A Quest for Short Identities

Which questions does automata theory ask algebra over and over again (but gets no answers so far)?

Mikhail Volkov

Ural Federal University, Ekaterinburg, Russia

GAIA, July 19, 2013

Finite Automata

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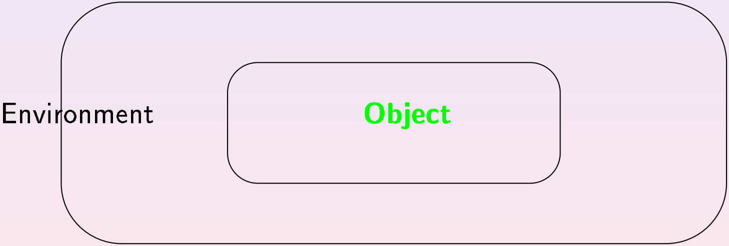

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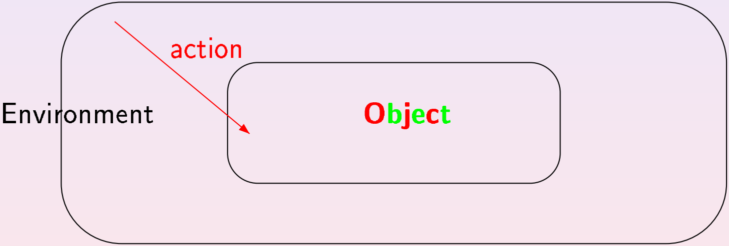

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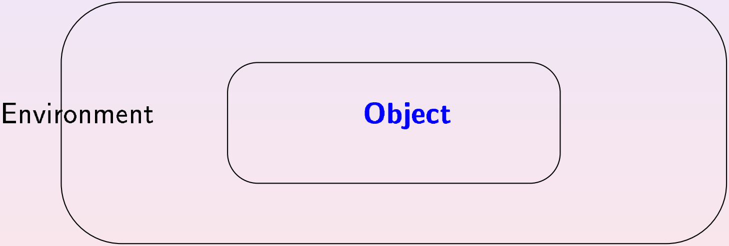

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This notion originates in the seminal work by Alan Turing ("On Computable Numbers, With an Application to the Entscheidungsproblem", Proc. London Math. Soc., Ser. 2, 42 (1936), 230–265).

"*The behavior of the computer at any moment is determined by the symbols which he is observing, and his state of mind at that moment*".

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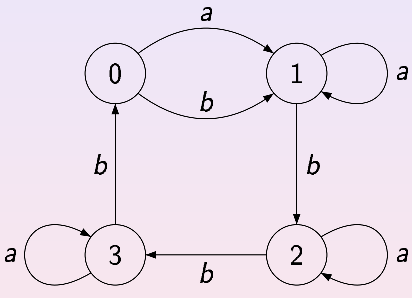

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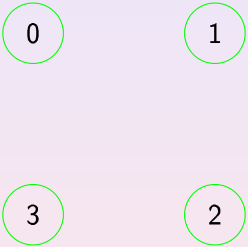

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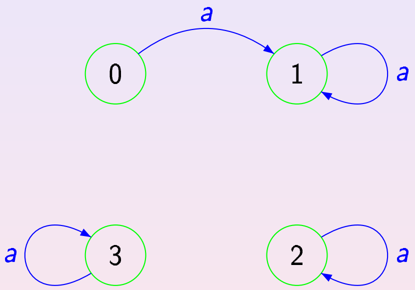

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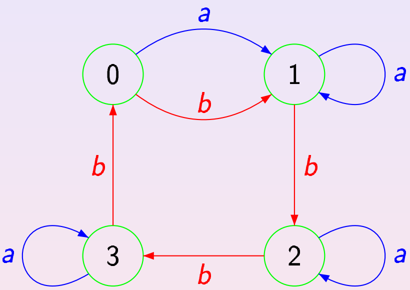

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Terminology and Notation

We consider complete deterministic finite automata (DFAs):

$$\mathscr{A} = \langle Q, \Sigma, \delta \rangle.$$

Here

- Q is the state set;
- Σ is the input alphabet;
- $\delta : Q \times \Sigma \to Q$ is the transition function.

