Synchronizing Automata – I

M. V. Volkov

Ural State University, Ekaterinburg, Russia

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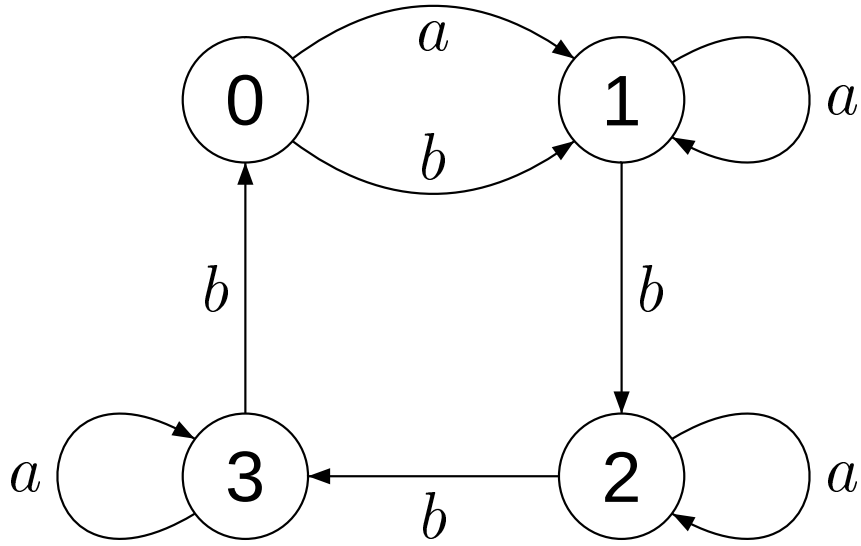
Other names:

- for automata: directable, cofinal, collapsible, etc;
- for words: directing, recurrent, synchronizing, etc.

Synchronizing automata



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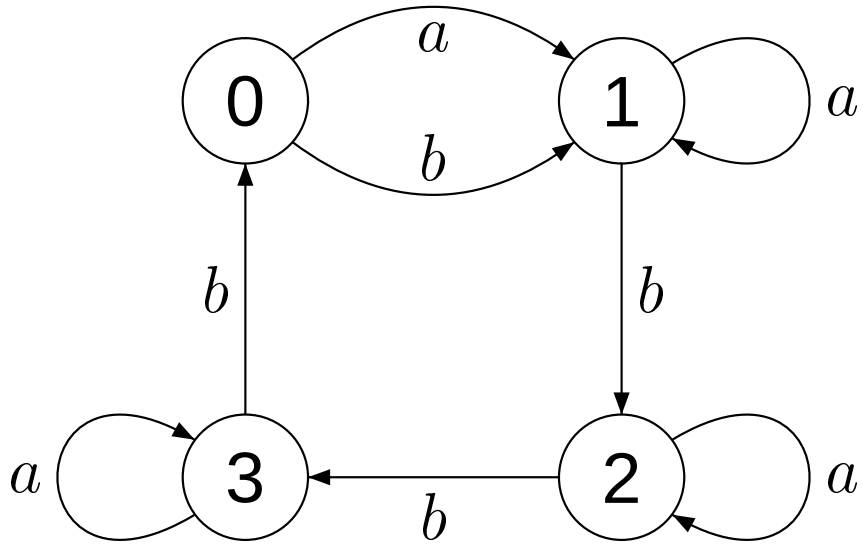

A reset word is $abbbabbba$. Applying it at any state brings the automaton to the state 1.

Synchronizing Automata

The notion was formalized in 1964 in a paper by Jan Černý (Poznámka k homogénnym eksperimentom s konečnými automatami, Matematicko-fyzikalny Časopis Slovensk. Akad. Vied, 14, no.3, 208–216 [in Slovak]) though implicitly it had been around since at least 1956 (see pre-proceedings).

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Think of a satellite which loops around the Moon and cannot be controlled from the Earth while “behind” the Moon (Černý’s original motivation).

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Example: A. E. Laemmel, B. Rudner, Study of the application of coding theory, Report PIBEP-69-034, Polytechnic Inst. Brooklyn, Dept. Electrophysics, Farmingdale, N.Y., 94 pp.; more examples in pre-proceedings.

Re-inventing by Engineers

Since the 60s and till the 90s synchronizing automata were considered as a useful tool for testing of reactive systems (first circuits, later protocols).

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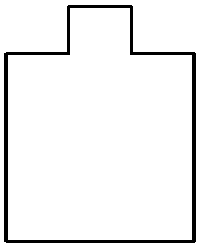
Suppose that one of the parts of a certain device has the following shape:



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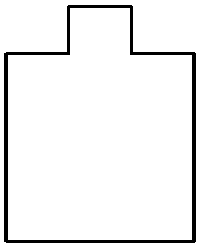

Such parts arrive at manufacturing sites in boxes and they need to be sorted and oriented before assembly.

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Suppose that prior the assembly the part should take the “bump-left” orientation (the second one in the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

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We put parts to be oriented on a conveyer belt which takes them to the assembly point and let the stream of the parts encounter a series of passive obstacles of two types (*high* and *low*) placed along the belt.

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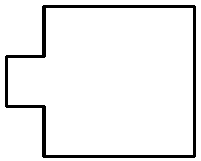
A high obstacle is high enough so that any part on the belt encounters this obstacle by its rightmost low angle.



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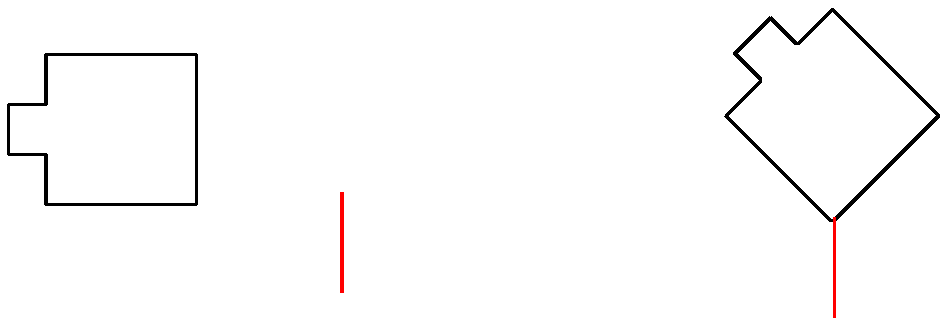

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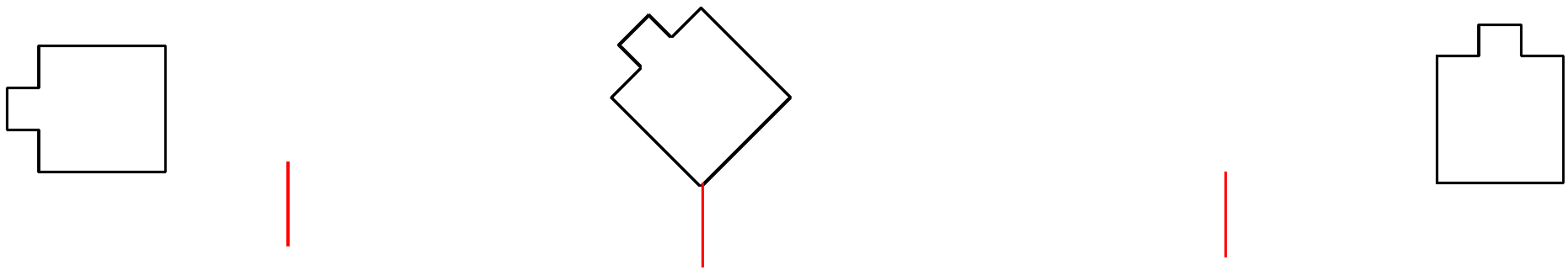

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A low obstacle has the same effect whenever the part is in the “bump-down” orientation; otherwise it does not touch the part which therefore passes by without changing the orientation.

