Synchronizing Automata – II

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Synchronizing automata – Recap

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$|Q\,.\,w| = 1$. Here $Q\,.\,v = \{\delta(q, v) \mid q \in Q\}$.

Any w with this property is said to be a *reset word* for the automaton.

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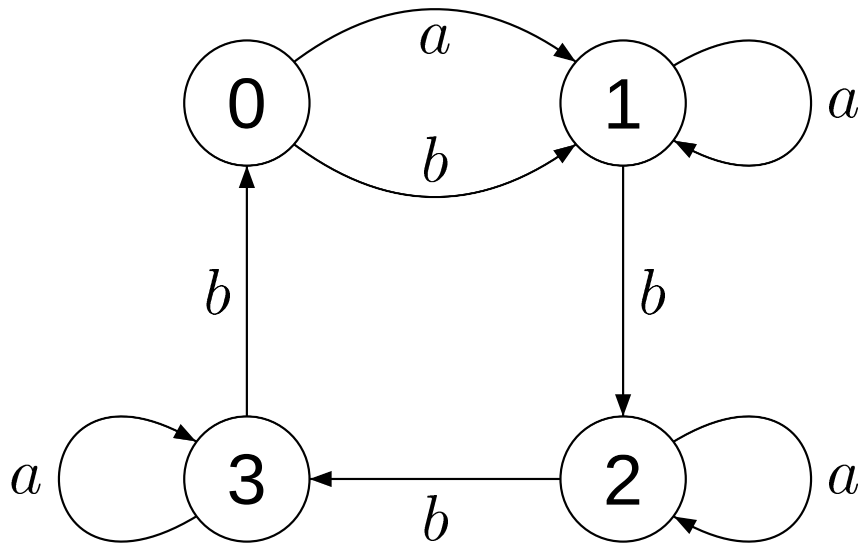

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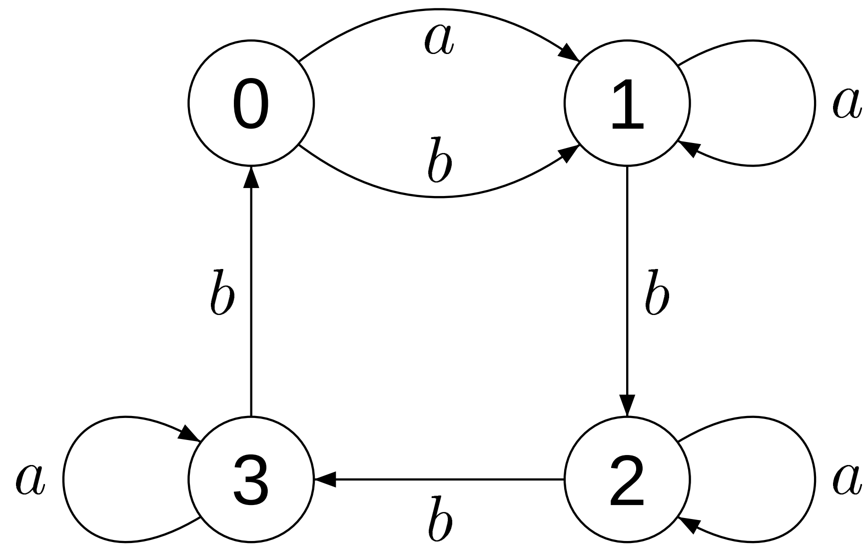

A reset word is $abbbabbba$. In fact, we have verified that this is the shortest reset word for this automaton.

The Černý automata

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The automaton in the previous slide is $\mathscr{C}_4$.

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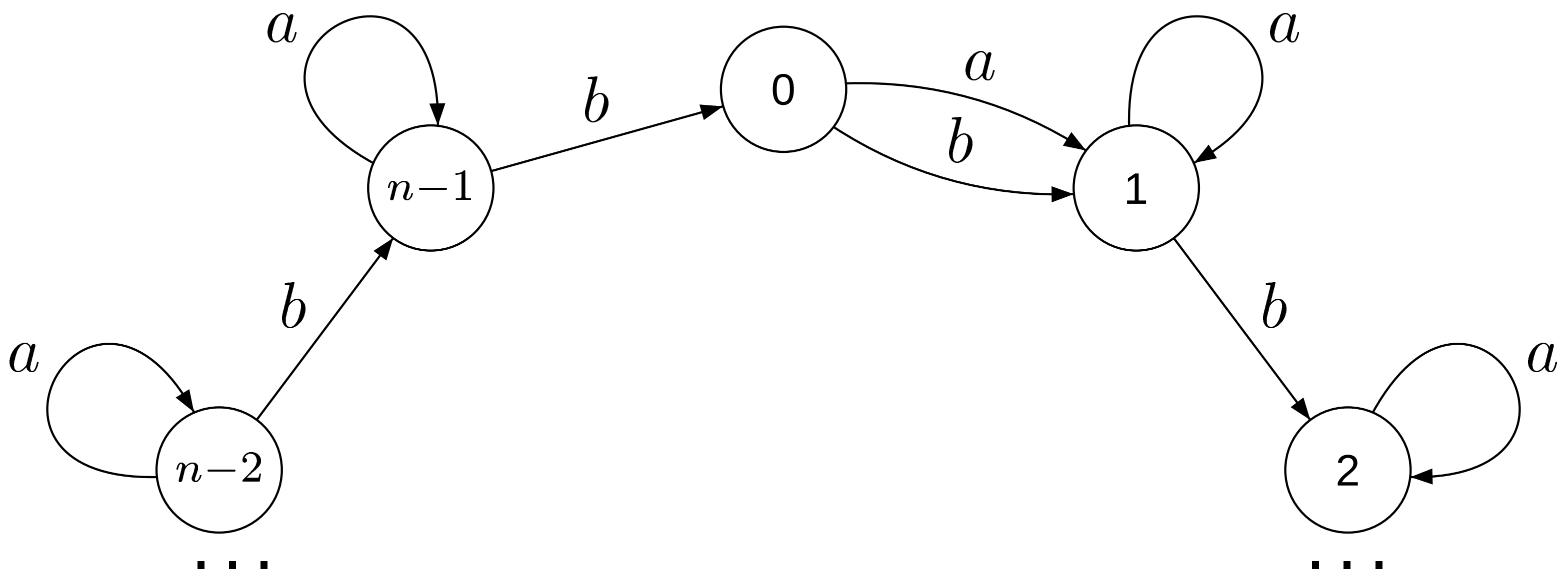
Here is a generic automaton from the Černý series:



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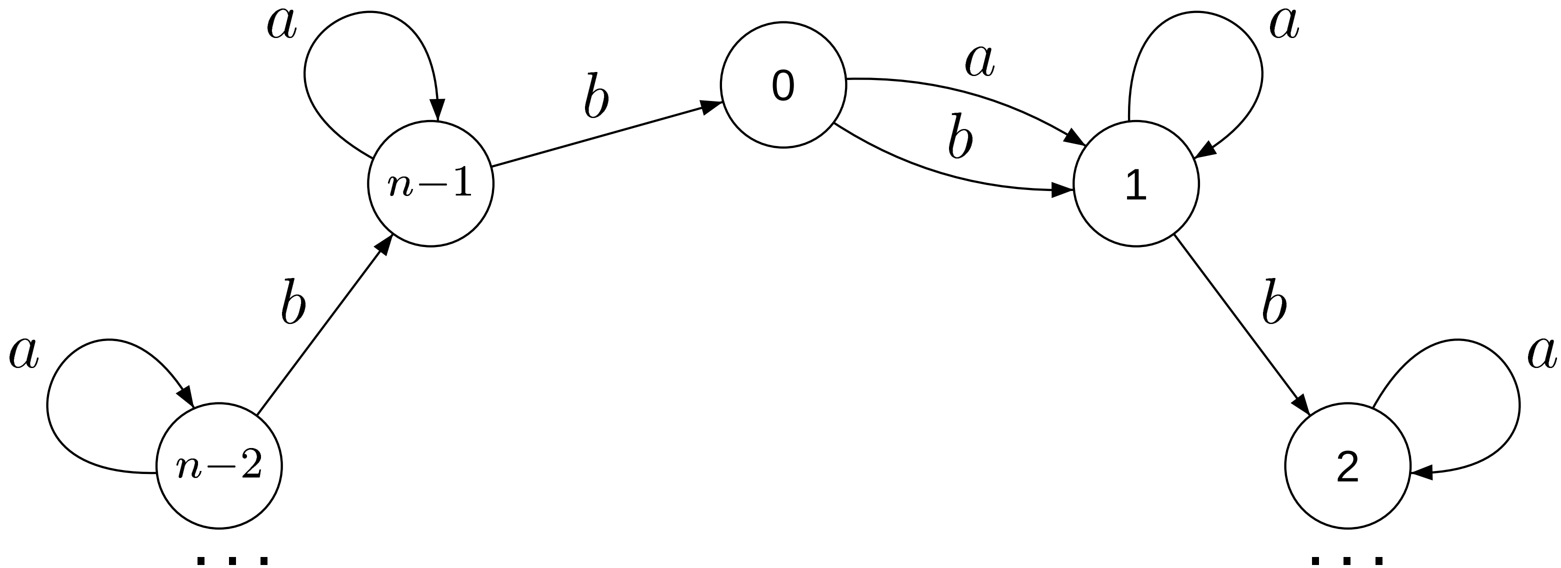

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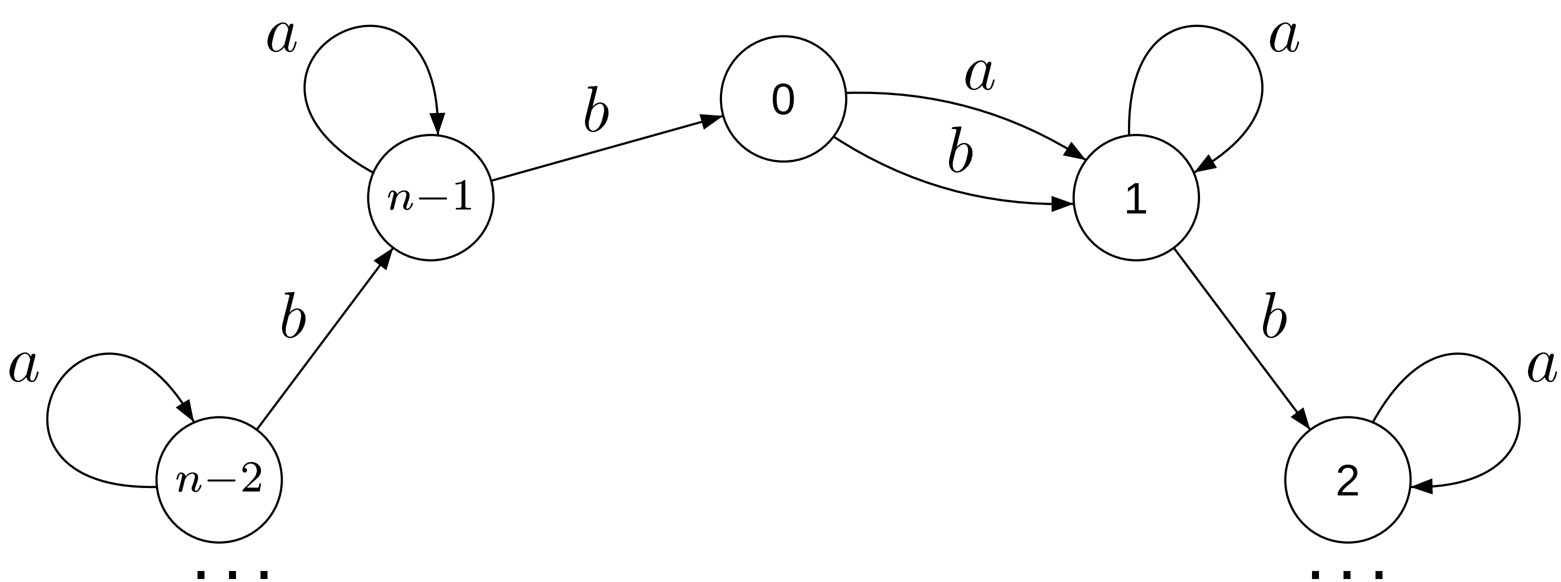

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$. As other results from Černý's paper of 1964, this nice series of automata has been rediscovered many times, see references in the pre-proceedings.

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- The only coin that remains at the end of the game is the golden coin G.

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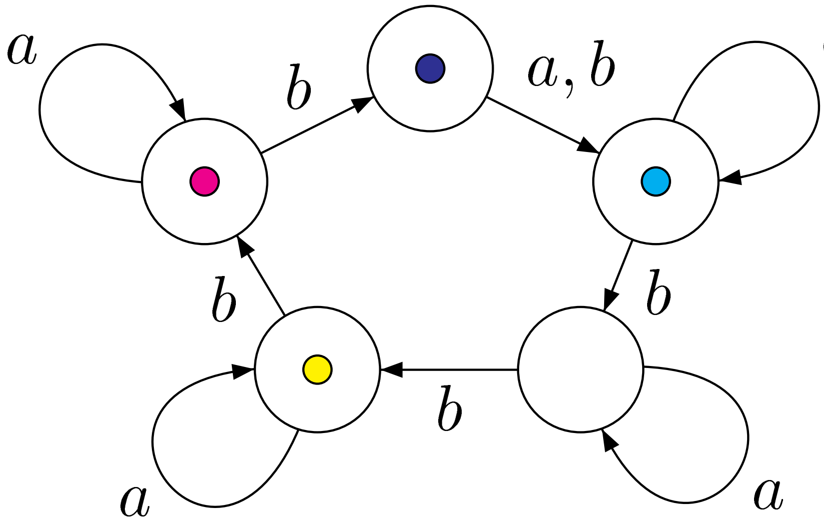

