Symmetries and Ordered Structures

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Endomorphisms of linear orders: the finite axiomatization problem and complexity of membership testing

Mikhail Volkov

Ural State University, Ekaterinburg, Russia

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Can finite linear orders and their endomorphisms be of any interest and any importance?

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- is related to my recent research.

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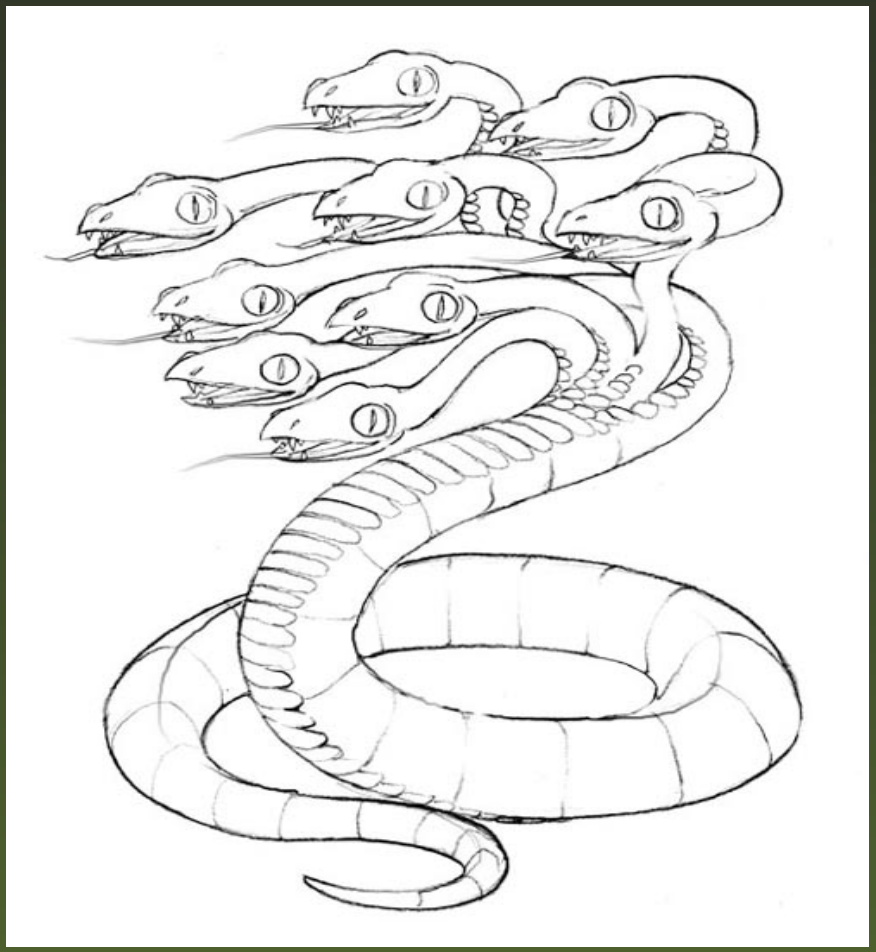

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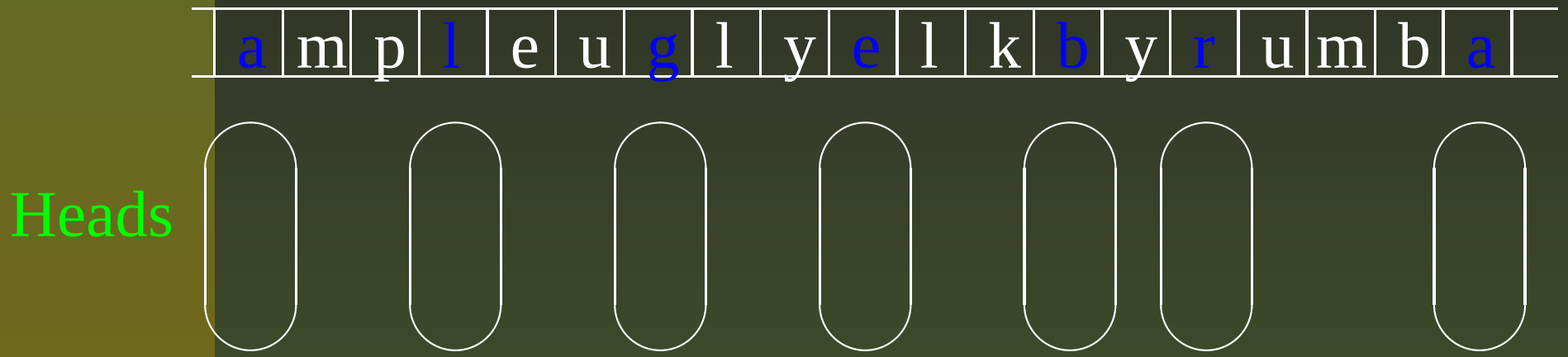

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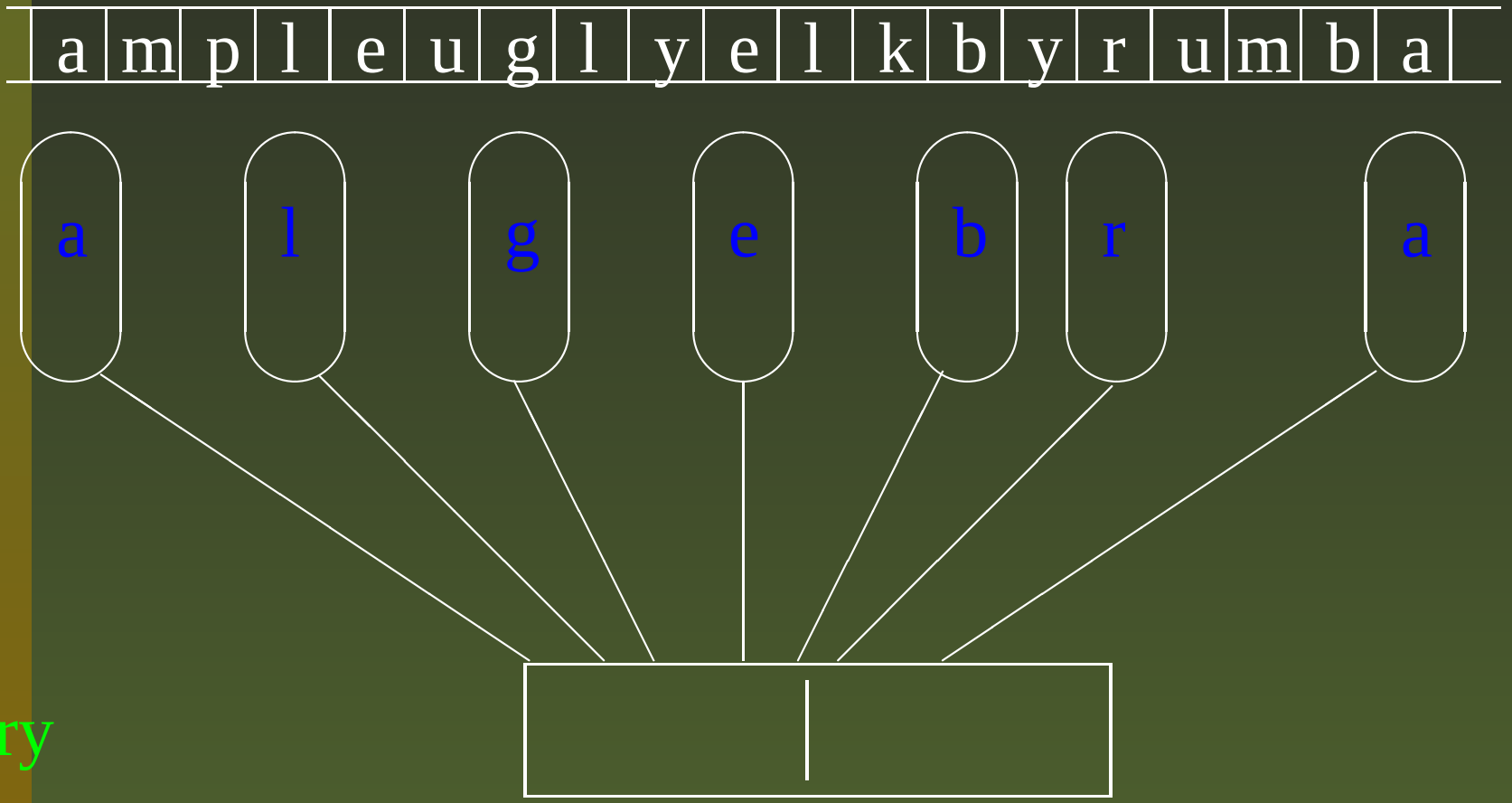