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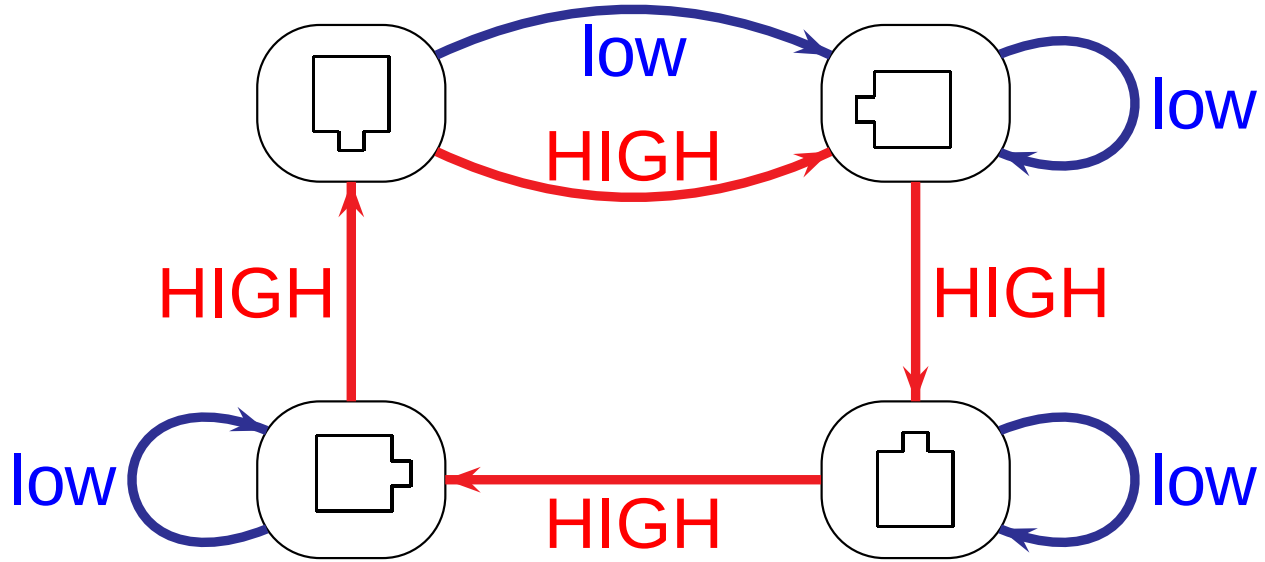
The following schema summarizes how the obstacles effect the orientation of the part in question:



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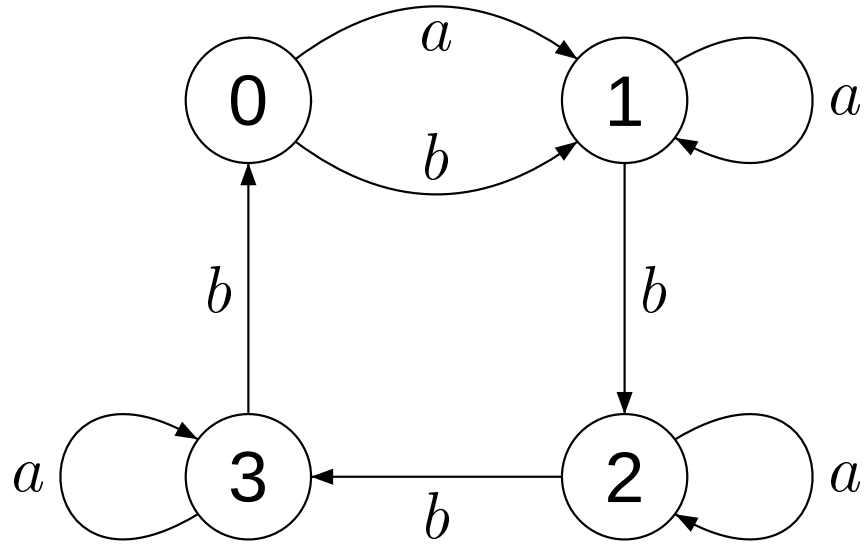

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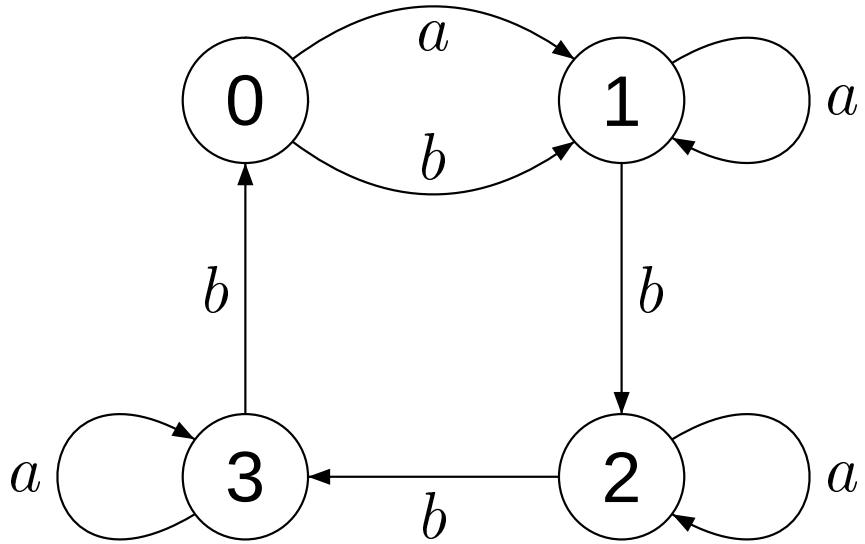

– this was our example of a synchronizing automaton, and we saw that $abbbabbba$ is a reset sequence of actions. Hence the series of obstacles

low-HIGH-HIGH-HIGH-low-HIGH-HIGH-HIGH-low

yields the desired sensorless orienter.

Possible Use in Biocomputing

In DNA-computing, there is a fast progressing work by Ehud Shapiro's group on "*soup of automata*" (Programmable and autonomous computing machine made of biomolecules, Nature 414, no.1 (November 22, 2001) 430–434; DNA molecule provides a computing machine with both data and fuel, Proc. National Acad. Sci. USA 100 (2003) 2191–2196, etc).

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- From the viewpoint of applications, real or yet imaginary, algorithmic issues are of crucial importance. We discuss them in Lecture I.
- Synchronizing automata constitute an interesting combinatorial object. Their studies are mainly motivated by the Černý conjecture. We discuss the Černý conjecture in Lecture II.
- Yet another mathematical motivation for studying synchronizing automata comes from symbolic dynamics. In Lecture III we present a recent breakthrough in the area—a (positive) solution to the Road Coloring Problem found by Trahtman.

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The *power automaton* $\mathcal{P}(\mathscr{A})$ of a given DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$:

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A $w \in \Sigma^*$ is a reset word for the DFA $\mathscr{A}$ iff w labels a path in $\mathcal{P}(\mathscr{A})$ starting at Q and ending at a singleton.

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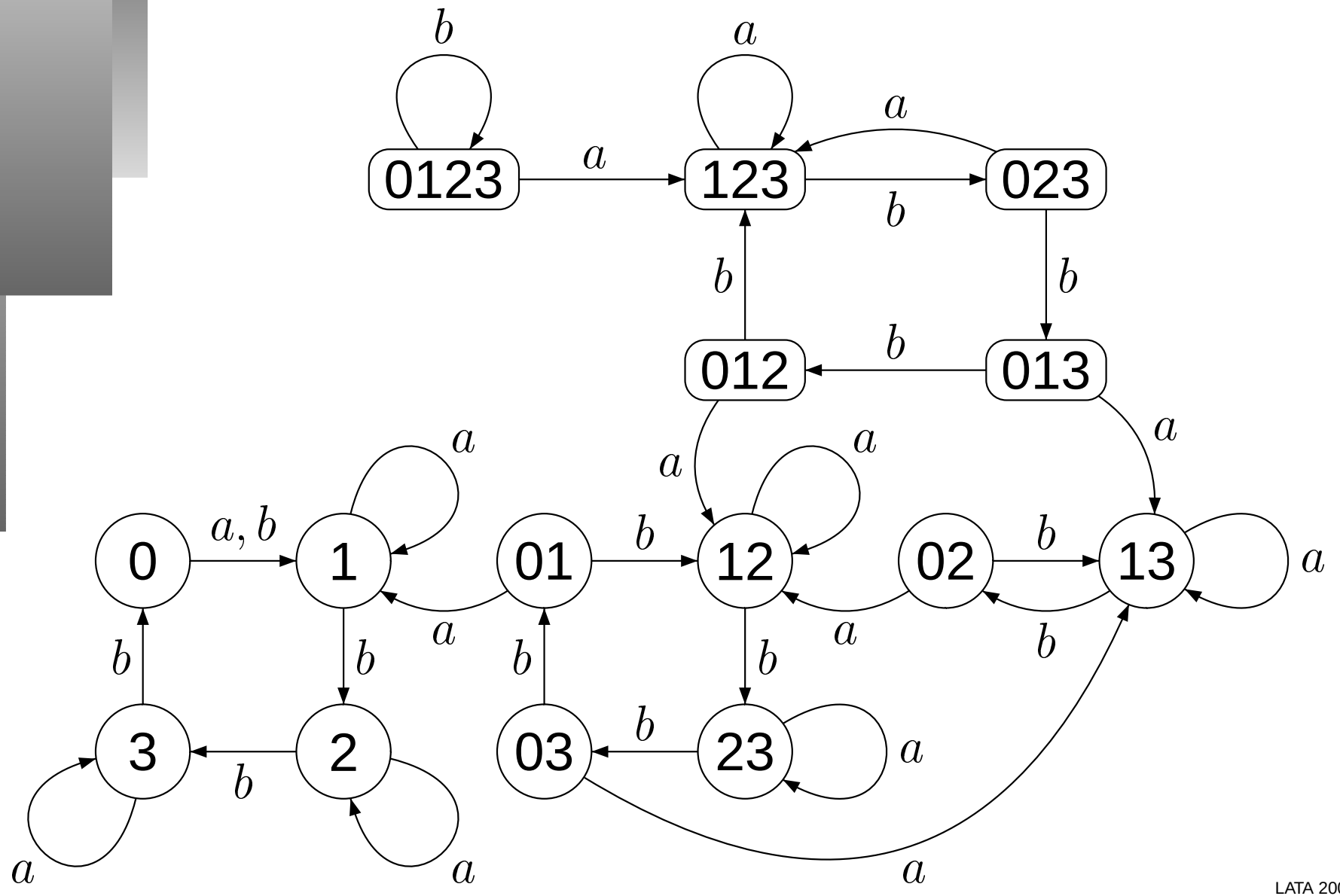