Σ^* stands for the set of all words over Σ including the empty word.
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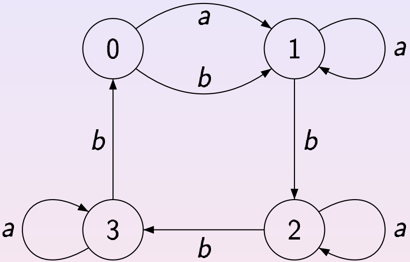

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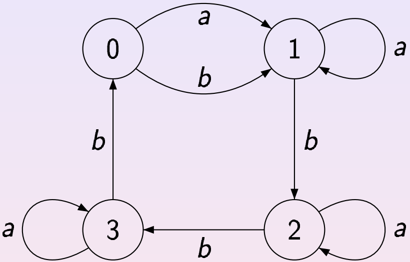

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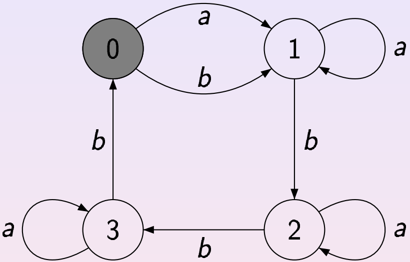

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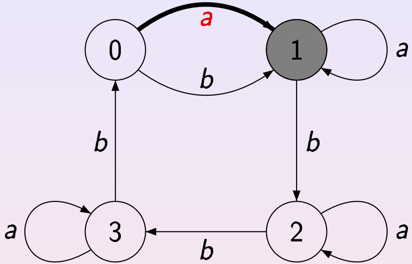

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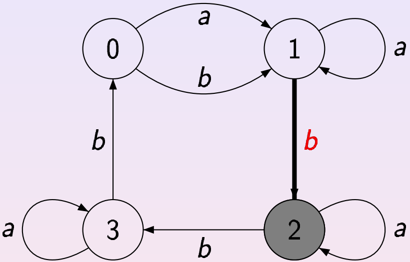

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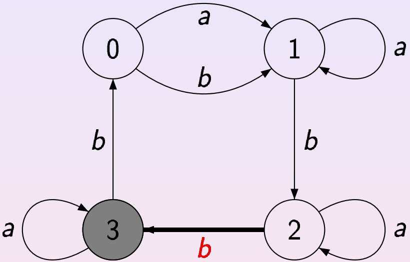

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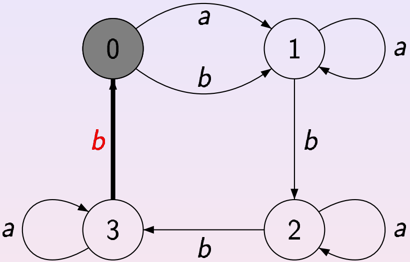

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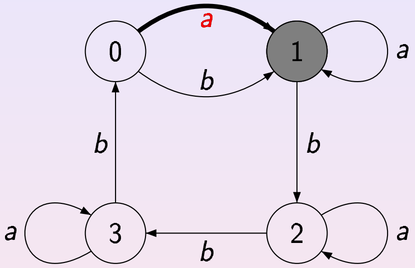

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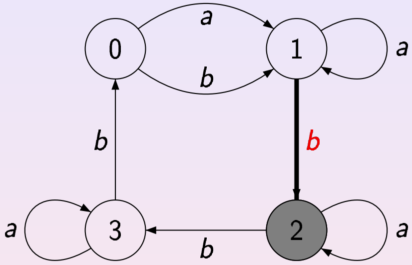

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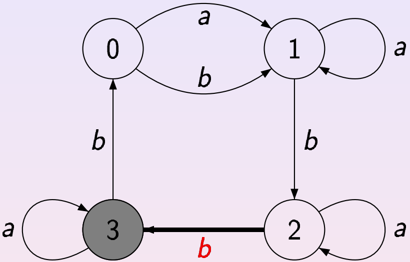

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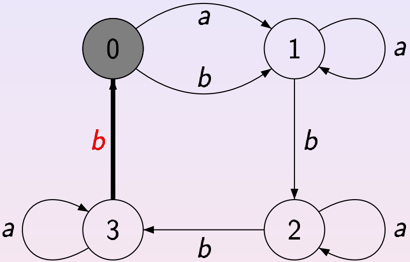

