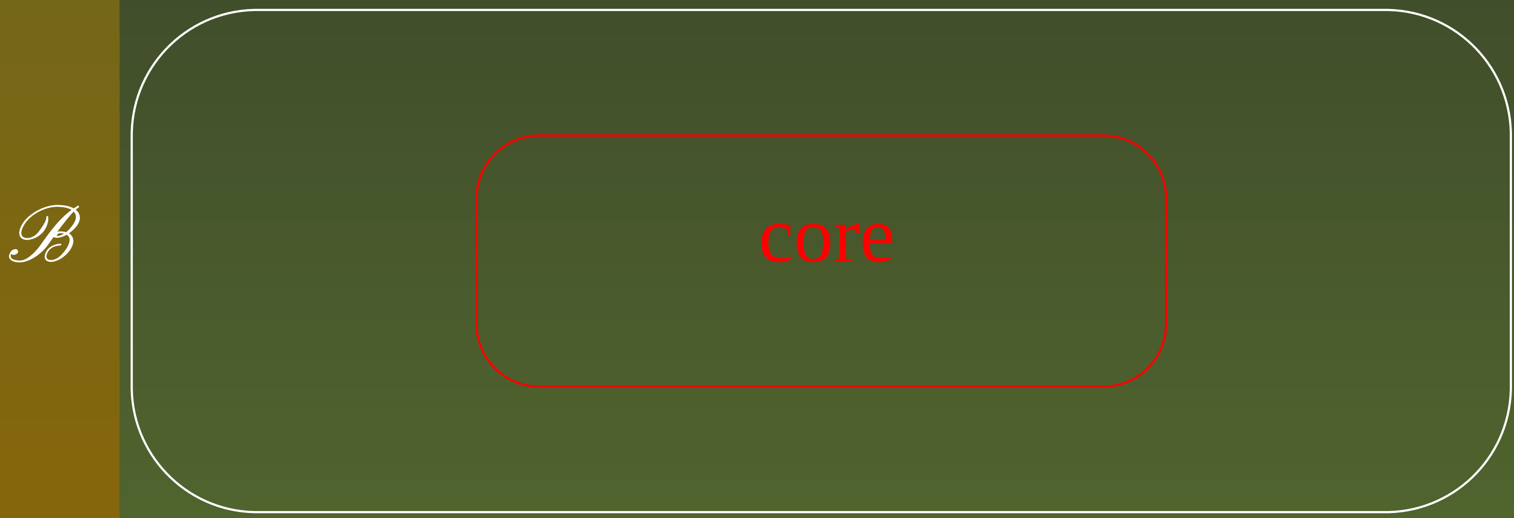

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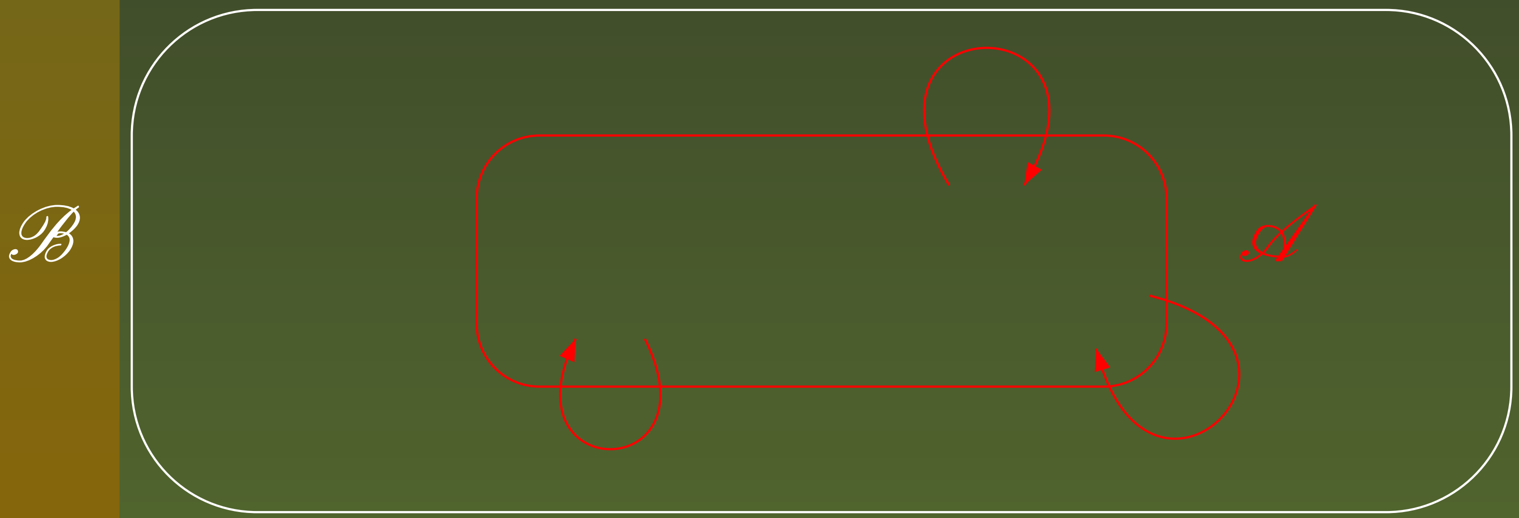
The total number of states subject to control does not exceed $3|w|(n-1)$. They form a core of the ‘small’ n-compressible DFA $\mathscr{A}$ which witness that w is not n-collapsing while ‘irrelevant’ parts of $\mathscr{B}$ are substituted by some tiny buffer automata adding at most $n+1$ extra states.



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- A significant progress in studying collapsing words has been achieved over the last 5 years. Still many intriguing questions remain widely open.
- The game interpretation underlying in this talk is not only entertaining but also provides some important insights. In particular, it was essential for finding a decision procedure for the property of being n-collapsing.
- I hope this talk did not make you collapse…