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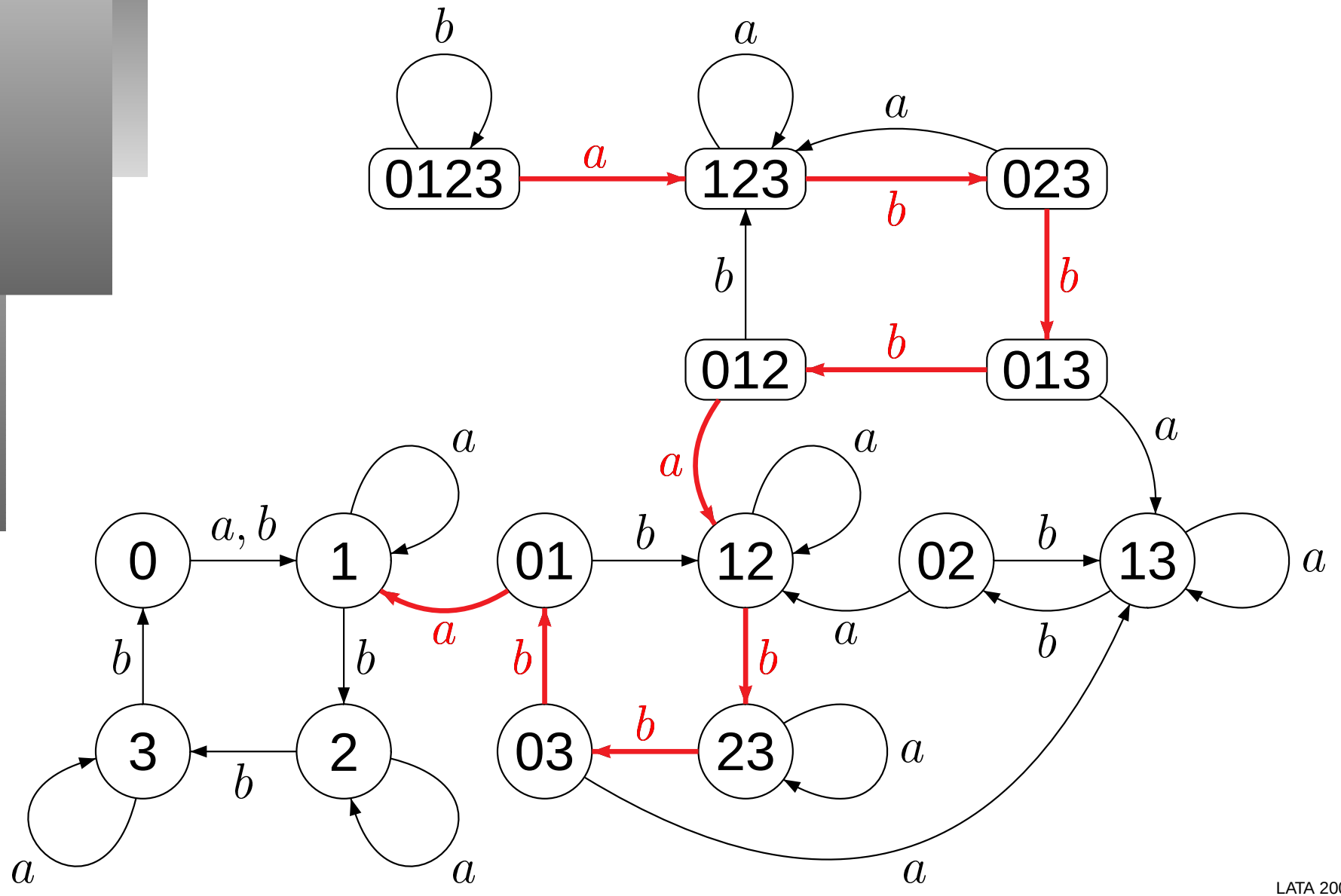

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The following result by Černý gives a polynomial algorithm:

Proposition. *A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is synchronizing iff for every $q, q' \in Q$ there exists a word $w \in \Sigma^*$ such that $\delta(q, w) = \delta(q', w)$.*

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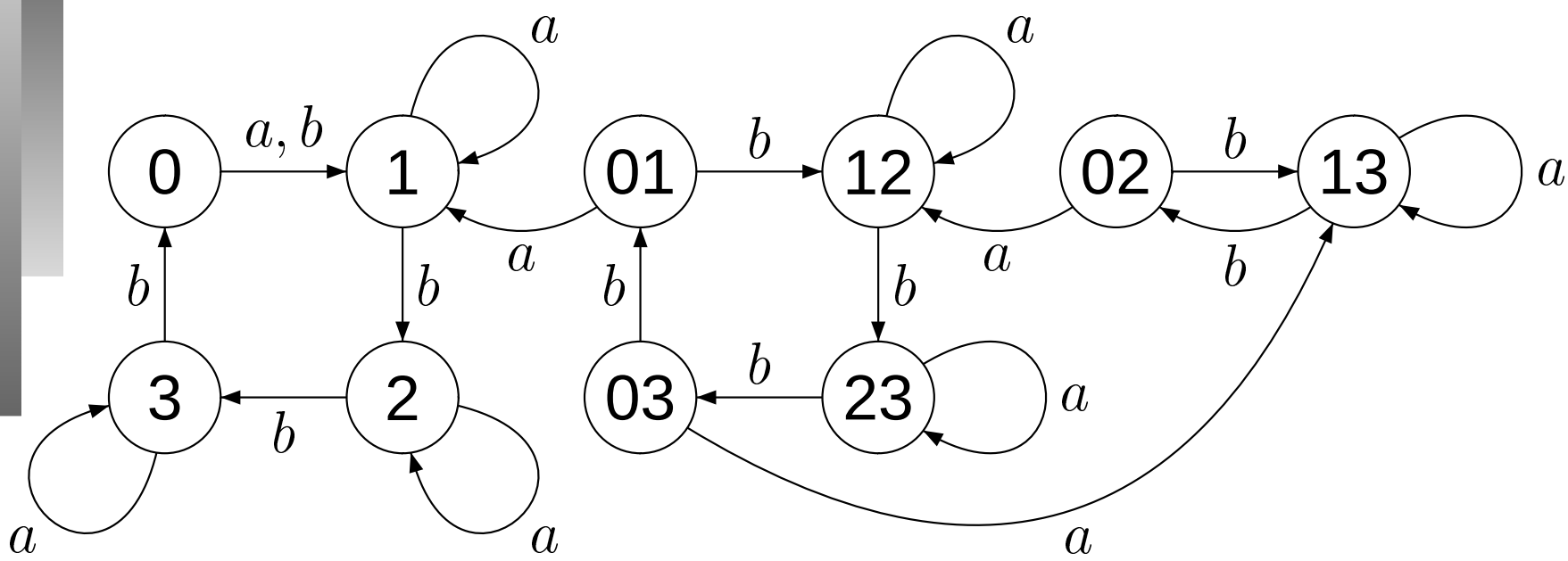

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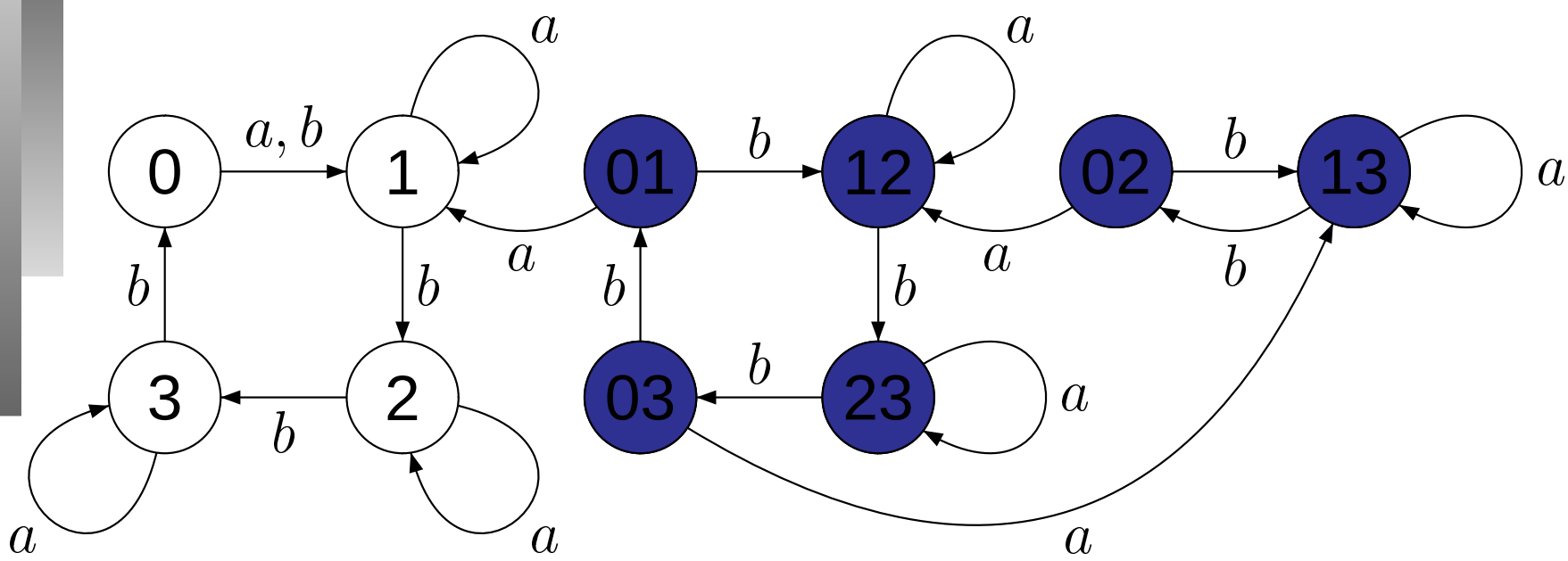