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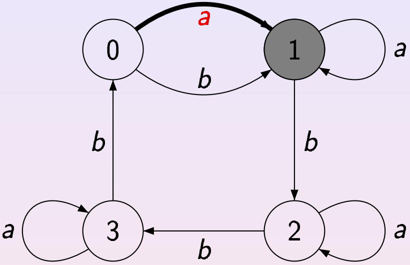

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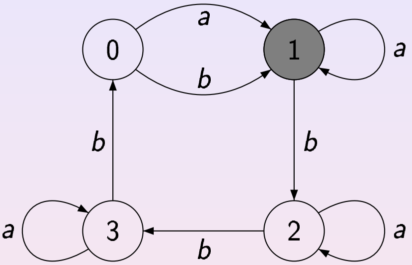

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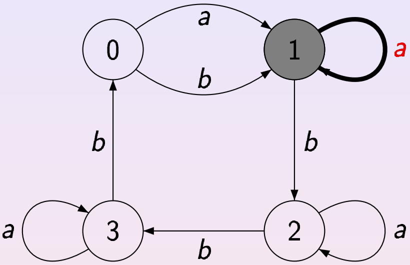

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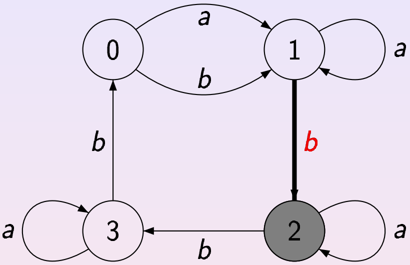

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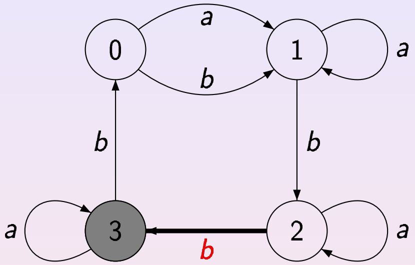

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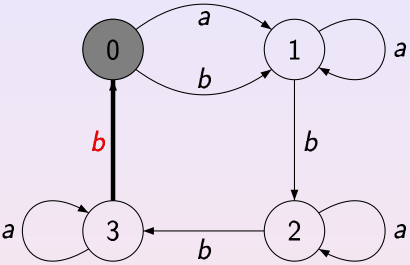

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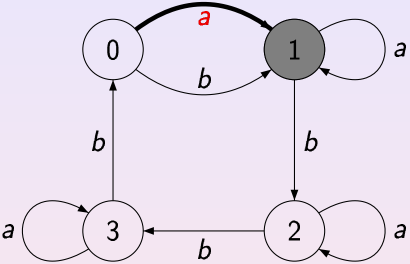

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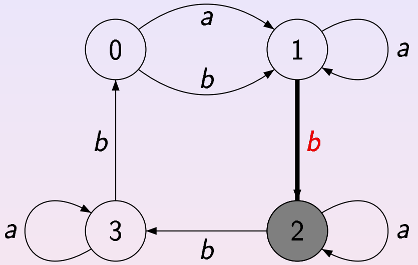

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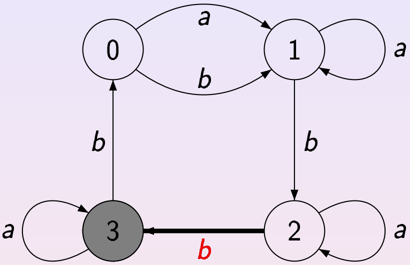

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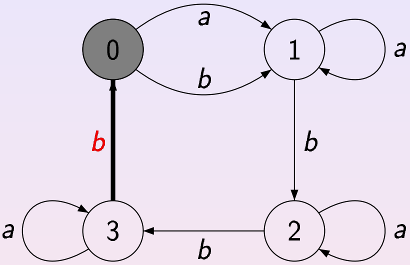

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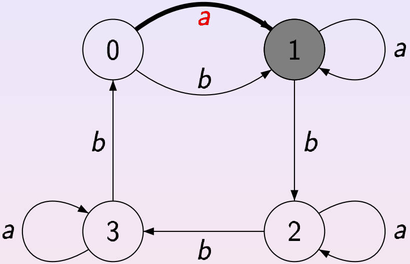

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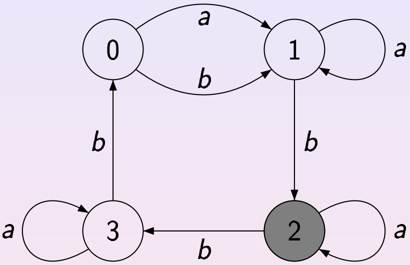