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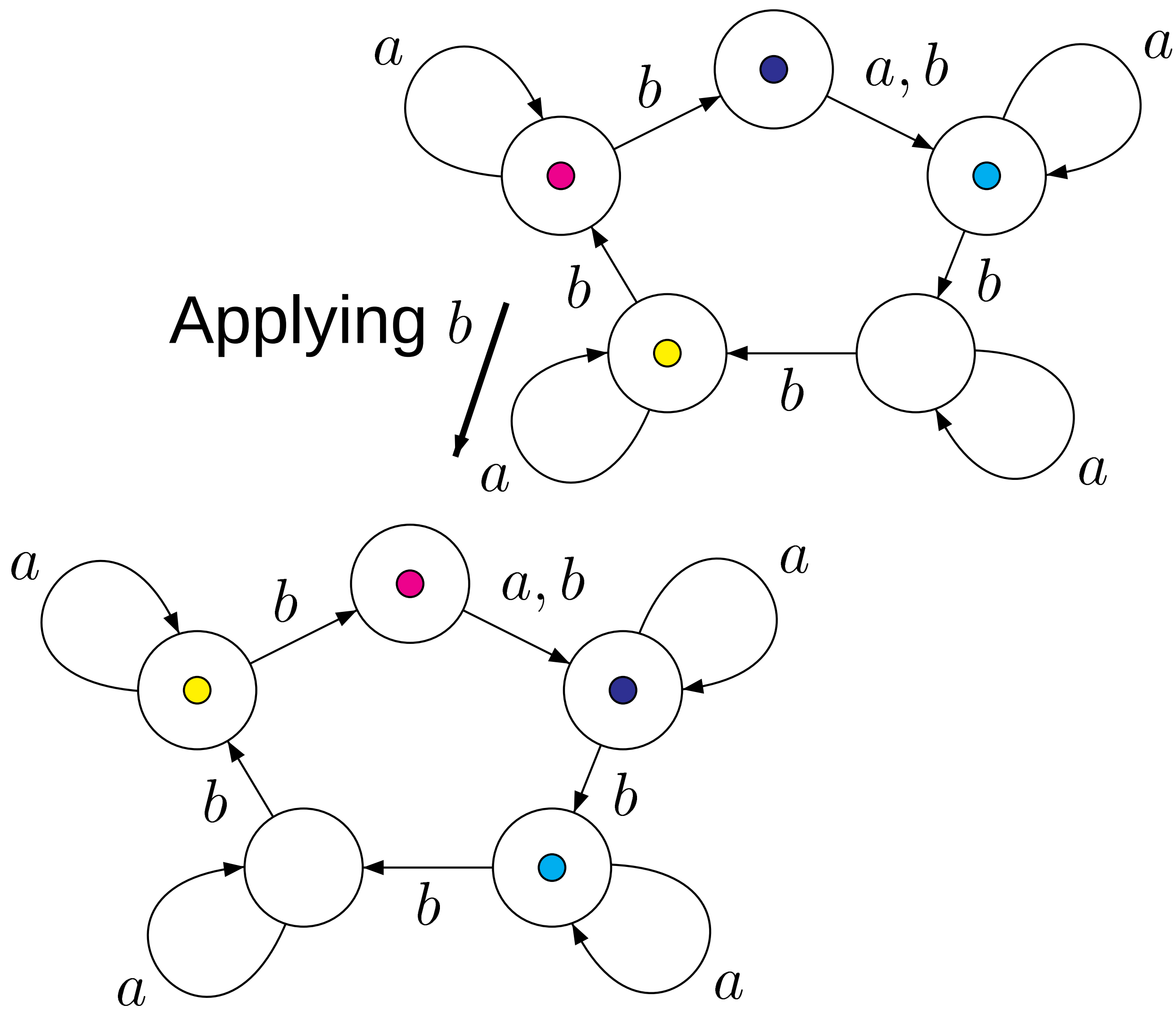

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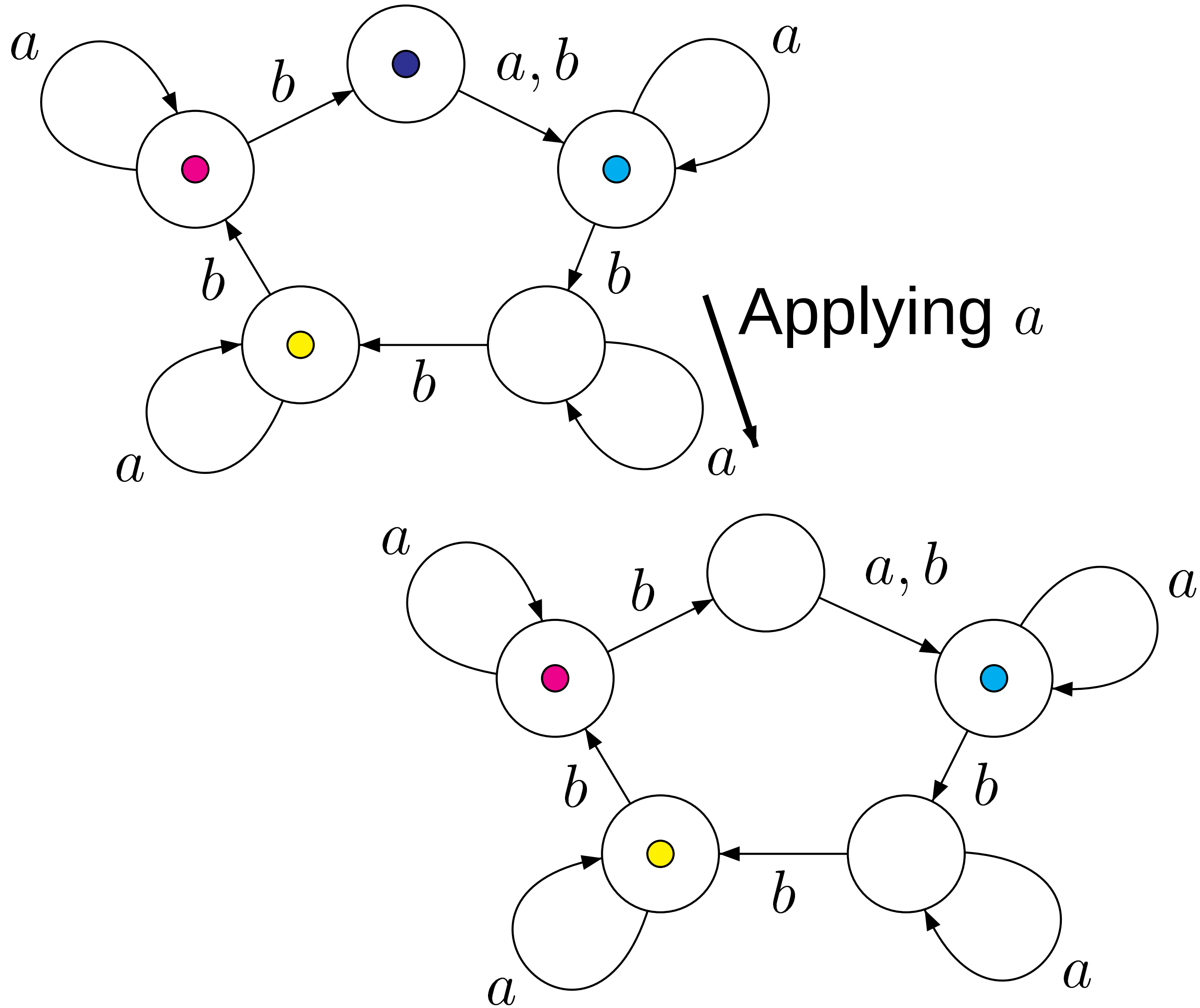

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(ii) for each $i = 1, \ldots, |w|$, the action of the i^{th} letter of w decreases the weight by 1 at most, that is, $1 \geq \mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)$.

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Then $$|w| = \sum_{i=1}^{|w|} 1 \geq \sum_{i=1}^{|w|} \big(\mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)\big) =$$

$$\mathrm{wg}(P_0) - \mathrm{wg}(P_{|w|}) \geq n(n-1) - (n-1) = (n-1)^2.$$

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$$\mathrm{wg}(C, P_i) = n \cdot d_i(C) + m_i(C)$$

where $m_i(C)$ is the residue of $n - s_i(C)$ modulo n and $d_i(C)$ is the number of steps from $s_i(C)$ to $s_i(G)$ in the 'main circle' of our automaton. (Recall that G stands for the golden coin G which is present in all positions.)