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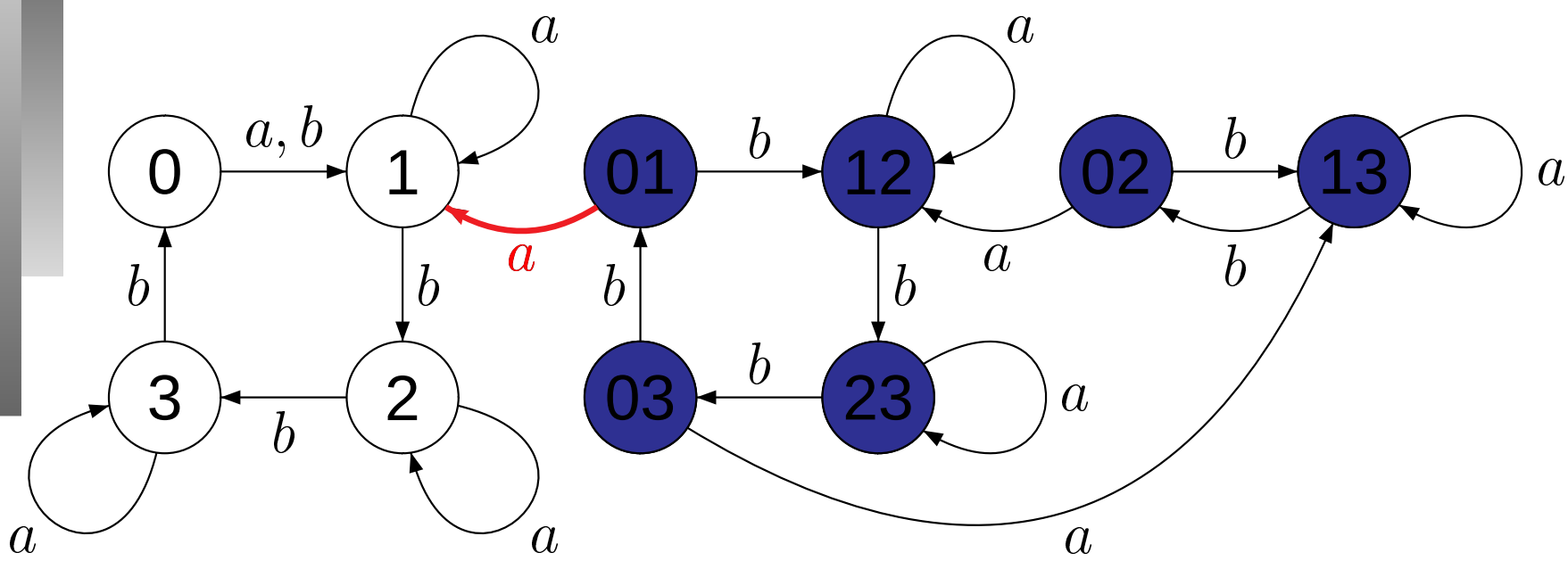

$$a,\ Q\,.\,a = \{1, 2, 3\}$$

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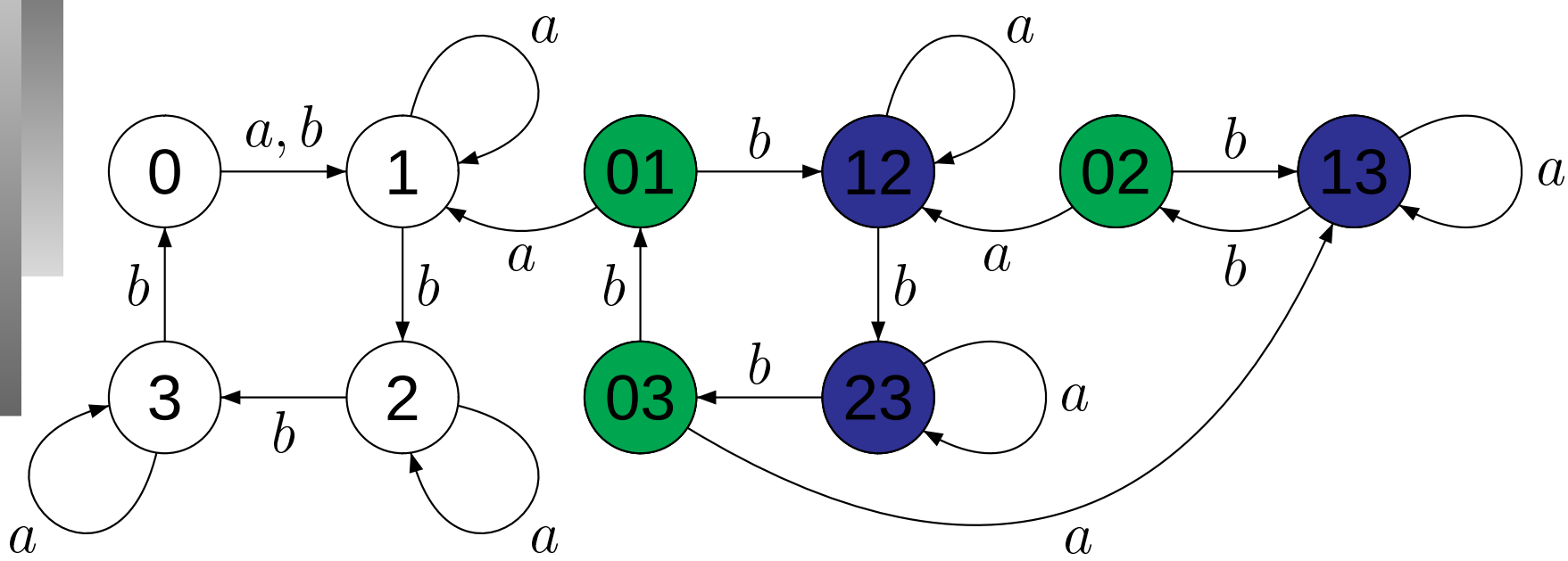

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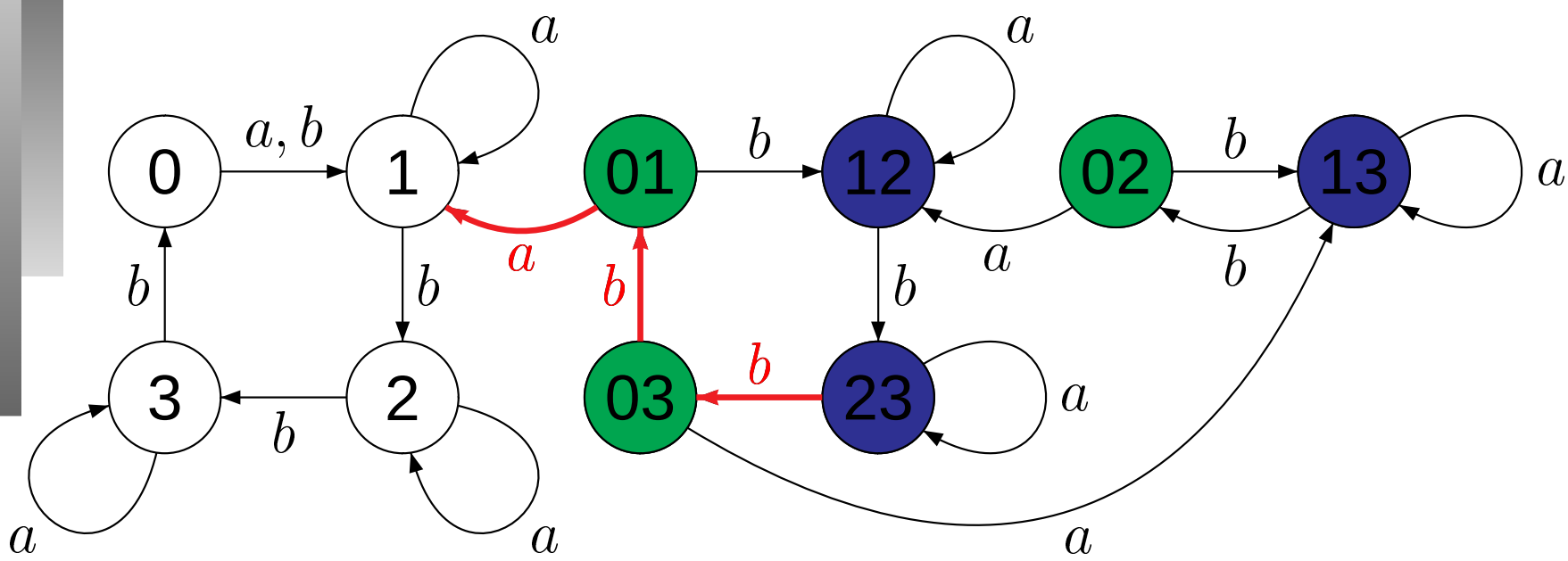