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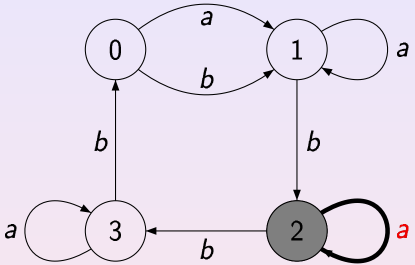

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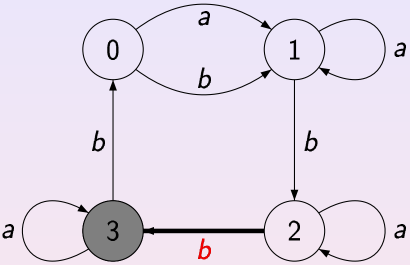

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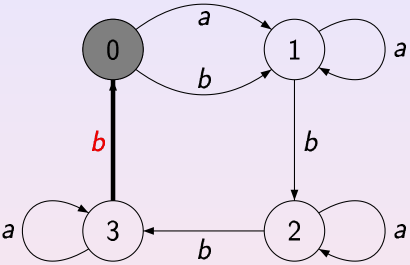

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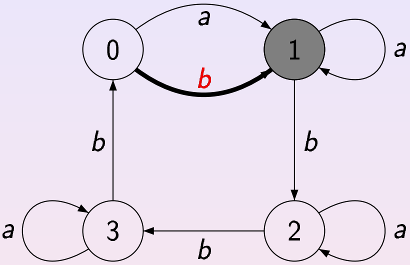

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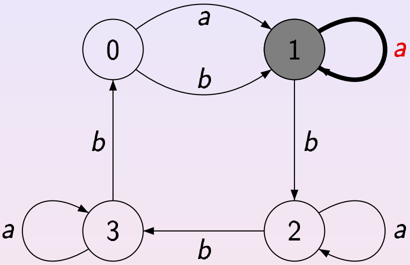

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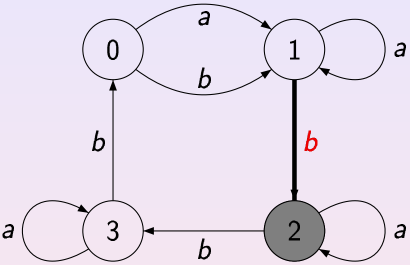

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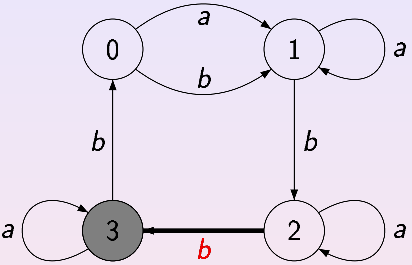

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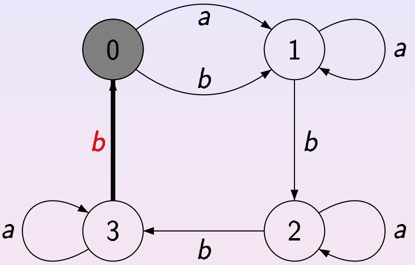

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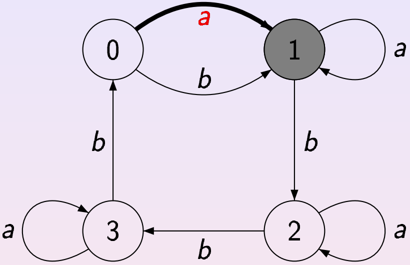

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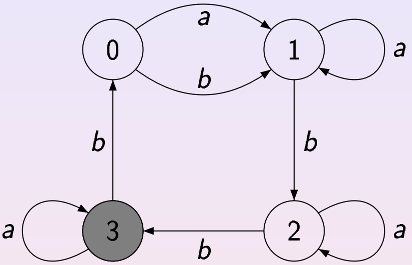

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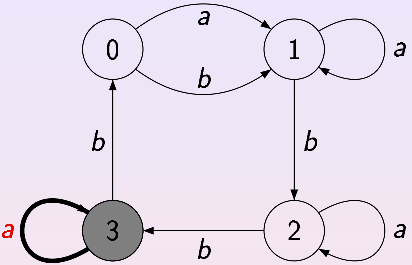