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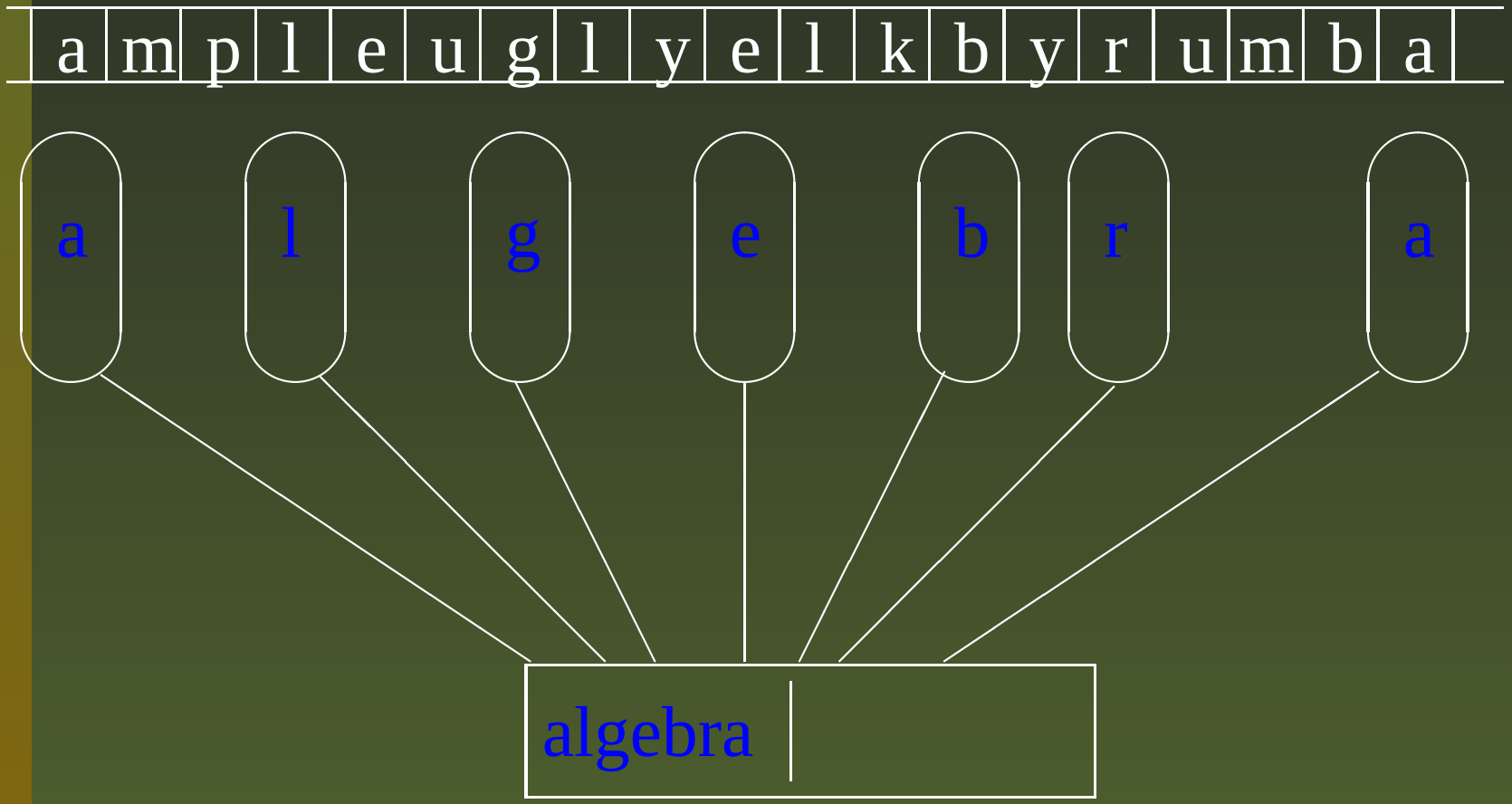

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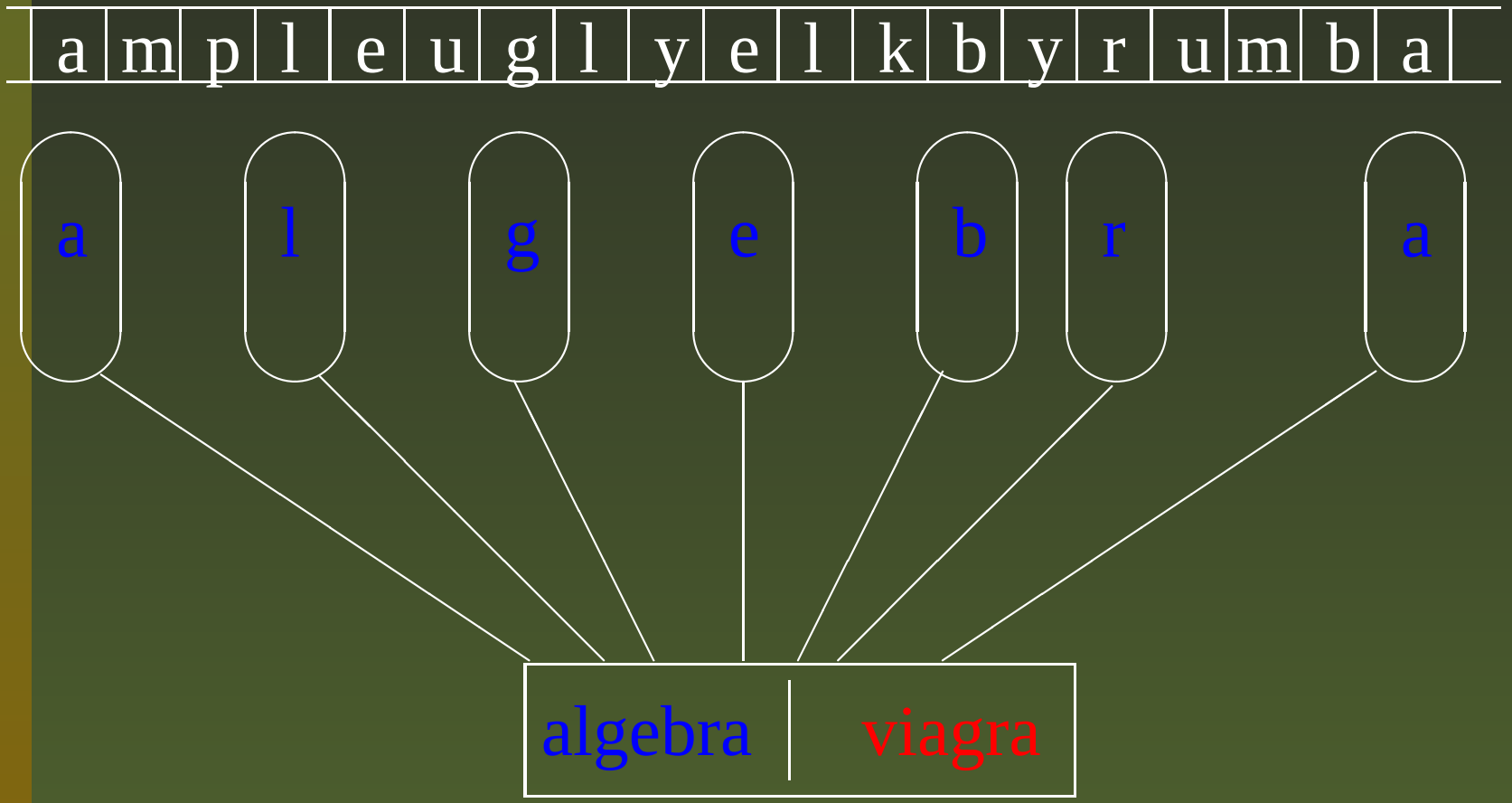