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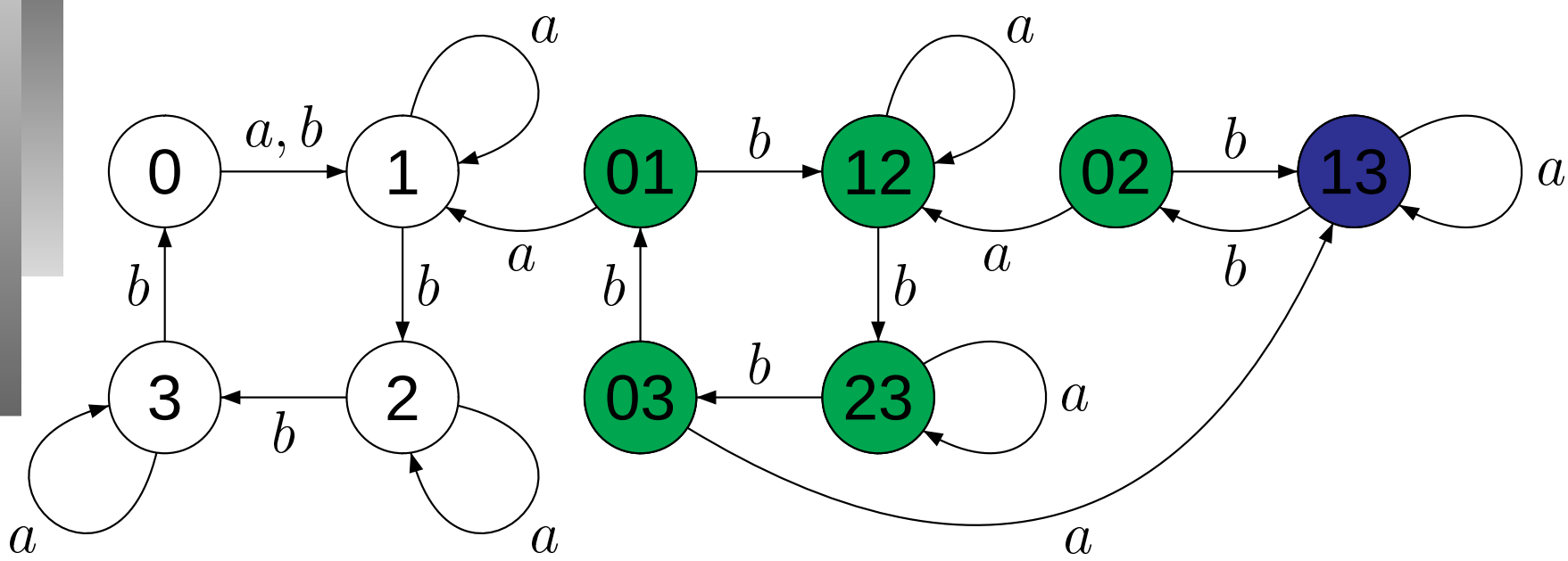

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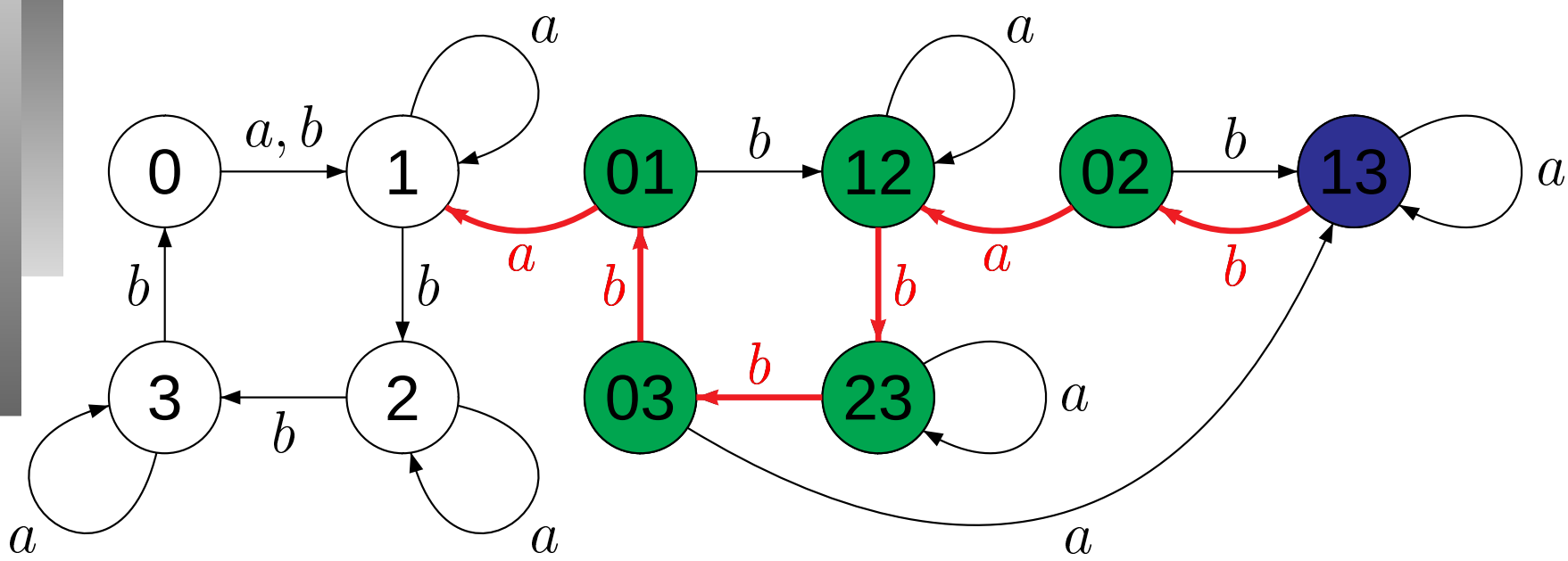

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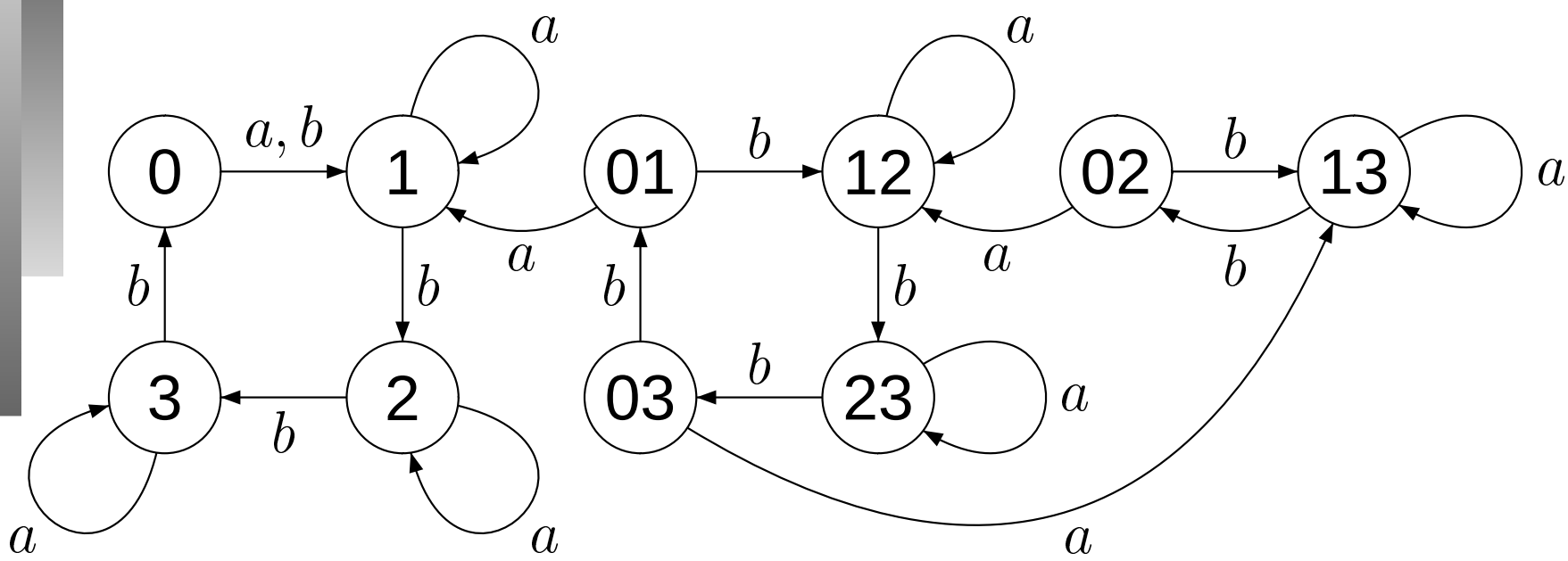

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Observe that the reset word constructed this way is of length 10 while we know a reset word of length 9 for this automaton.