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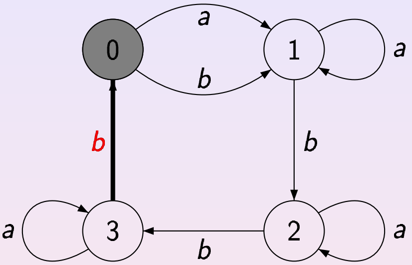

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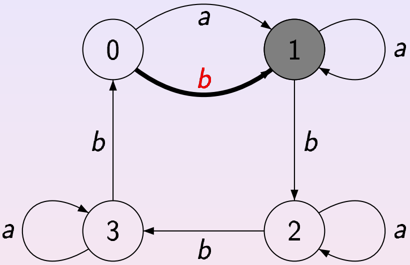

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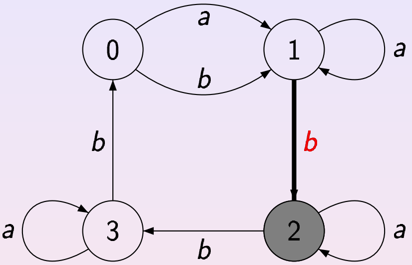

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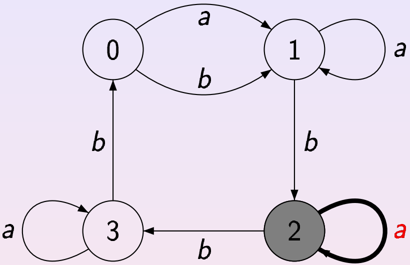

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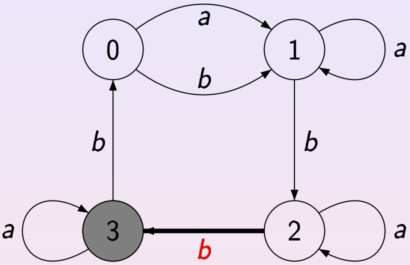

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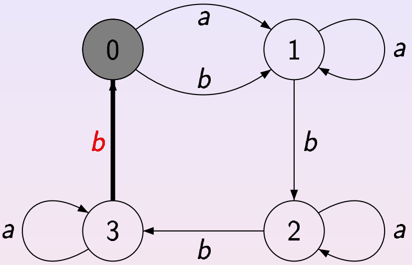

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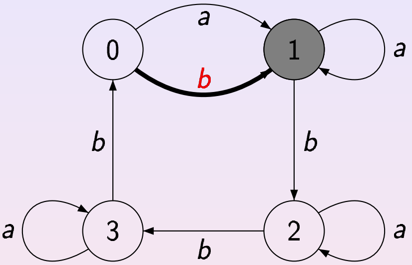

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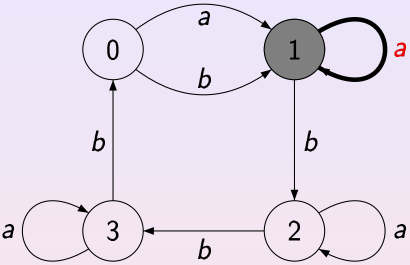

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Černý's Paper

The notion was formalized in 1964 in a paper by Jan Černý (Poznámka k homogénnym eksperimentom s konečnými automatami, Matematicko-fyzikalny Časopis Slovensk. Akad. Vied, 14, no.3, 208–216 [in Slovak]) though implicitly it had been around since at least 1956.

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- The notion was very natural by itself and fitted fairly well in what was considered as the mainstream of automata theory in the early 1960s.
- It also naturally arises in the framework of variable-length codes (such as Huffman codes), see, e.g., S. Even, “Test for synchronizability of finite automata and variable length codes”, IEEE Trans. Inform. Theory, 10 (1964), 185-189.
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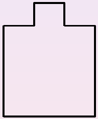

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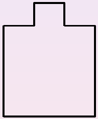

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Assume that only four initial orientations of the part shown above are possible, namely, the following ones:

Suppose that prior the assembly the part should take the "bump-left" orientation (the second one in the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

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We put parts to be oriented on a conveyer belt which takes them to the assembly point and let the stream of the parts encounter a series of passive obstacles of two types (tall and low) placed along the belt.