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A language $L \subseteq \Sigma^*$ is said to be *recognized* by a hydra automaton $\mathscr{H}$ if $\mathscr{H}$ accepts exactly words that are members of L. Such languages are called *piecewise testable*.

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Simon's hierarchy of piecewise testable languages:

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Syntactic monoids

For a language $L \subseteq \Sigma^*$ its *syntactic congruence* $\sim_L$ is defined by

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For a regular language L, the syntactic monoid $M(L)$ can be also defined as the *transition monoid* of the *minimal automaton* of L.

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Thus, whenever L is "given", so is $M(L)$.

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Theorem 1. (Imre Simon, 1972) *A language L is piecewise testable if and only if its syntactic monoid $M(L)$ is $\mathscr{J}$-trivial.*

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- Very efficient: There are polynomial time algorithms to verify if the syntactic monoid $M(L)$ is $\mathscr{J}$-trivial when presented the minimal automaton of L. Such a description of $M(L)$ is much more compact than the Cayley table — recall that the transition monoid of an automaton with n states may consist of as many as n^n elements!

Simon vs. Schützenberger

Compare with Schützenberger's theorem (1966) that provides an algebraic characterization of star-free languages: *a language L can be defined by a star-free expression* (that is, involving only Boolean operations and products but not Kleene's star) *if and only if the syntactic monoid $M(L)$ has only trivial subgroups*.

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Again a very natural language property is related to a natural semigroup property that can be verified in time $O(|M(L)|^2)$. On the other hand, the problem of deciding whether or not $M(L)$ has only trivial subgroups from the minimal automaton of L is PSPACE-complete!

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- Endomorphisms of linear orders — Higgins, 1997.

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Simon's theorem means that the pseudovariety **J** of all finite $\mathscr{J}$-trivial monoids is generated by syntactic monoids of piecewise testable languages.

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Let $\mathbf{J}_h$ denote the pseudovariety of finite monoids generated by the syntactic monoids of piecewise testable languages of height $\leq h$. We have

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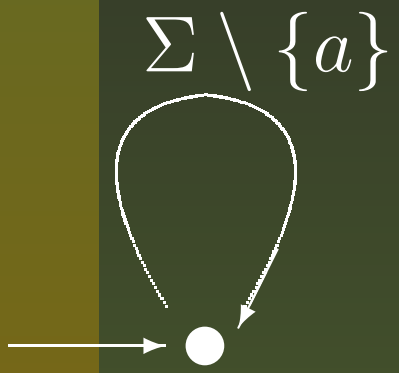

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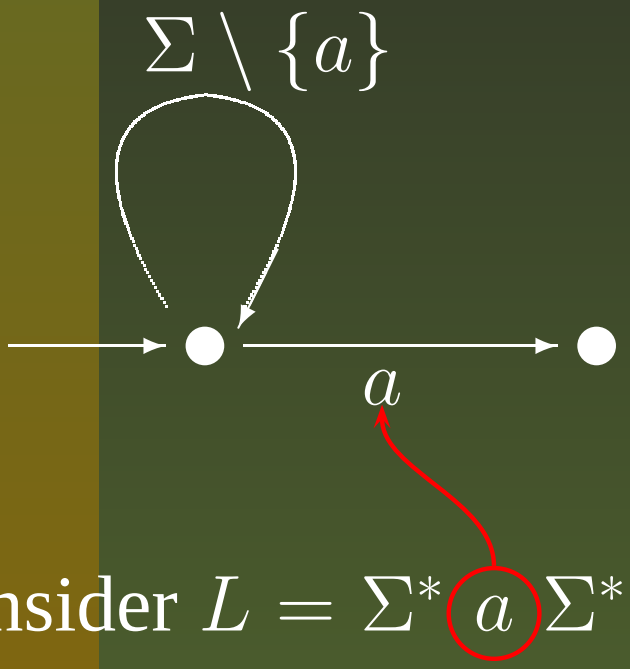

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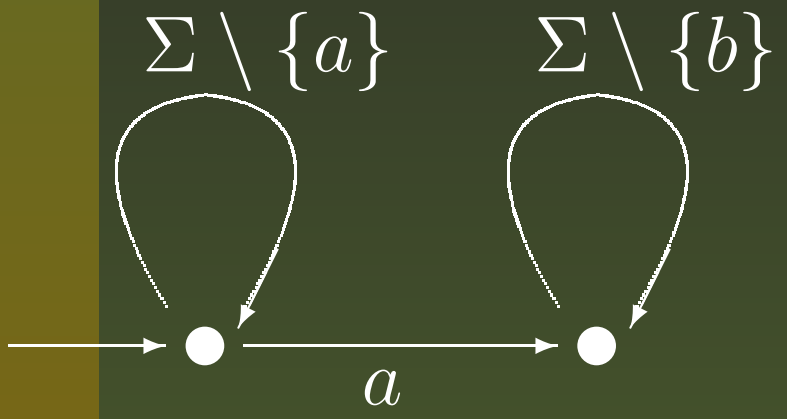