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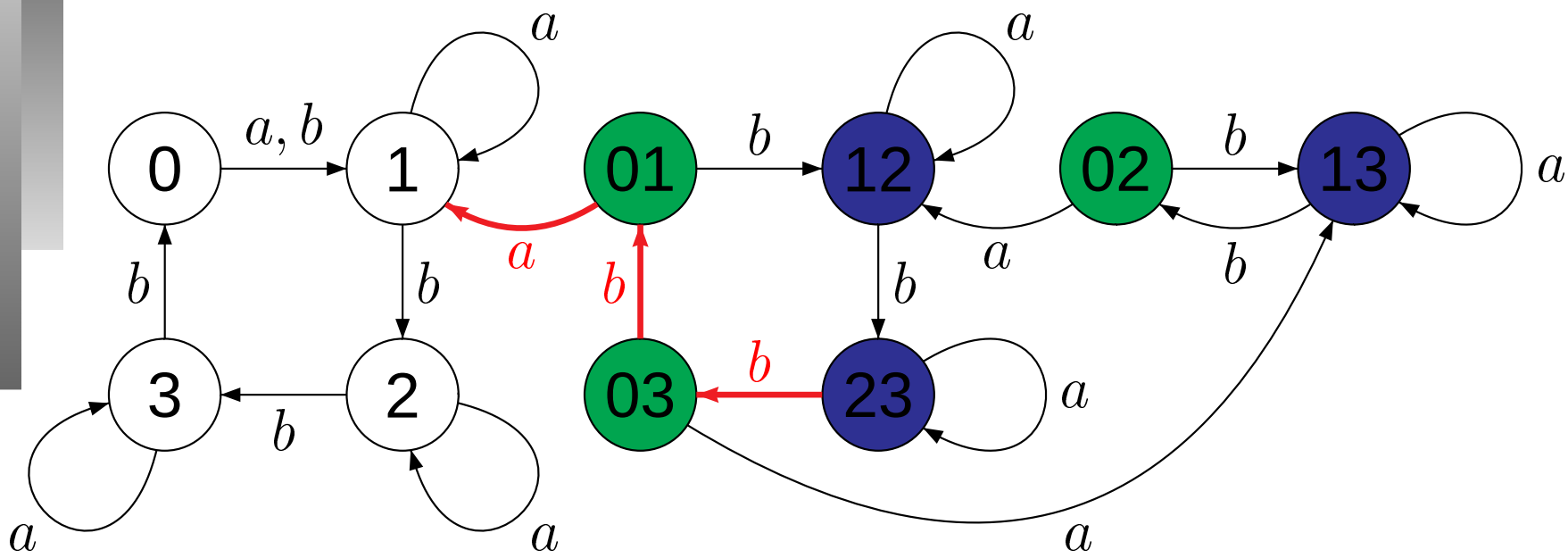
Studying Greedy Algorithm



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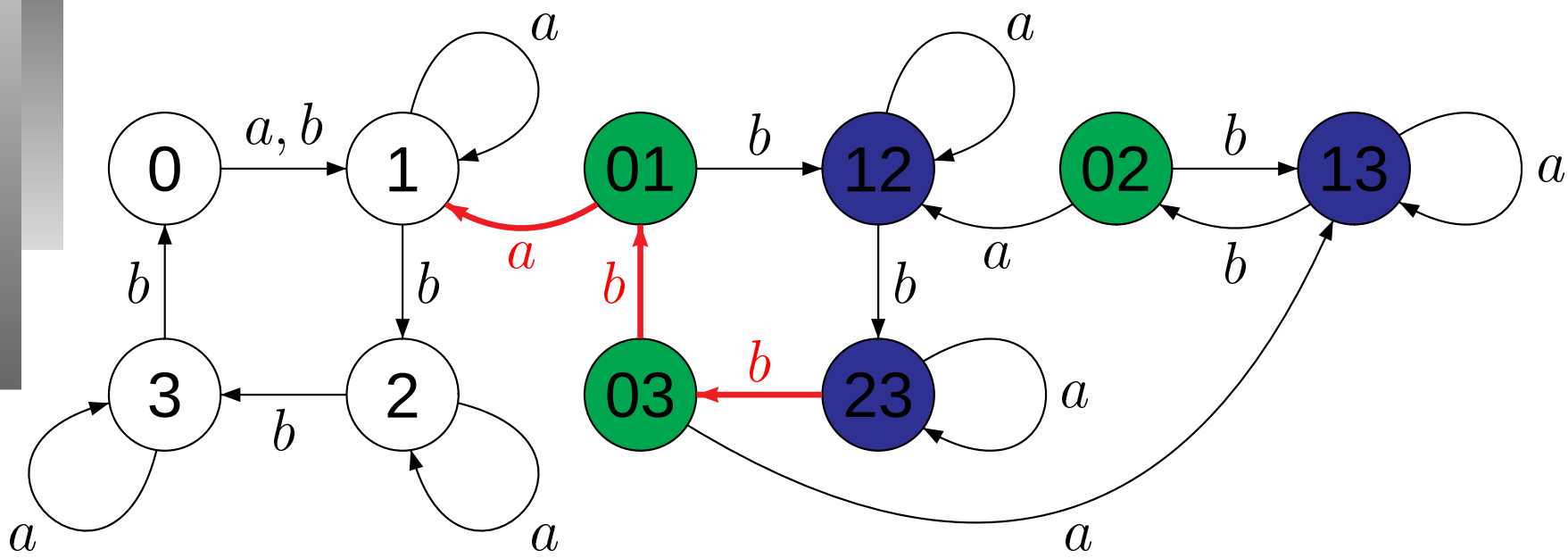

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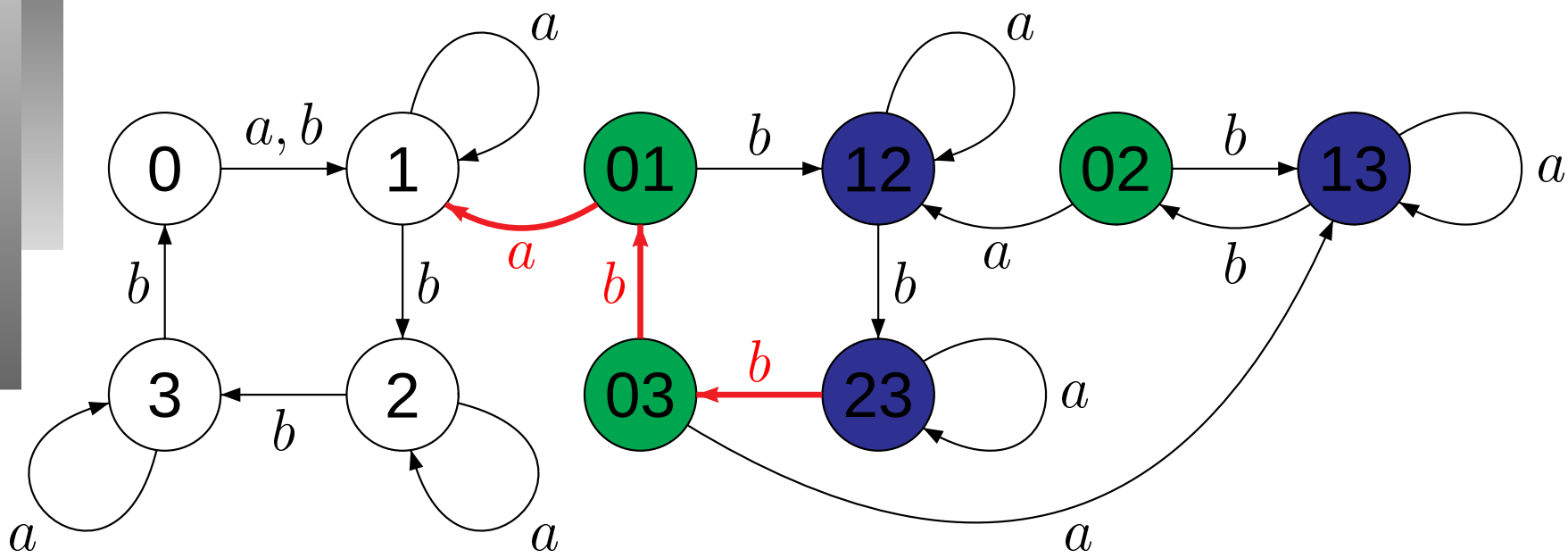

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Consider a generic step of the algorithm at which states to be compressed form a set P with $|P| = k > 1$ and let $v = a_1 \cdots a_\ell$ with $a_i \in \Sigma$, $i = 1, \ldots, \ell$, be a word of minimum length such that $|P \,.\, v| < k$.