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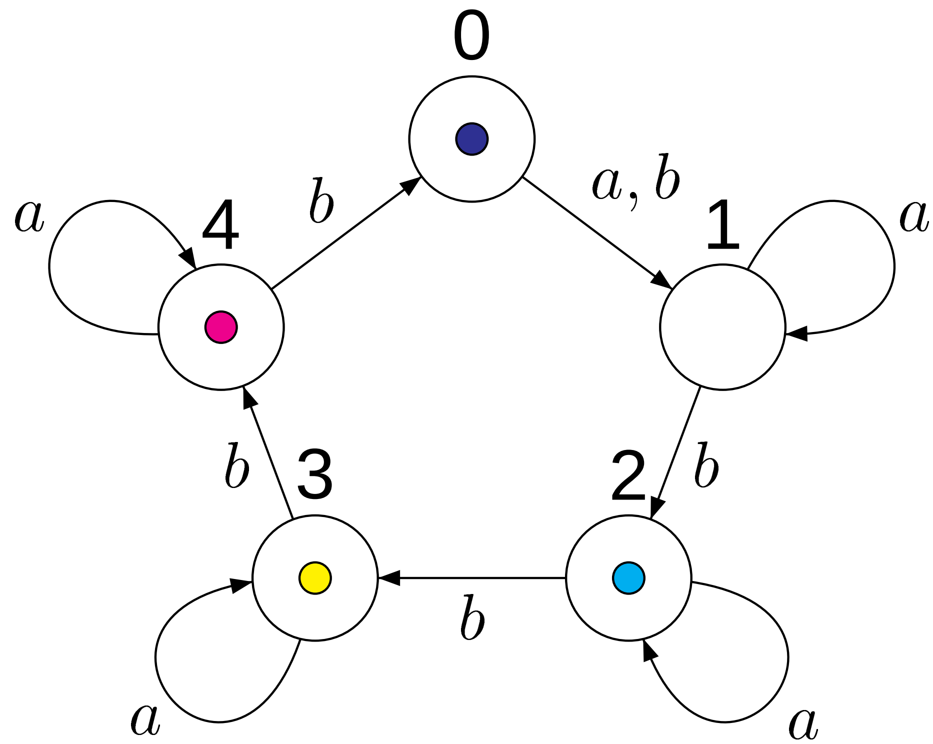

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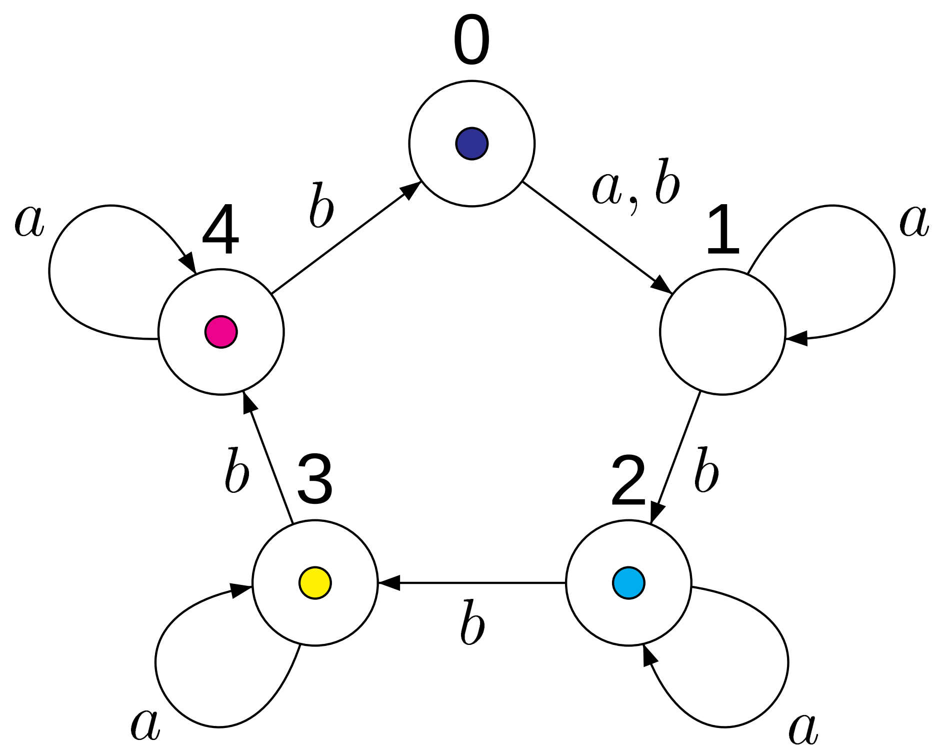

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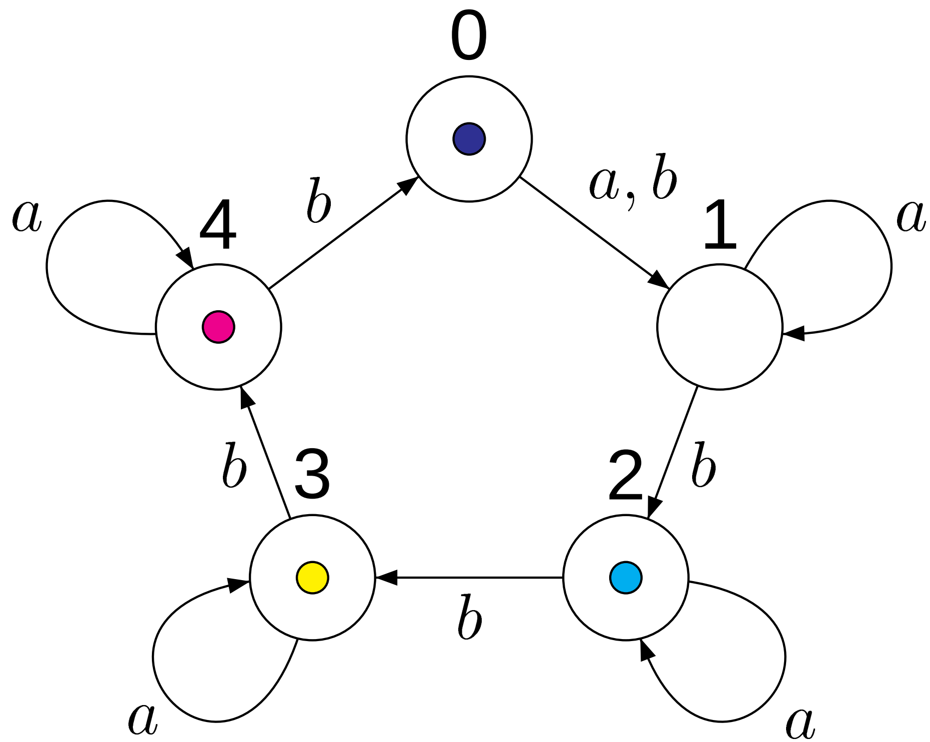

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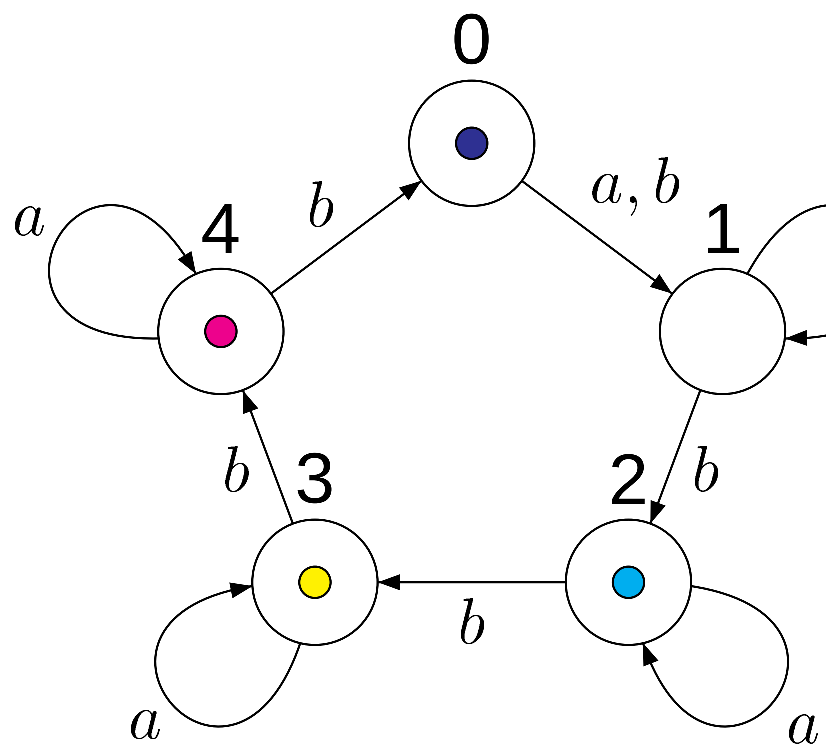