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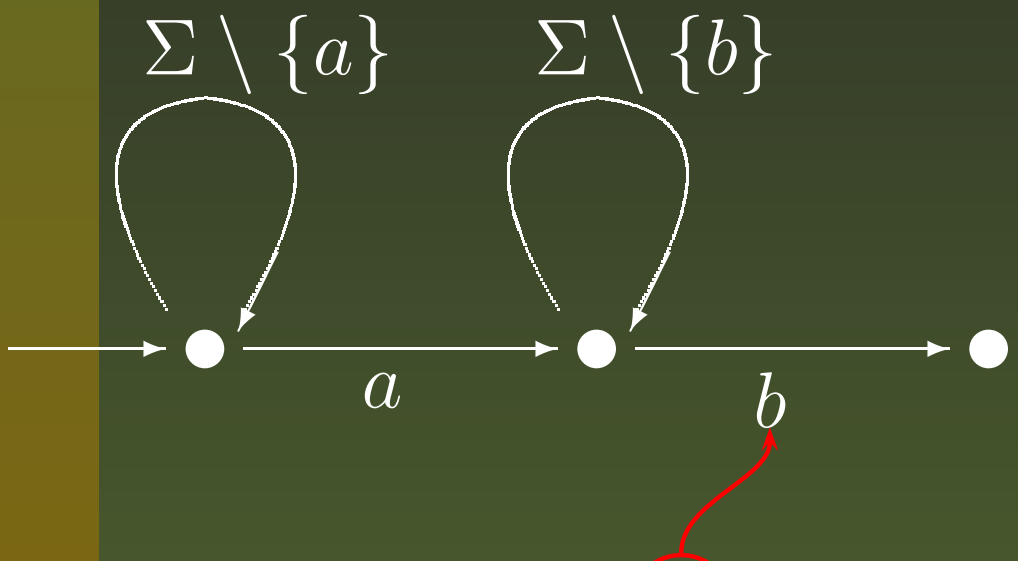

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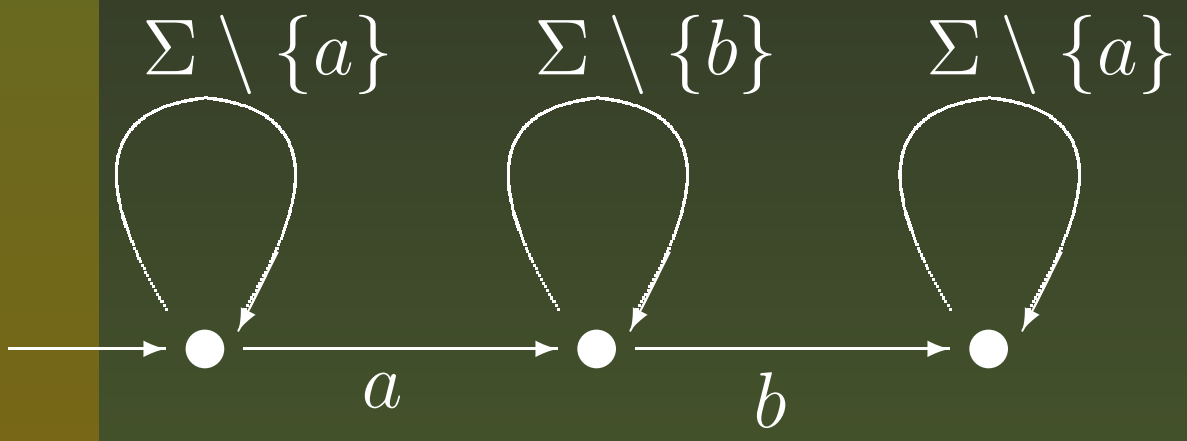

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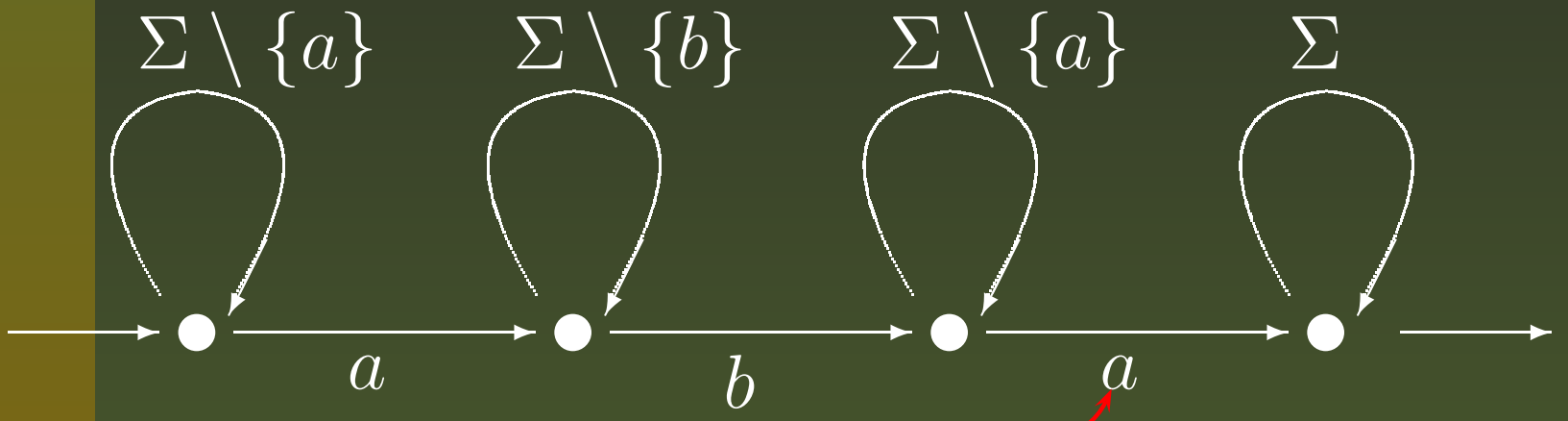

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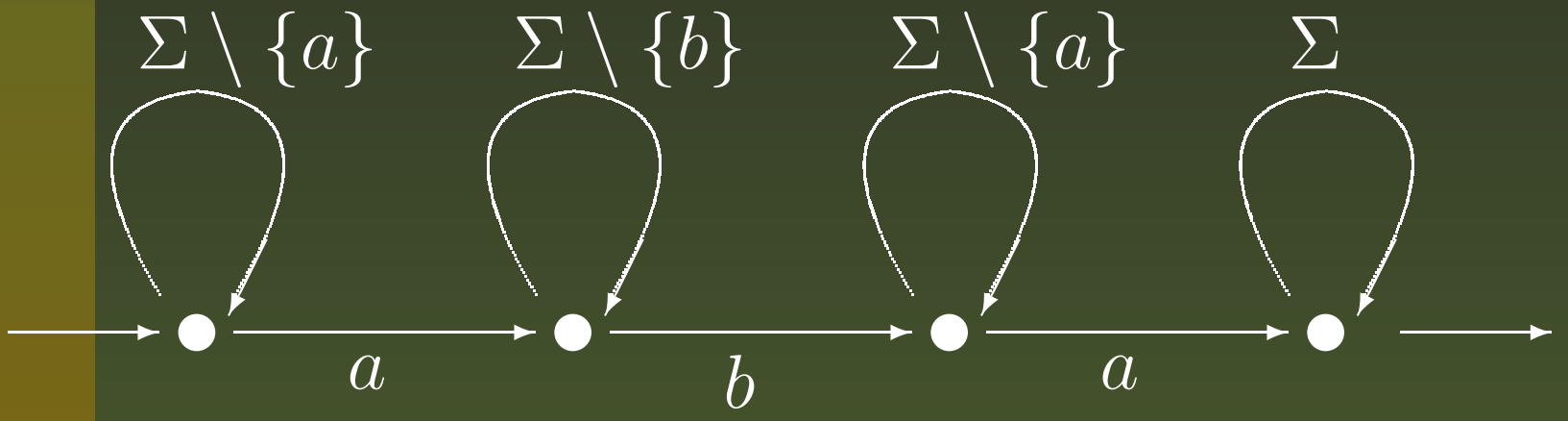