A tall obstacle is tall enough so that any part on the belt encounters this obstacle by its rightmost low angle.

Being curried by the belt, the part then is forced to turn 90° clockwise.

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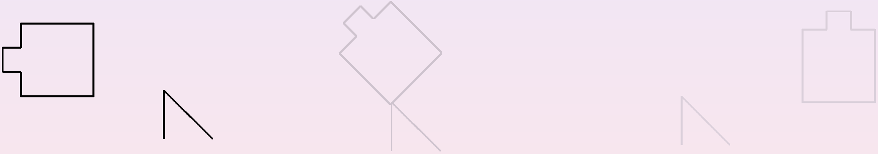

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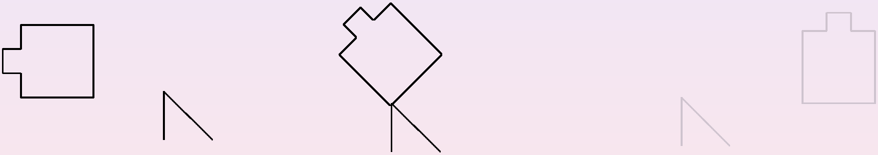

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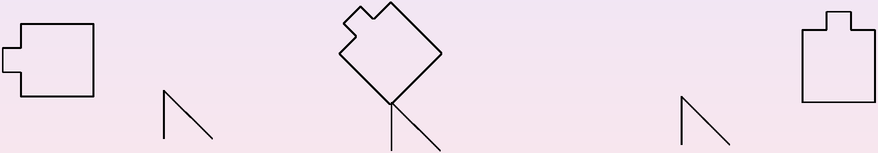

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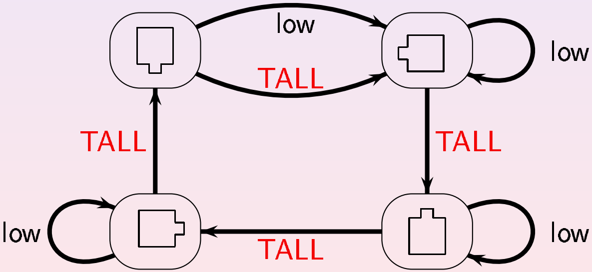

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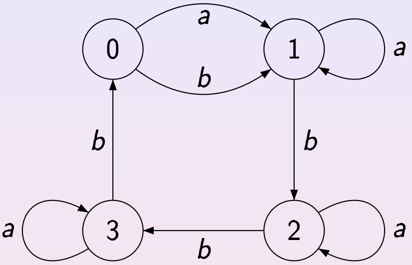

– this was our example of a synchronizing automaton, and we saw that *abbbabbba* is a reset sequence of actions. Hence the series of obstacles

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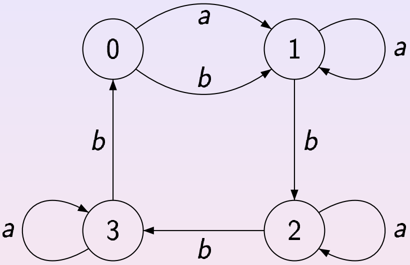

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A substitution on a finite alphabet X is a map $\sigma : X \to X^+$; the substitution is said to be of constant length if all words $\sigma(x)$, $x \in X$, have the same length. One says that σ satisfies the coincidence condition if there exist positive integers m and k such that all words $\sigma^k(x)$ have the same letter in the m-th position. For an example, consider the substitution τ on $X = \{0, 1, 2\}$ defined by $0 \mapsto 11,\ 1 \mapsto 12,\ 2 \mapsto 20$. Calculate the iterations of τ up to τ^4:

Thus, τ satisfies the coincidence condition (with $k = 4$, $m = 7$). The coincidence condition completely characterizes the constant length substitutions that give rise to dynamical systems measure-theoretically isomorphic to a translation on a compact Abelian group (Dekking, 1978).