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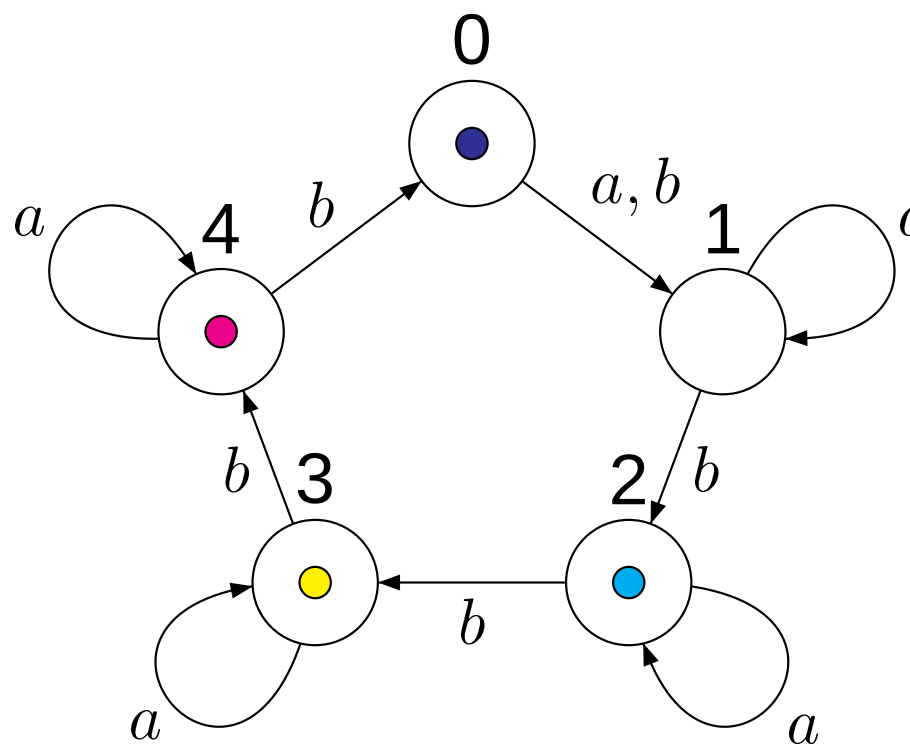

Assume that the yellow coin is the golden one. Then its weight is $5 - 3 = 2$. The weight of the cyan coin is $5 \cdot 1 + (5 - 2) = 8$. The weight of the blue coin is $5 \cdot 3 + 0 = 15$.

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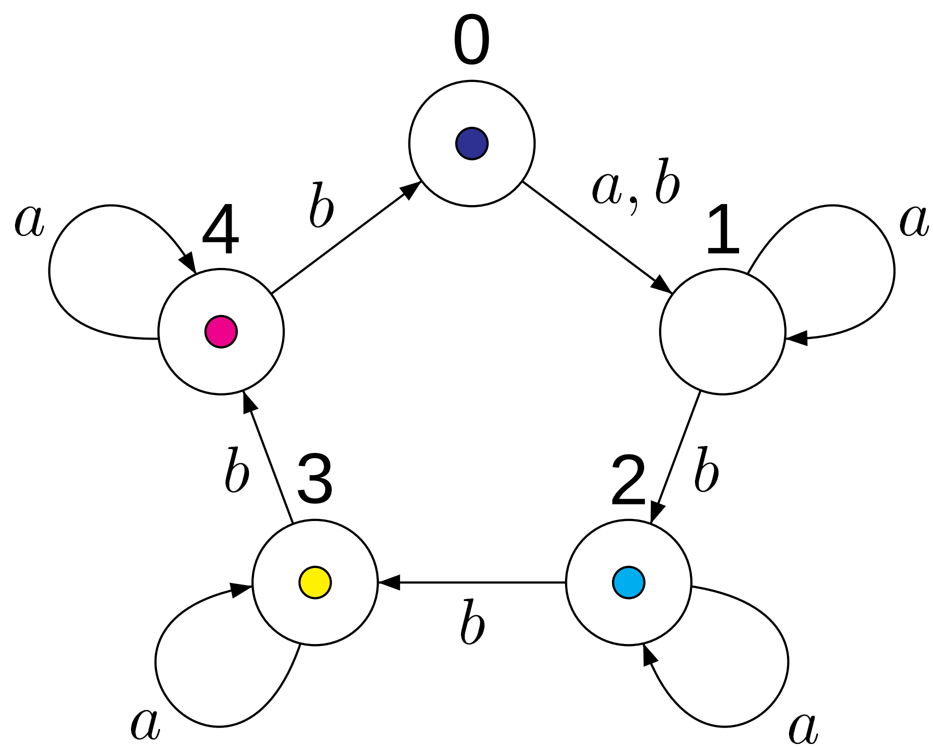

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In the initial position all states are covered with coins. Consider the coin C that covers the state $s_0(G) + 1 \pmod{n}$, that is the state in one step clockwise after the state covered with the golden coin. Then $d_0(C) = n - 1$ whence $\mathrm{wg}(C, P_0) = n \cdot (n-1) + m_0(C) \geq n(n-1)$. Since the weight of a position is not less that the weight of any coin in this position, we have $\mathrm{wg}(P_0) \geq n(n-1)$.

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$$\mathrm{wg}(P_i) \geq \mathrm{wg}(C', P_i) = n \cdot d_i(C') + n - 1 = n \cdot (d_{i-1}(C) - 1) + n - 1$$

$$= n \cdot d_{i-1}(C) - 1 = \mathrm{wg}(C, P_{i-1}) - 1 = \mathrm{wg}(P_{i-1}) - 1.$$

The Černý conjecture

Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, $n = 2, 3, \ldots$, yields the inequality $C(n) \geq (n-1)^2$.

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$$(n-1)^2 \leq C(n) \leq \frac{n^3 - n}{6}.$$