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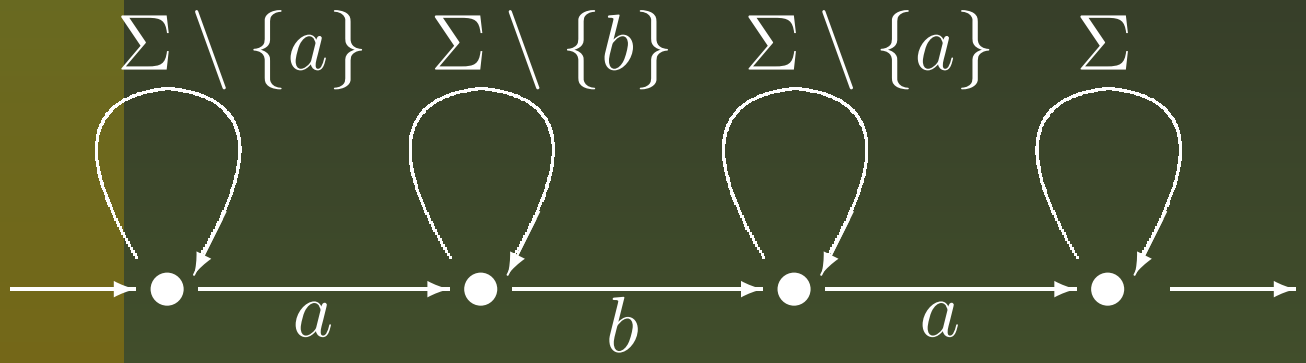

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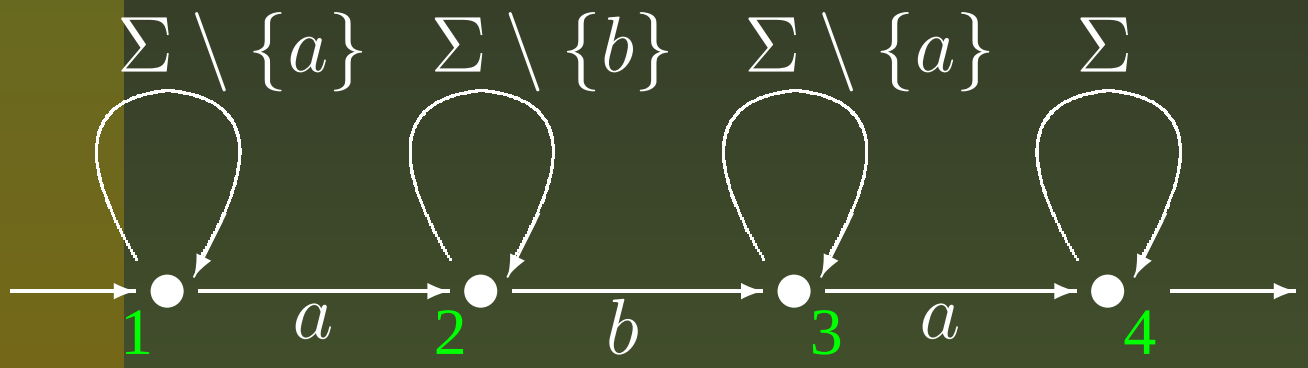

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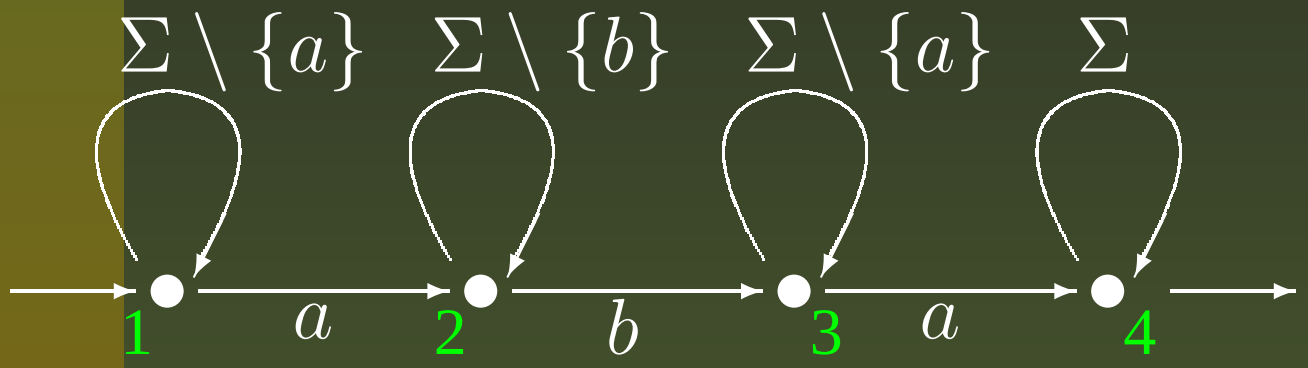

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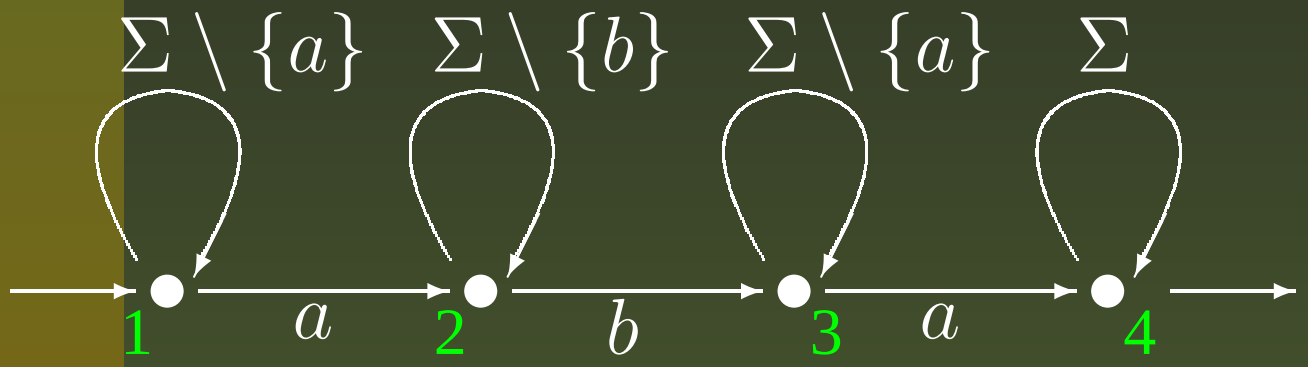

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Recall that a transformation α of a linear order is said to be *extensive* if $i \leq i\alpha$ for every i in the domain of α.

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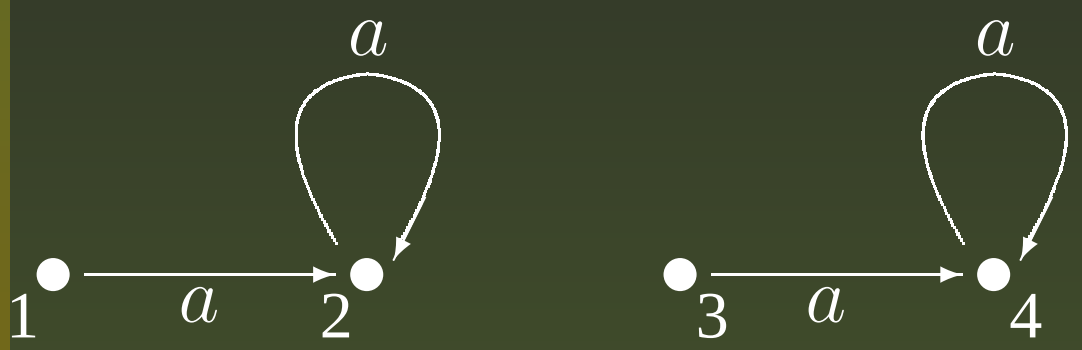
For instance, this is the one induced by a:



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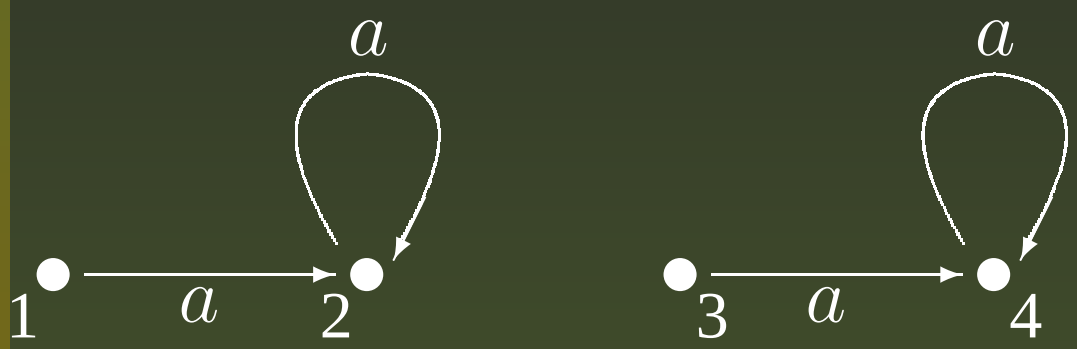

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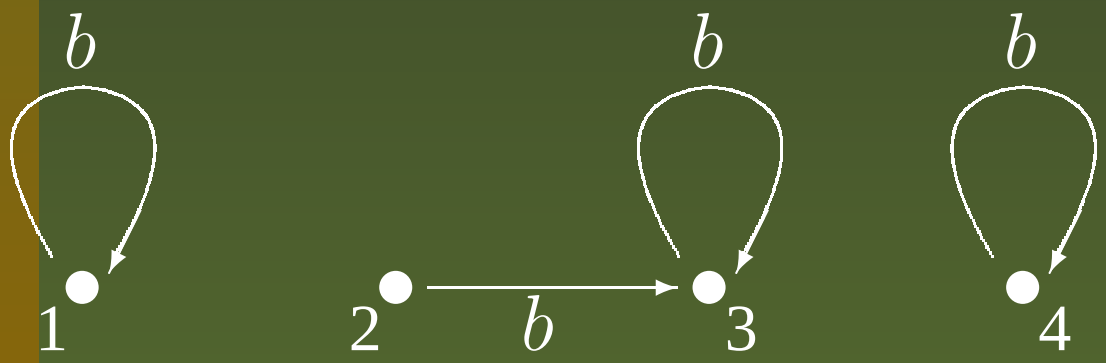

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Hence the transition monoid of the deterministic automaton recognizing the language $L = \Sigma^* a \Sigma^* b \Sigma^* a \Sigma^*$ consists of extensive endomorphisms of the chain $1 < 2 < 3 < 4$.

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we end up in the monoid $\mathscr{C}_{h+1}$. Therefore the pseudovariety $\mathbf{J}_h$ is contained in the pseudovariety $\operatorname{pvar} \mathscr{C}_{h+1}$ generated by $\mathscr{C}_{h+1}$.

Endomorphisms of linear orders

Theorem 2. ($\sim$, 2003) *For every $h = 1, 2, \ldots,$ the monoid $\mathscr{C}_{h+1}$ generates the pseudovariety $\mathbf{J}_h$.*

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Corollary. (Jean-Eric Pin, 1984) *For each $h = 1, 2, \ldots,$ the membership problem for the pseudovariety $\mathbf{J}_h$ is decidable, and hence, given a piecewise testable language, its height can be algorithmically determined.*

Recognizing height

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Can we do better?

Recognizing height via identities

Lemma. (Eilenberg-Schützenberger, 1976) *Every pseudovariety generated by a single finite monoid is equational, that is, it consists precisely of finite monoids satisfying a certain system of usual monoid identities.*

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d) *The monoids* $\mathscr{C}_n$ *with* $n > 4$ *are nonfinitely based.*

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Thus, there is an efficient algorithm to check if a given piecewise testable language can be recognized by a hydra automaton with 1, 2 or 3 heads, but this approach fails for larger numbers of heads.

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One may conclude that the optimal number of heads is equal to 3!

Conclusion

What we have seen is just a sample from a rather big area in which natural combinatorial properties of languages lead to certain classes of transformation monoids of finite linear orders (endomorphisms, partial endomorphisms, partial automorphisms, extensive mappings, etc).

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These properties define 4 monoids of partial transformations of $\{1, 2, \ldots, n\}$: $\mathscr{T}_n$, $\mathscr{I}_n$, $\mathscr{PO}_n$, $\mathscr{PE}_n$.