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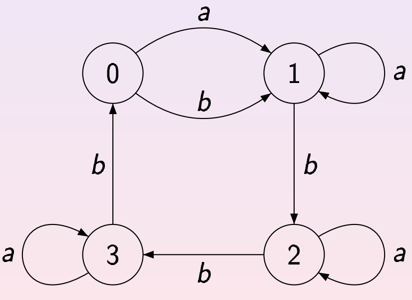

induces the substitution $0 \mapsto 11,\ 1 \mapsto 12,\ 2 \mapsto 23,\ 3 \mapsto 30.$

Conversely, each substitution $\sigma : X \to X^+$ such that all words $\sigma(x)$, $x \in X$, have length ℓ gives rise to a DFA for which X is the state set and which has ℓ input letters $a_1, \dots, a_\ell$ acting on X as follows: $x\,.\,a_i$ is the symbol in the i-th position of the word $\sigma(x)$.

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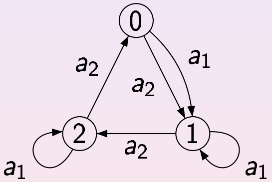

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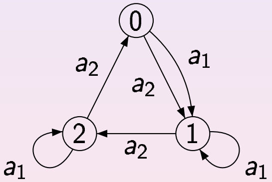

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Černý Conjecture

The Černý conjecture is the claim that every synchronizing automaton with n states possesses a reset word of length $(n-1)^2$. The validity of the conjecture is the main open problem of the area and is arguably one of the most long-standing open problems in combinatorial theory of finite automata.

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Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

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An Algebraic Viewpoint

One may treat DFAs as unary algebras since each letter of the input alphabet defines a unary operation on the state set.

Terms in the language of such unary algebras are expressions of the form $x \,.\, w$, where x is a variable and w is a word over an alphabet Σ. Identities of unary algebras can be of the form either $x \,.\, u = x \,.\, v$ (homotypical identities) or $x \,.\, u = y \,.\, v$ with $x \neq y$ (heterotypical identities).

Clearly, if $\mathscr{A}$ is a synchronizing automaton and w is its reset word, then $\mathscr{A}$ satisfies the identity $x \,.\, w = y \,.\, w$. Conversely, if $\mathscr{A}$ satisfies a heterotypical identity, $x \,.\, u = y \,.\, v$ say, then substituting y for x we get $y \,.\, u = y \,.\, v$ whence $x \,.\, u = y \,.\, u$. We conclude that u is a reset word for $\mathscr{A}$.

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Algebraic Reformulation

Thus, synchronizing automata = unary algebras satisfying heterotypical identities.

The Černý conjecture is thus just the claim that if a unary algebra on an n-element base set satisfies a heterotypical identity, then the algebra satisfies such an identity with one of the terms involved of length at most $(n-1)^2$.

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Black-Box Version

Consider now a black-box synchronizing automaton $\mathscr{A} = \langle Q, \Sigma, \textbf{?} \rangle$.

How to synchronize such an automaton?

Motivation: real computational devices are composites made from many finite automata, each a with relatively small number of states. We need an input signal which would simultaneously reset all those automata and which could be generated without analyzing the structure of each particular component of the device.

In particular, Yacob Benenson *et al*'s "*soup of automata*", see "Programmable and autonomous computing machine made of biomolecules", Nature 414 (2001), 430–434; "DNA molecule provides a computing machine with both data and fuel", Proc. National Acad. Sci. USA 100 (2003), 2191–2196, is a solution containing 3×10^{12} DNA-based automata per μl that work in parallel on different inputs (DNA strands). One has to feed the automata with a common reset word in order to get them ready for a new use.

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Consider now a black-box synchronizing automaton $\mathscr{A} = \langle Q, \Sigma, \mathbf{?} \rangle$. (We know the state set and the input alphabet but we have no idea about the transition function.)

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Algebraic Applications

Universal reset words also admit various algebraic applications.

Reinhard Pöschel *et al* ("Identities in full transformation semigroups", Algebra Universalis 31 (1994), 580–588) used them to find identities separating the full transformation semigroup $\mathbb{T}_n$ from its proper subsemigroups;

Jorge Almeida and $\sim$ ("Profinite identities for finite semigroups whose subgroups belong to a given pseudovariety", J. Algebra and its Applications 2 (2003), 137–163) applied them to construct idempotents in the least ideal of the free profinite semigroup.

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A brute force method relies on the fact that a synchronizing automaton with n states has a reset word of length at most $\frac{n^3-n}{6}$ (Pin-Frankl, 1983). Therefore, given a finite alphabet Σ, one can concatenate all words over Σ of length up to $\frac{n^3-n}{6}$ and get a word that resets all synchronizing automata with n states and input alphabet Σ. This idea is due to Masami Ito and Jürgen Duske "On cofinal and definite automata", Acta Cybernetica 6 (1983), 181–189.