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- non-locality: prefixes of optimal solutions need not be optimal (that's why the greedy algorithm fails;
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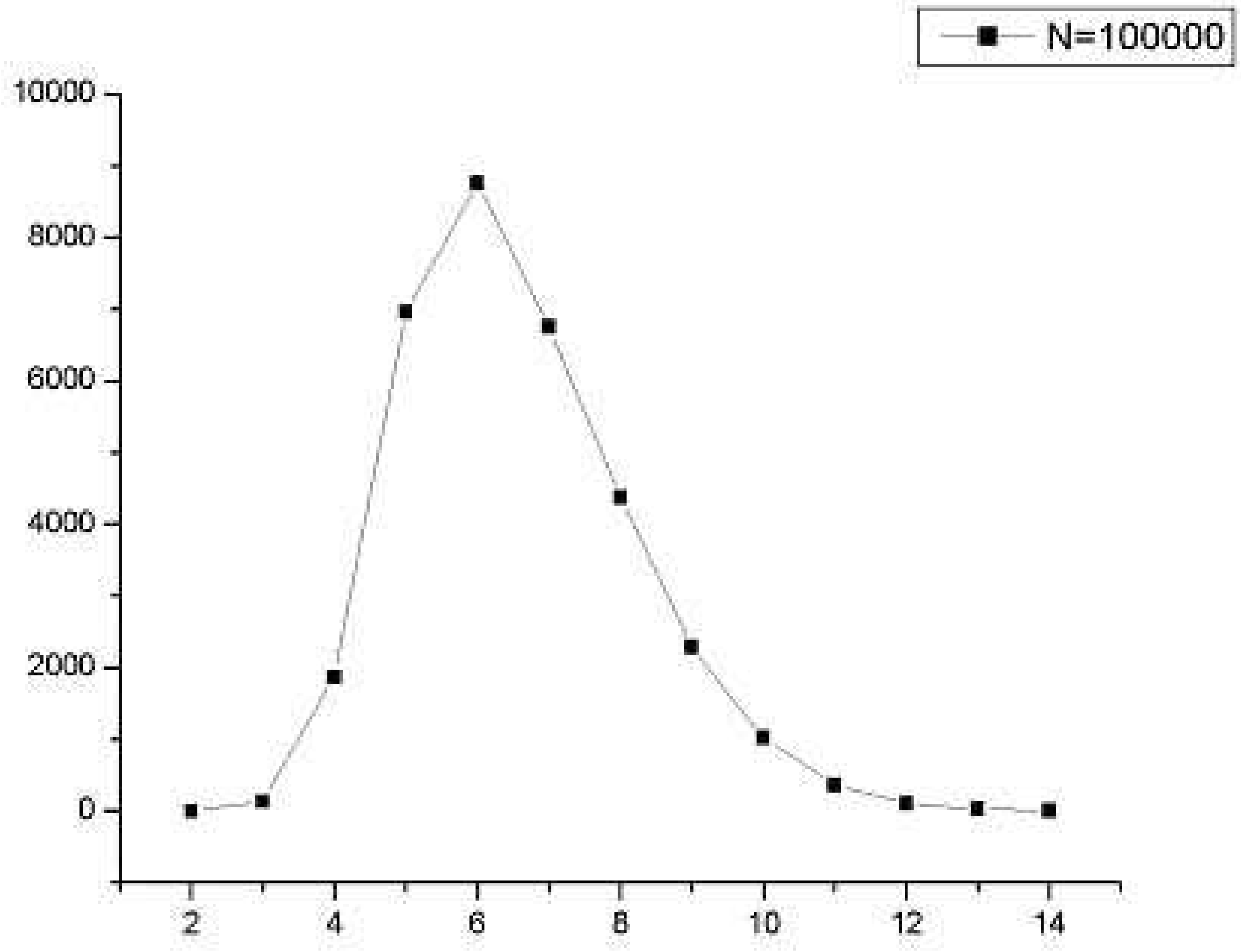
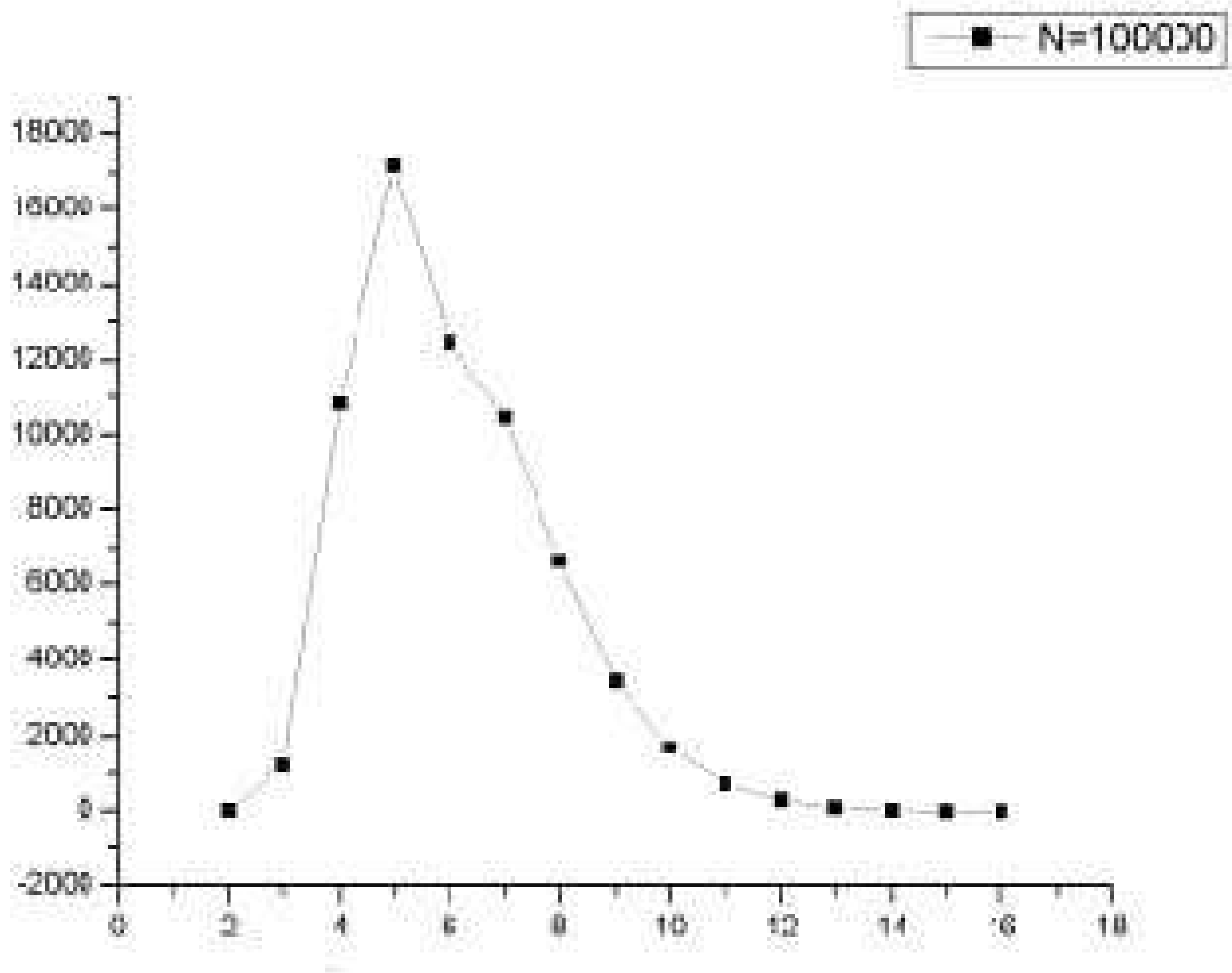
Yet another reason: "slowly" synchronizing automata turn out to be extremely rare. The only known infinite series of n-state synchronizing automata with shortest reset words of length $(n-1)^2$ is the Černý series $\mathscr{C}_n$, $n = 2, 3, \dots$, with a few sporadic examples for $n \leq 6$.

20-State Experiment



30-State Experiment



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Thus, "slowly" synchronizing automata cannot be discovered via a random sampling.

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A synchronizing automaton with n states *reaches the Černý bound* if the minimum length of its reset words is $(n-1)^2$.

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A synchronizing automaton with n states *reaches the Černý bound* if the minimum length of its reset words is $(n-1)^2$. We present here all known proper synchronizing automata beyond the Černý series $\mathscr{C}_n$, $n = 3, 4, \dots$, that reach the Černý bound.

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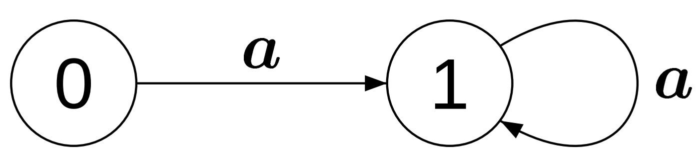
For the sake of completeness, we start with $n = 2$:



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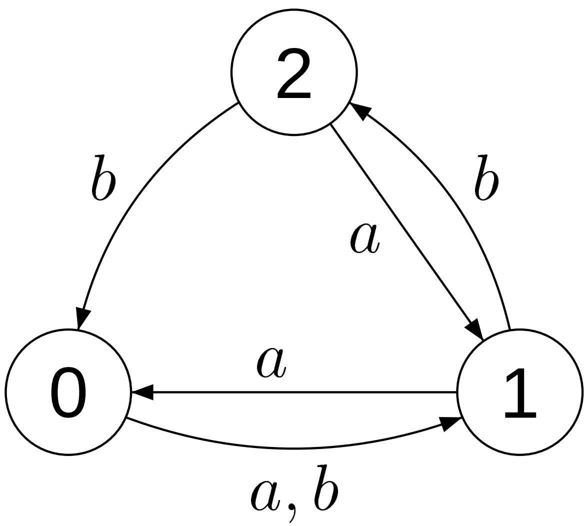