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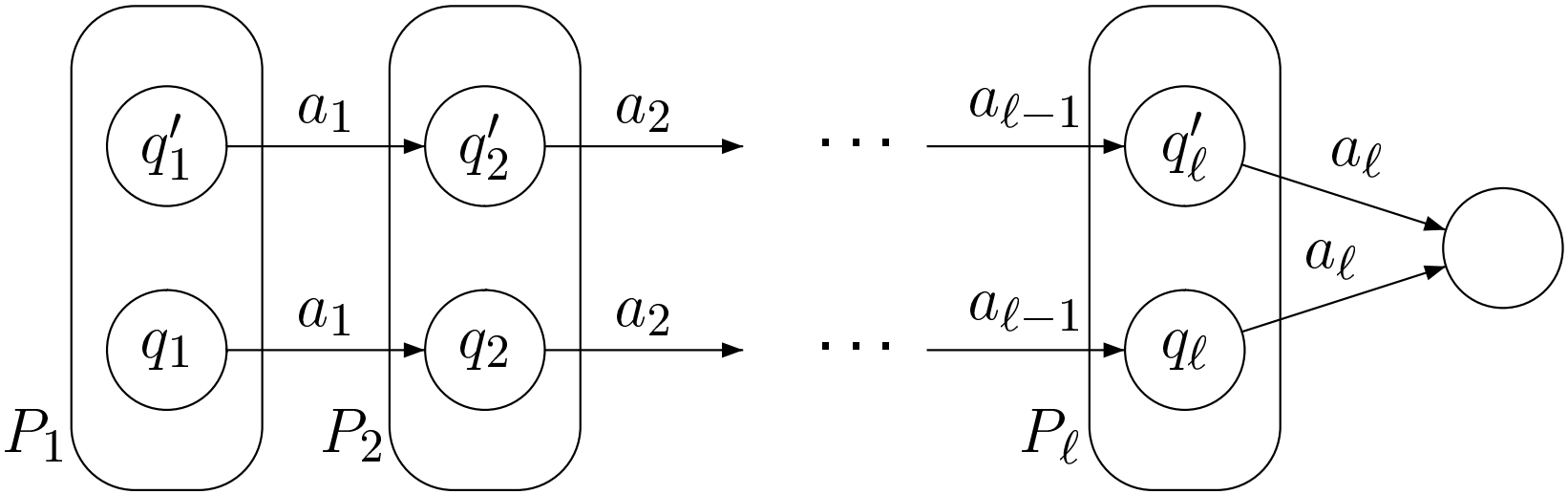

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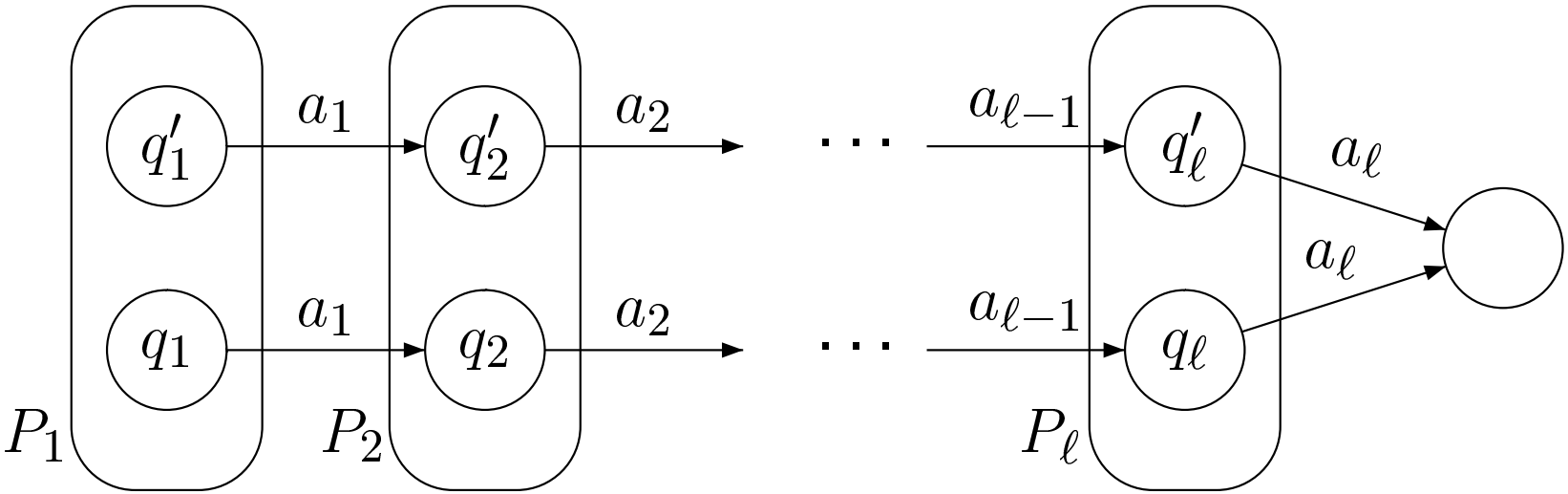

The condition that v is a word of minimum length with $|P \,.\, v| < |P|$ implies $R_i \not\subseteq P_j$ for $1 \leq j < i \leq \ell$.

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A construction: fix a $(k-2)$-subset W of Q, list all $\binom{n-k+2}{2}$ 2-subsets of $Q \setminus W$ and let T_i be the union of W with the i^{th} 2-subset in the list.

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We identify Q with $\{1, 2, \ldots, n\}$ and assign to each k-subset $I = \{i_1, \ldots, i_k\}$ the following polynomial $D(I)$ in variables $x_{i_1}, \ldots, x_{i_k}$ over the field of rationals.

Combinatorial Configuration

$$I = \{i_1, \ldots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

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- the polynomials $D(T_1), \ldots, D(T_s)$ (derived from the "standard" sequence) generate the linear space spanned by all polynomials of the form $D(I)$.

Studying Greedy Algorithm

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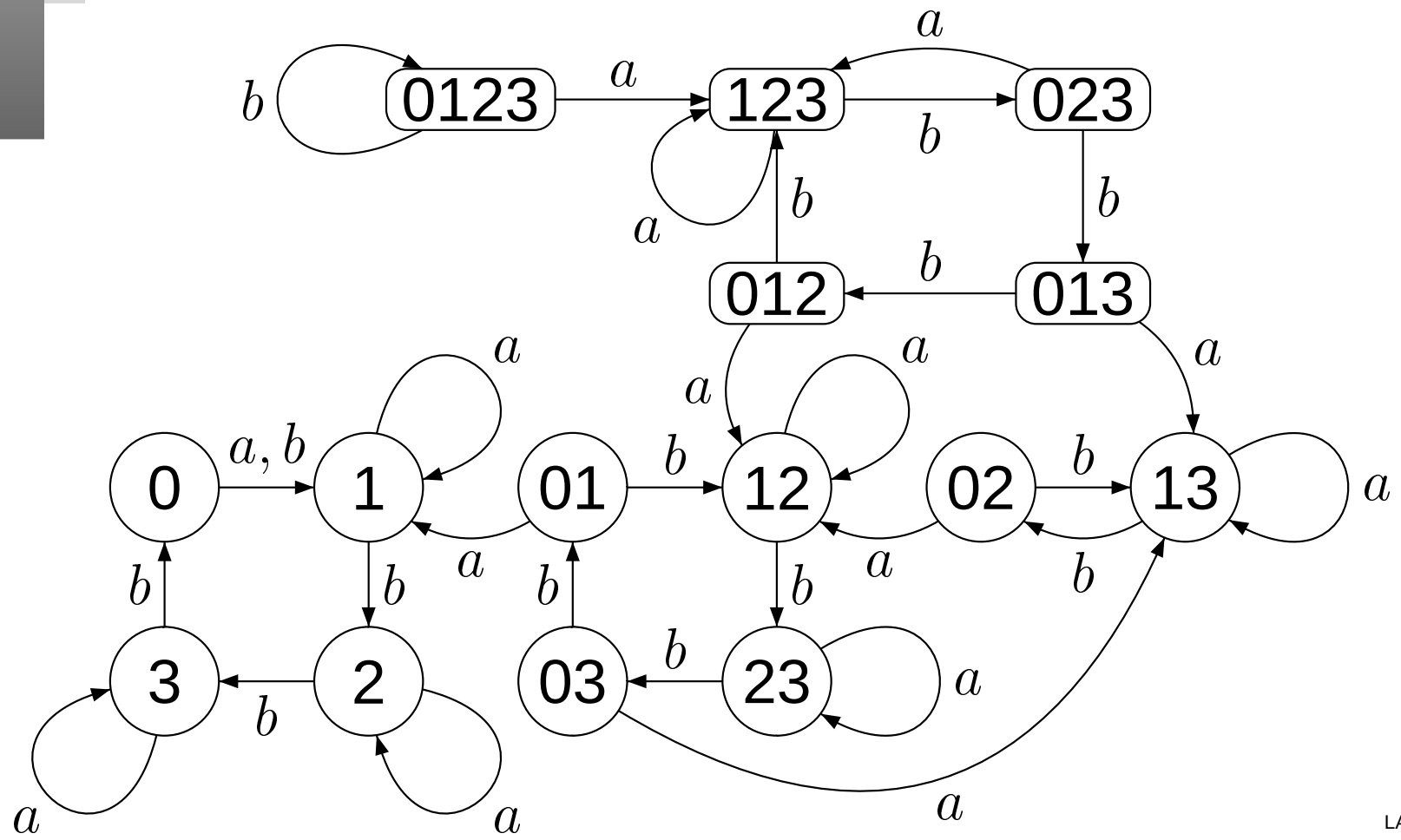