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- $\mathscr{T}_n$
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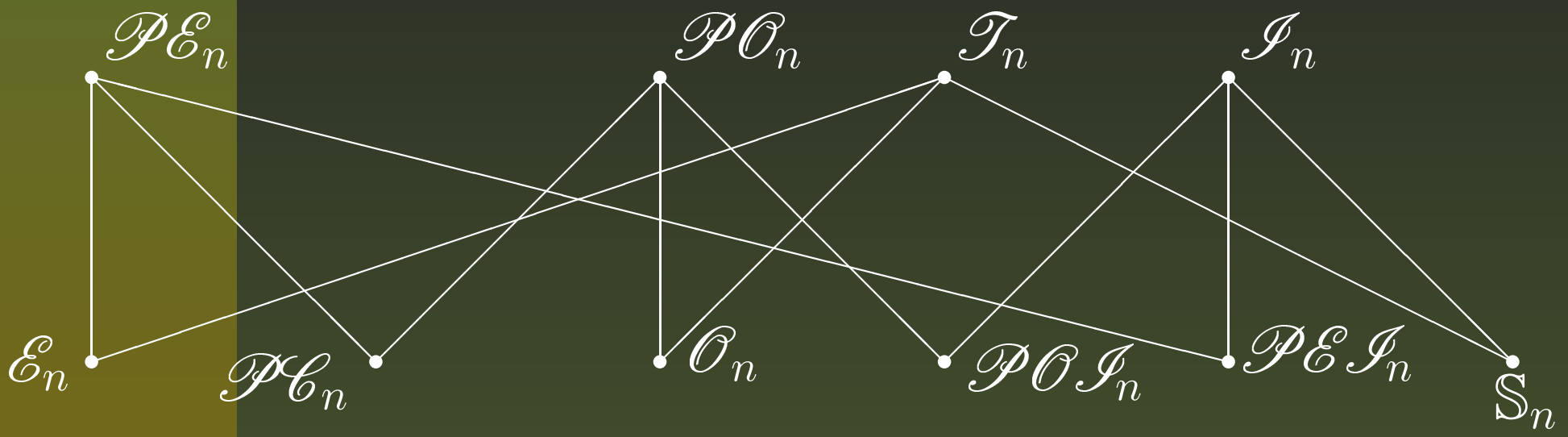
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$\mathscr{POI}_n$
$\mathscr{PEI}_n$
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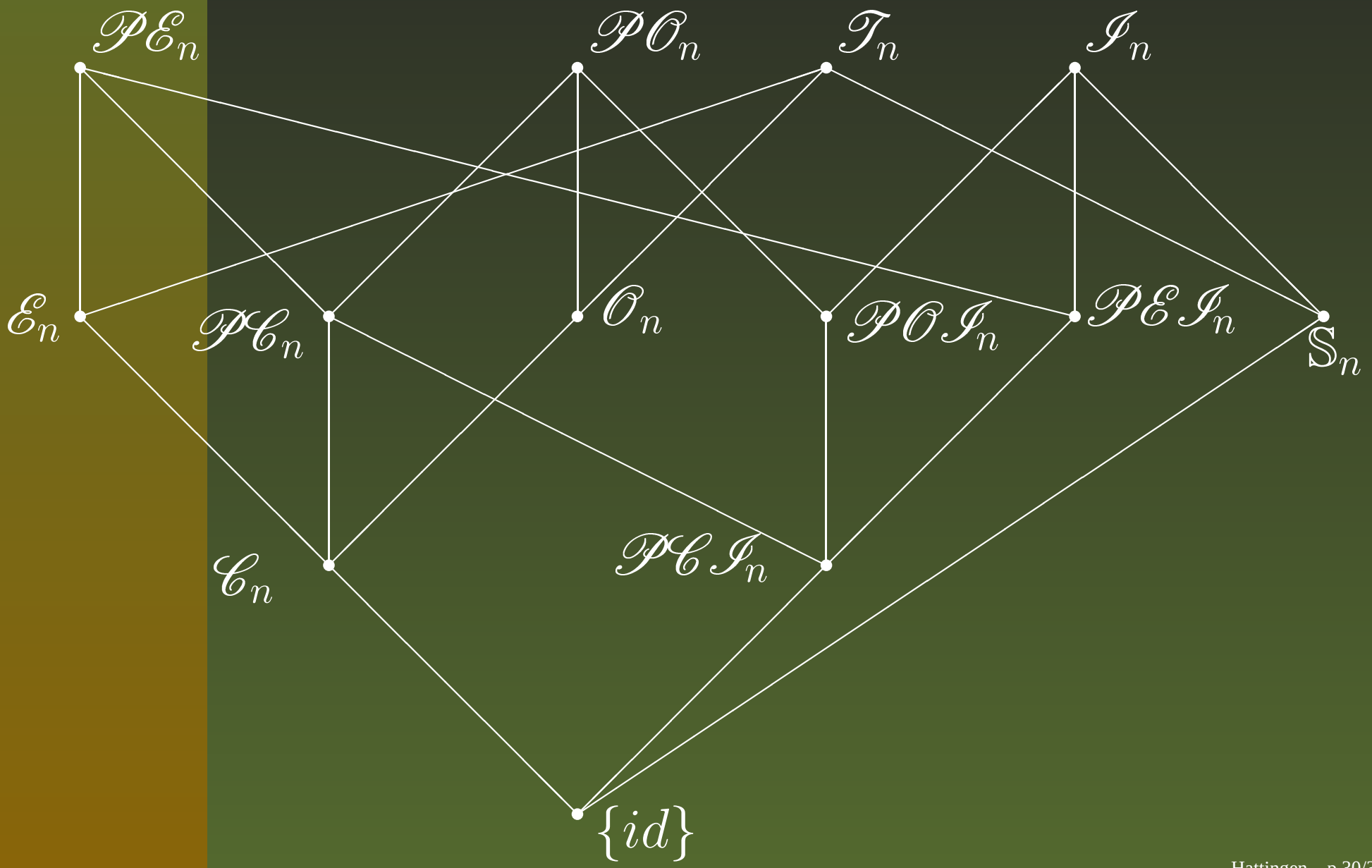