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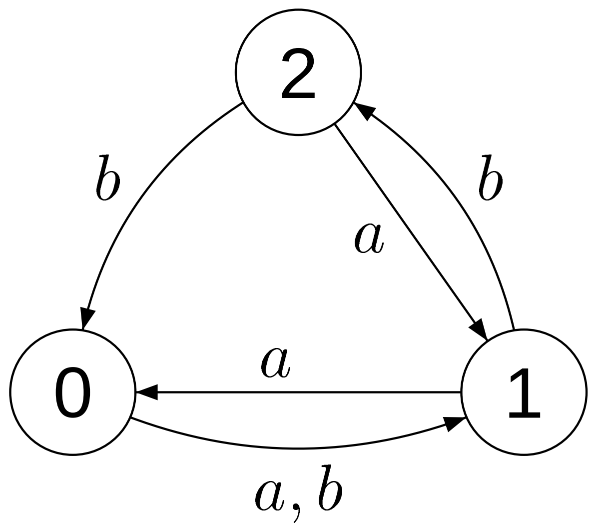

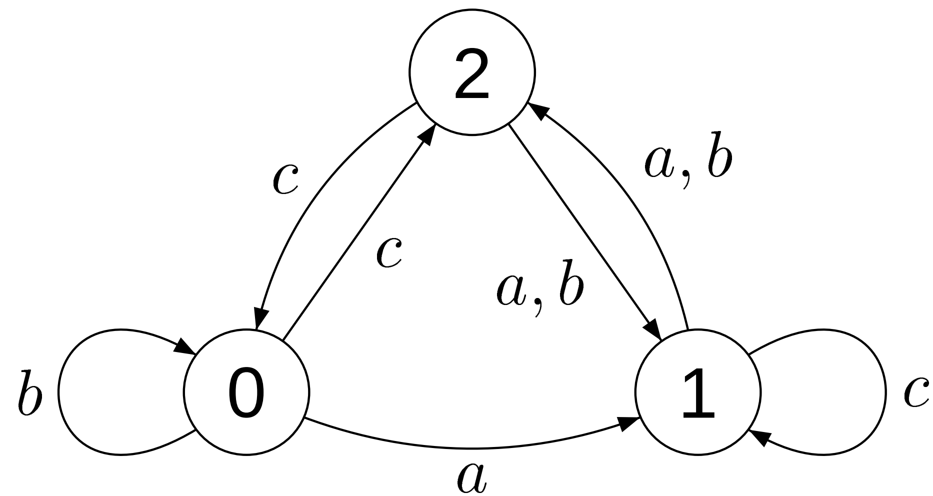

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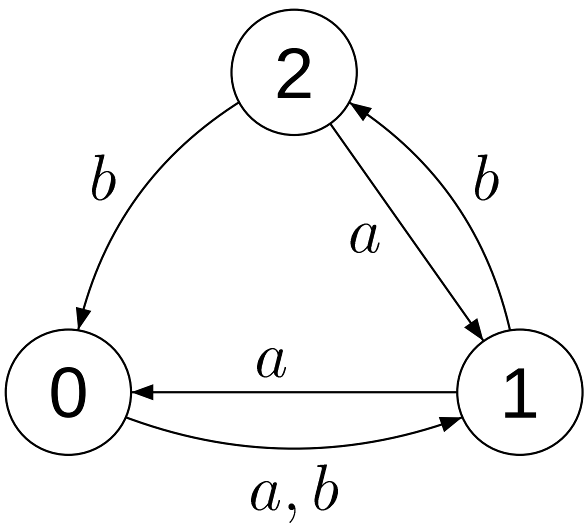

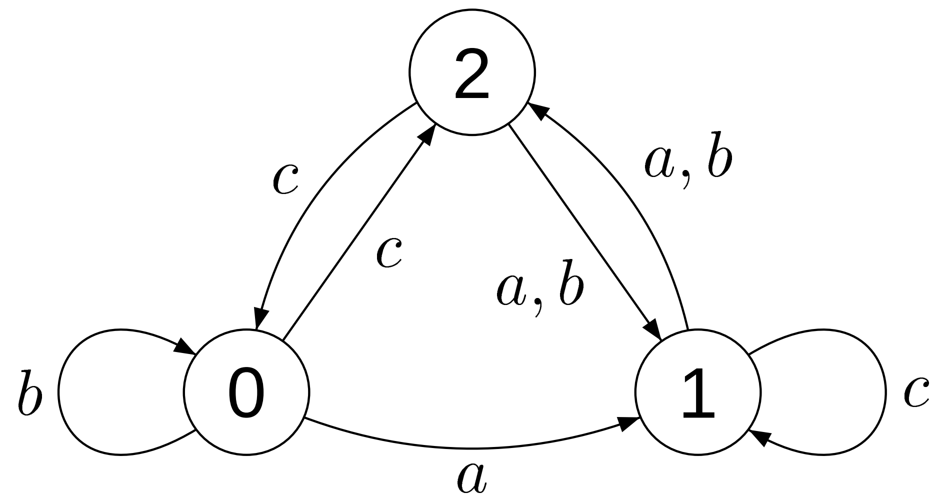

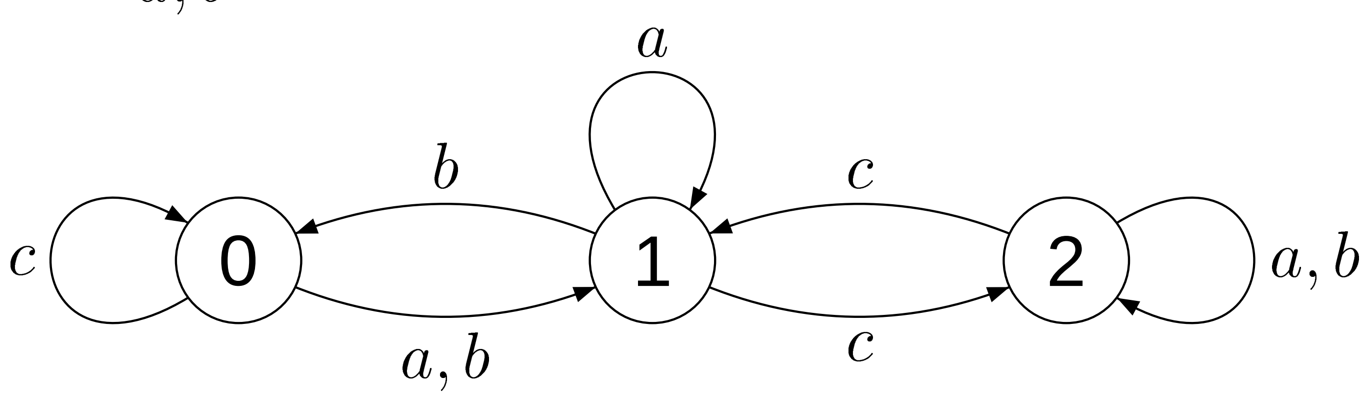

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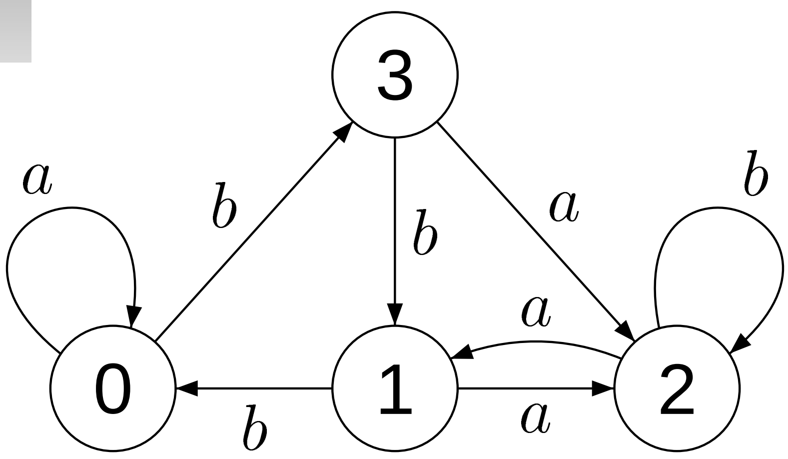