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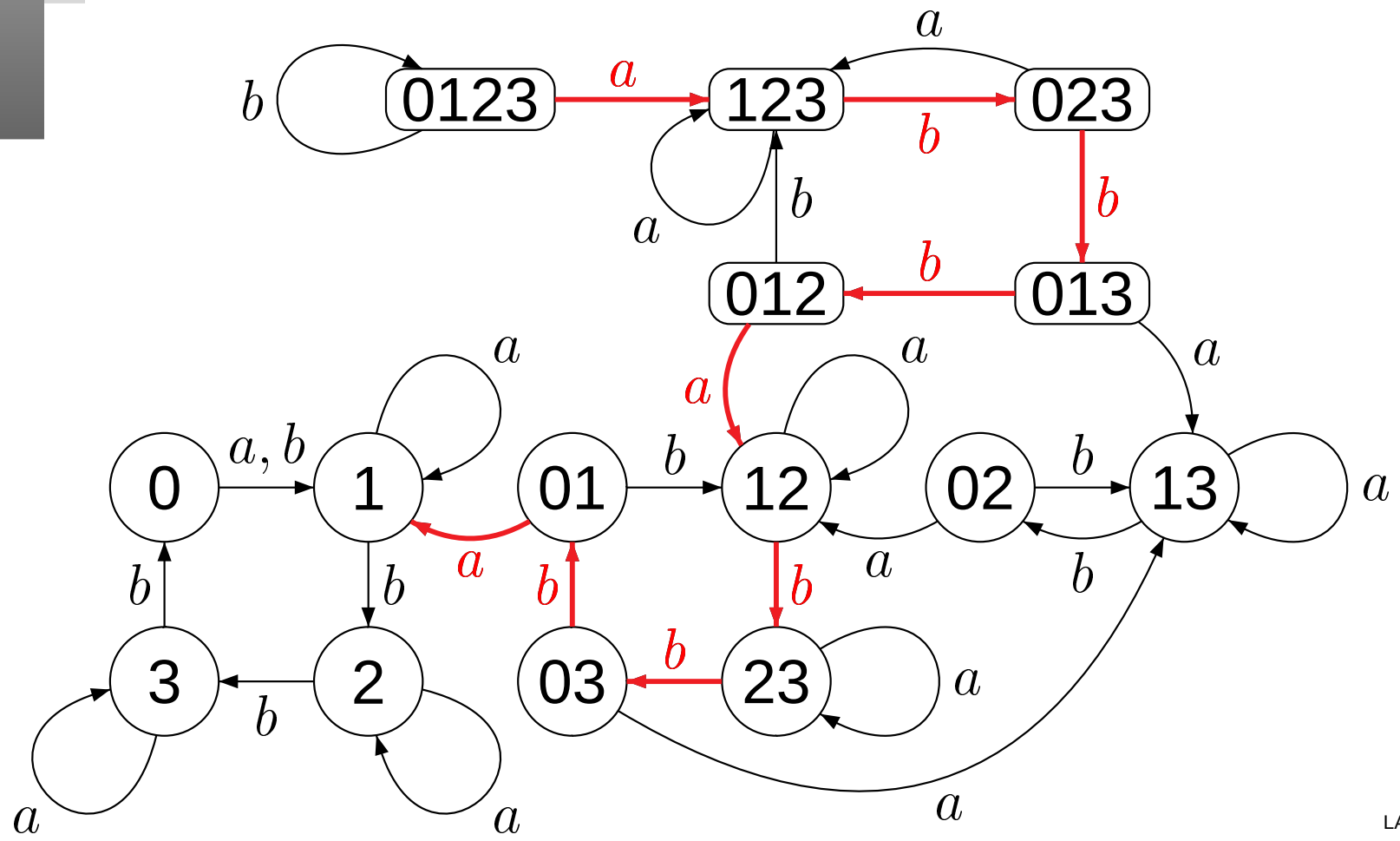

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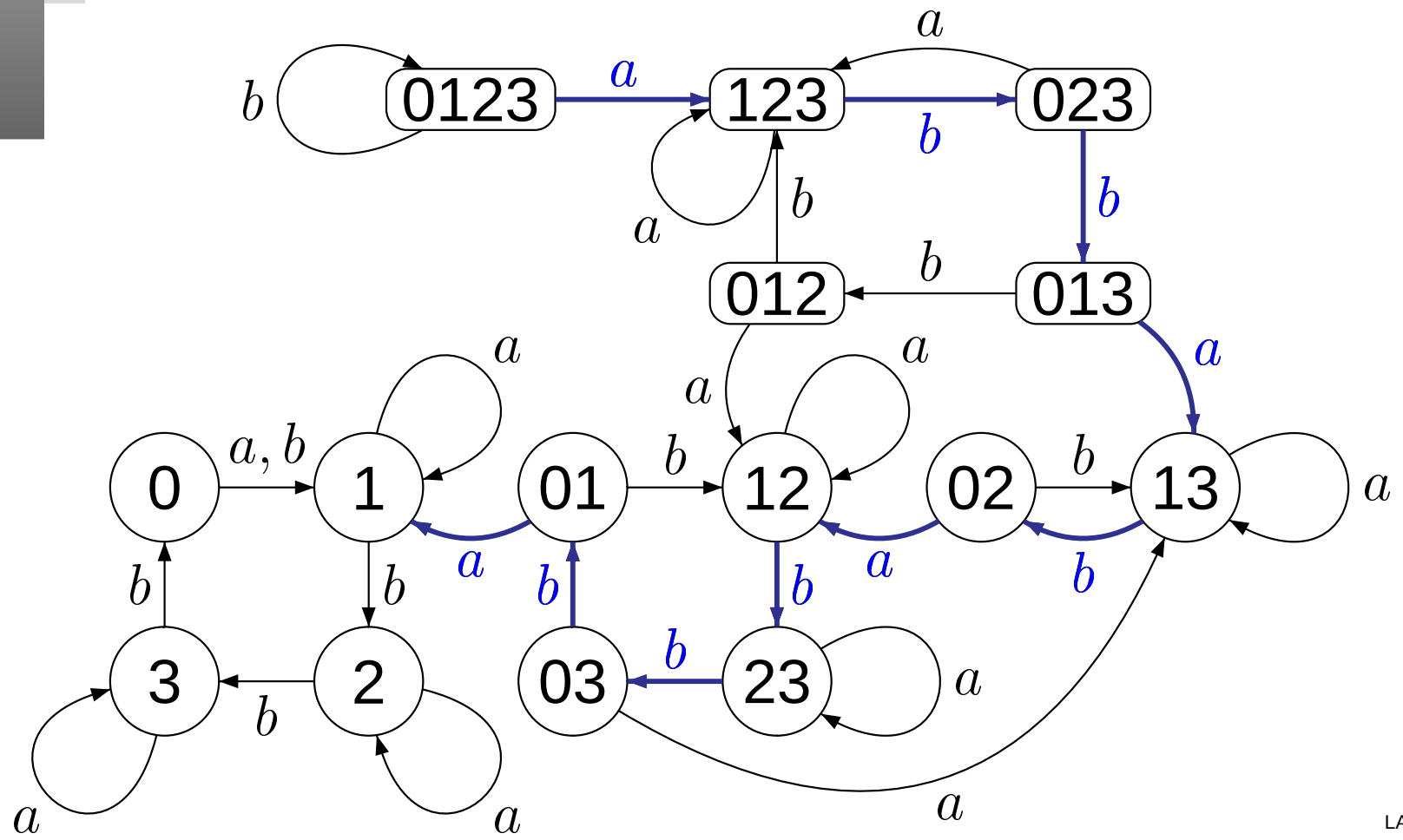

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The behaviour of the greedy algorithm on average is not yet understood; practically it behaves rather well.

Under standard assumptions (like NP $\neq$ coNP) no polynomial algorithm, even non-deterministic, can find the minimum length of reset words for synchronizing automata, see the pre-proceedings.

Studying Greedy Algorithm

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For instance (though it is not very likely), it may exists a polynomial algorithm that, given a synchronizing automaton, produces a reset word whose length either is minimum possible or exceeds the minimum possible exactly by one!