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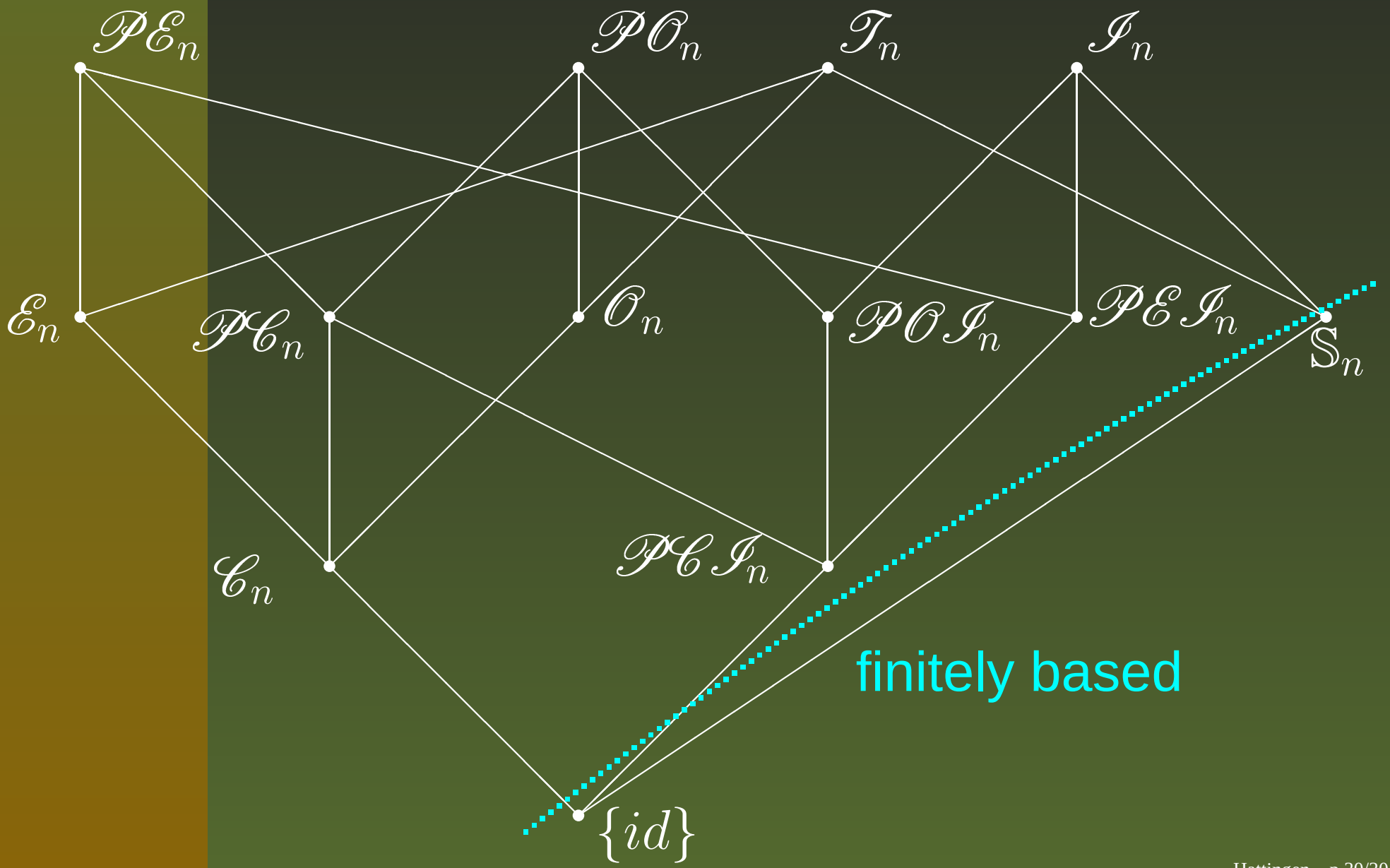
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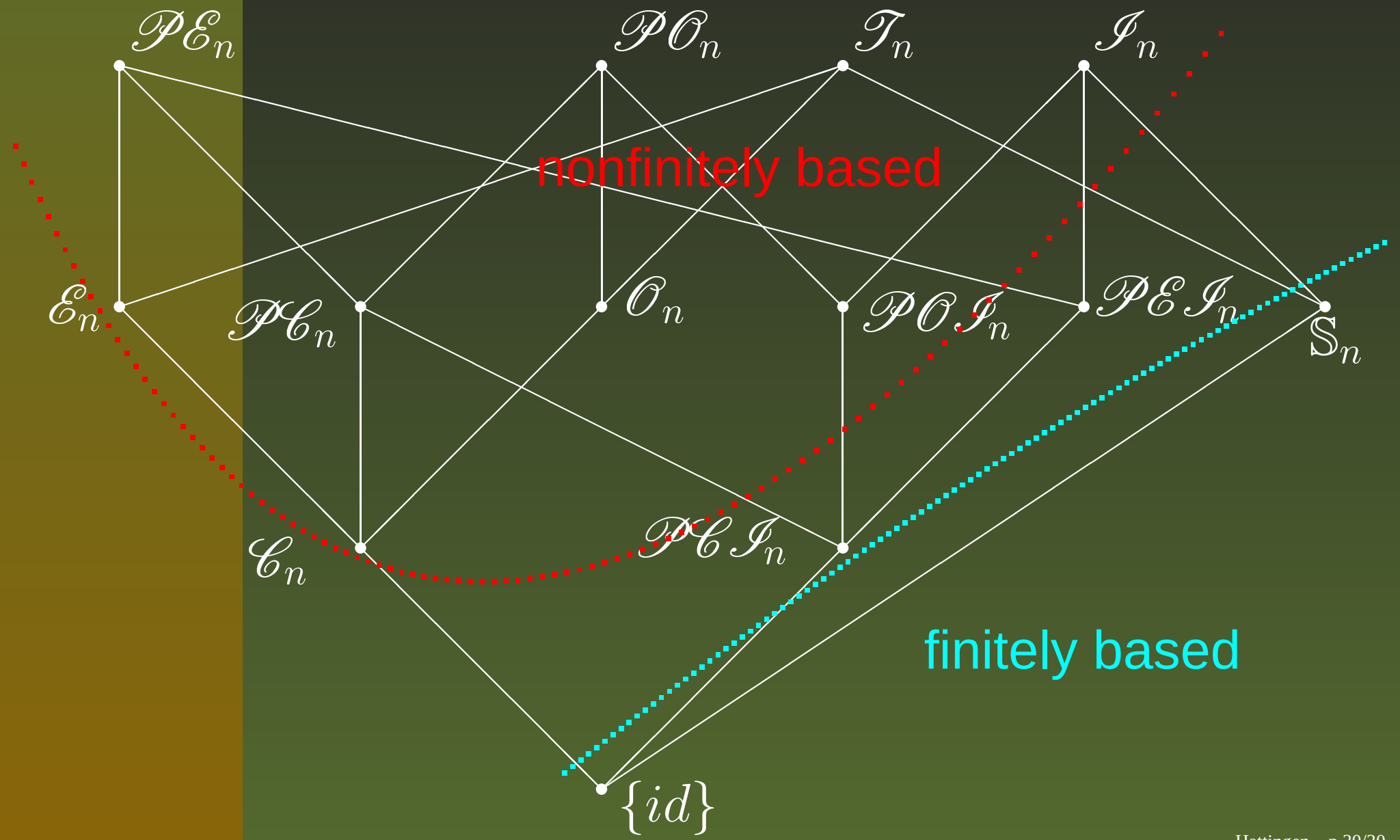
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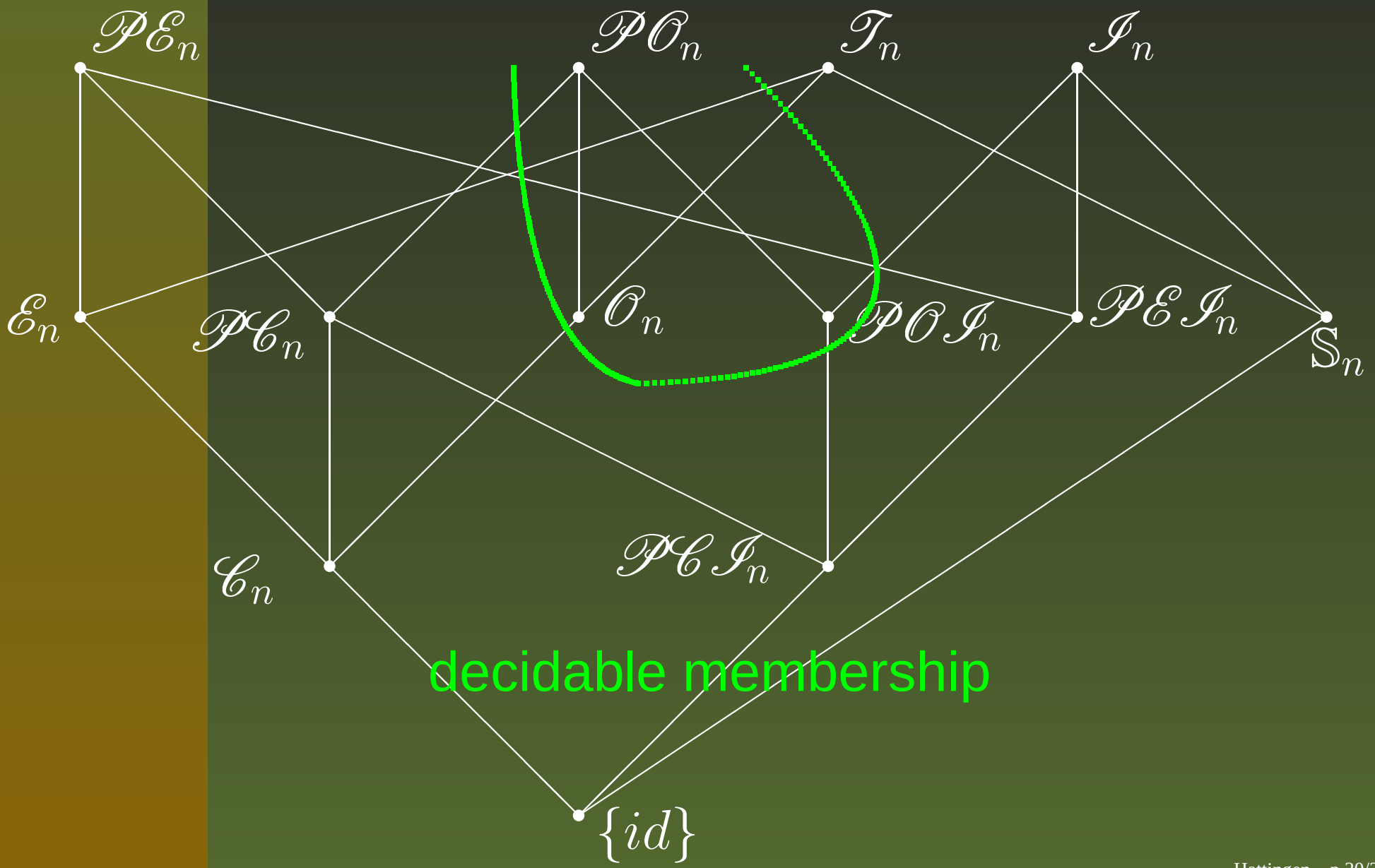
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