If the Černý conjecture holds true, it suffices to concatenate all words over Σ of length up to $(n-1)^2$. An accurate concatenation (based on the DeBruijn graph) yields a universal reset word of length $|\Sigma|^{(n-1)^2} + n^2 - 2n$.

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On the other hand, it is proved that such a word cannot have length less than $|\Sigma|^{n-1} + n - 2$.

Define the *universal Černý function* $UC(t, n)$ as the minimum length of a word that resets all synchronizing automata with n states and t input letters. In terms of this function, our current knowledge can be summarized in the following line:

$$t^{n-1} + n - 2 \leq UC(t, n) \leq t^{\frac{n^2-n}{2}} + o(t^{\frac{n^2-n}{2}}).$$

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On the other hand, it is proved that such a word cannot have length less than $|\Sigma|^{n-1}+n-2$.

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Algebraic Reformulation

Again, it should be clear that universal reset words correspond to heterotypical identities that hold in every unary algebra with given size of the base set.

Therefore, the problem of evaluating the universal Černý function $UC(t, n)$ is nothing but the problem of finding a heterotypical identity of minimum length which holds in all n-element algebras with t unary operations that satisfy a heterotypical identity.

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Separating Words by Automata

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA in which we fix an initial state $q_0 \in Q$ and a set of final states $F \subseteq Q$. We say that $\mathscr{A}$ accepts a word $w \in \Sigma^*$ if $q_0 \,.\, w \in F$, that is, the directed path starting at q_0 and labeled w ends at a state in F. Otherwise $\mathscr{A}$ rejects w.

For instance, the above automaton accepts the word *aabb* but rejects the word *bbaa*.

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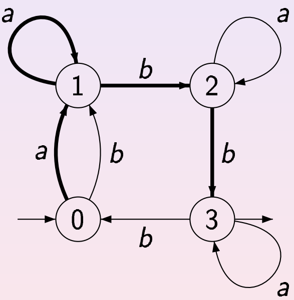

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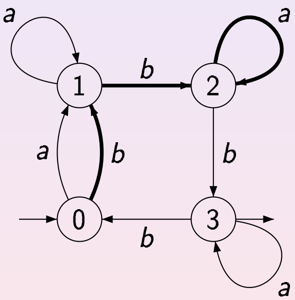

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Let $S(n) = \max \mathrm{sep}(u, v)$ where u and v are distinct words of length at most n. The Separating Words Problem is to determine good upper and lower bounds on $S(n)$. It was introduced by Paweł Goralčik and Vačlav Koubek ("On discerning words by automata", Lect. Notes Comput. Sci. 226 (1986), 116–122), who proved

$$S(n) = o(n).$$

The best upper bound so far is due to Robson ("Separating words with machines and groups", RAIRO Inform. Théor. App., 30 (1996), 81–86), who obtained

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Algebraic Viewpoint

Observe that if distinct words u, v are such that the identity $u = v$ holds true in the full transformation semigroup $\mathbb{T}_k$, then u and v cannot be separated by any automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ with at most k states. Indeed, the transition semigroup of $\mathscr{A}$, that is, the semigroup of transformations of the set Q induced by the words in Σ^*, embeds in $\mathbb{T}_k$ whence u and v act the same on Q and are simultaneously accepted or rejected.

Hence short identities in $\mathbb{T}_k$ may be used to produce lower bounds for the Separating Words Problem.

All known lower bounds for $S(n)$ are of magnitude $\Omega(\log n)$. They correspond to the following one-letter identity of $\mathbb{T}_k$:

$$x^{k-1} = x^{k-1+\mathrm{lcm}(1,2,\dots,k)}.$$

It is known that $\mathrm{lcm}(1, 2, \dots, k)$ grows faster than 2^k so that k is logarithmic in $\mathrm{lcm}(1, 2, \dots, k)$.

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Again, there is a one-letter identity of length $\mathrm{lcm}(1, 2, \ldots, k)$ which is exponential of k. However, at least for some k there are shorter identities, for instance, the identity

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of length 32 holds in $\mathbb{S}_5$ while $\mathrm{lcm}(1,2,3,4,5) = 60$.
Further, the identity

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GAIA, July 19, 2013