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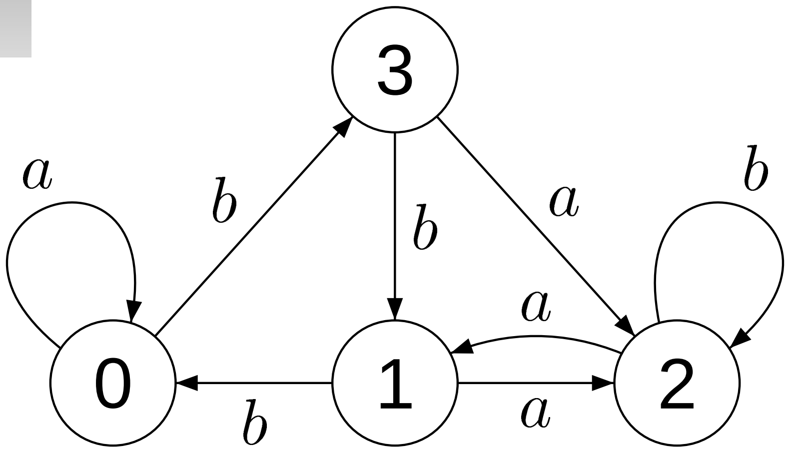

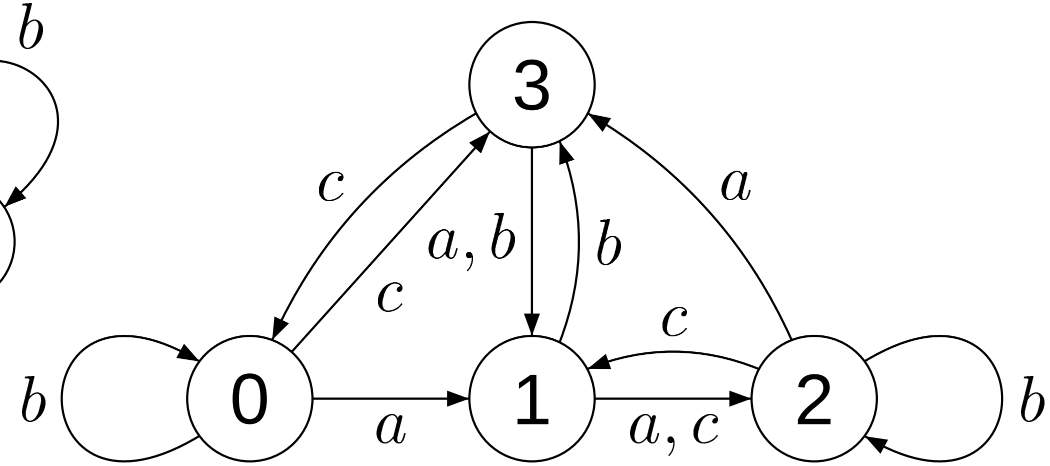

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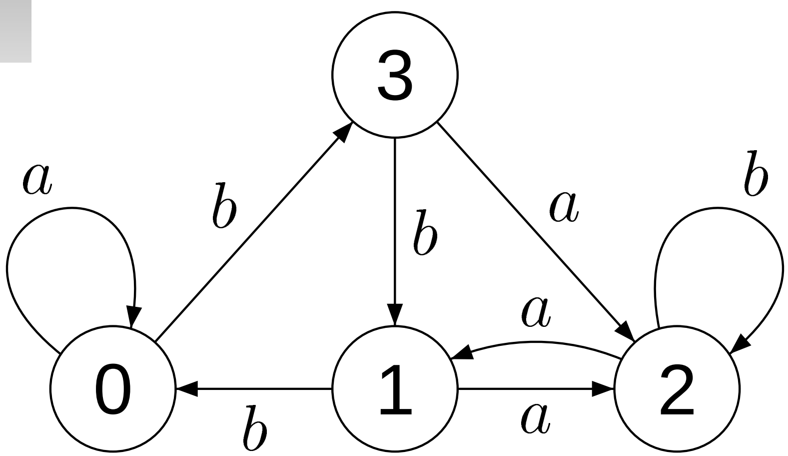

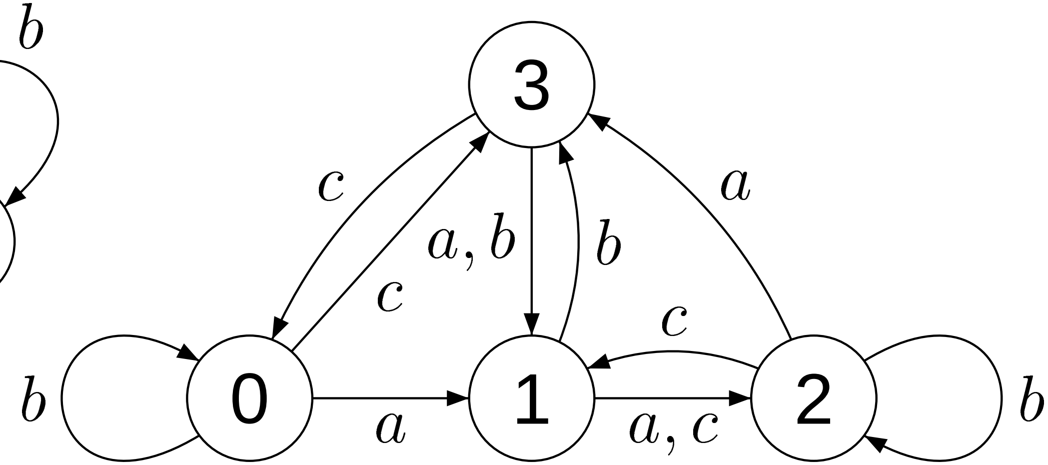

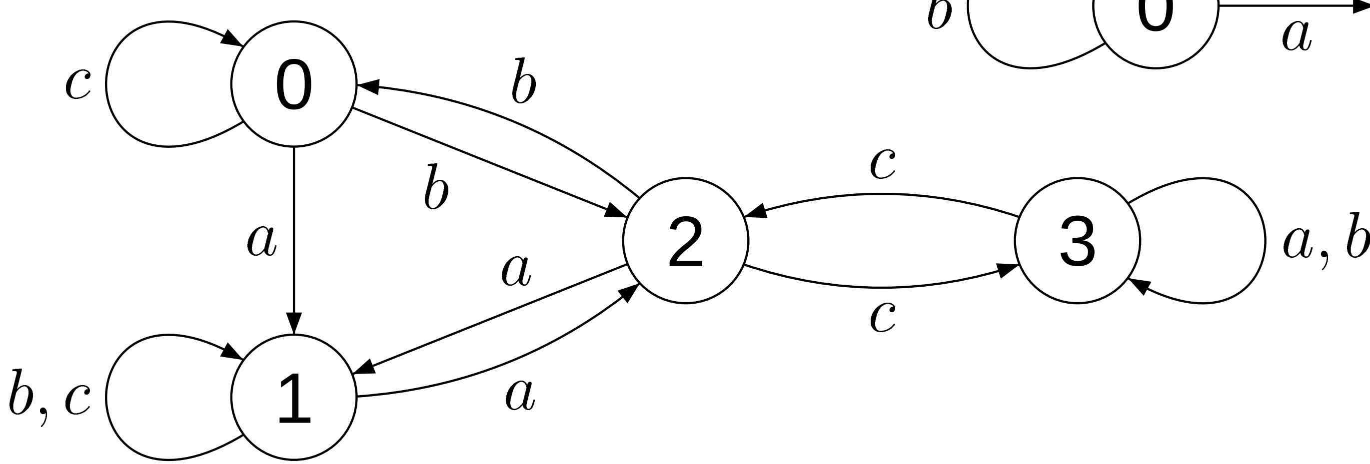

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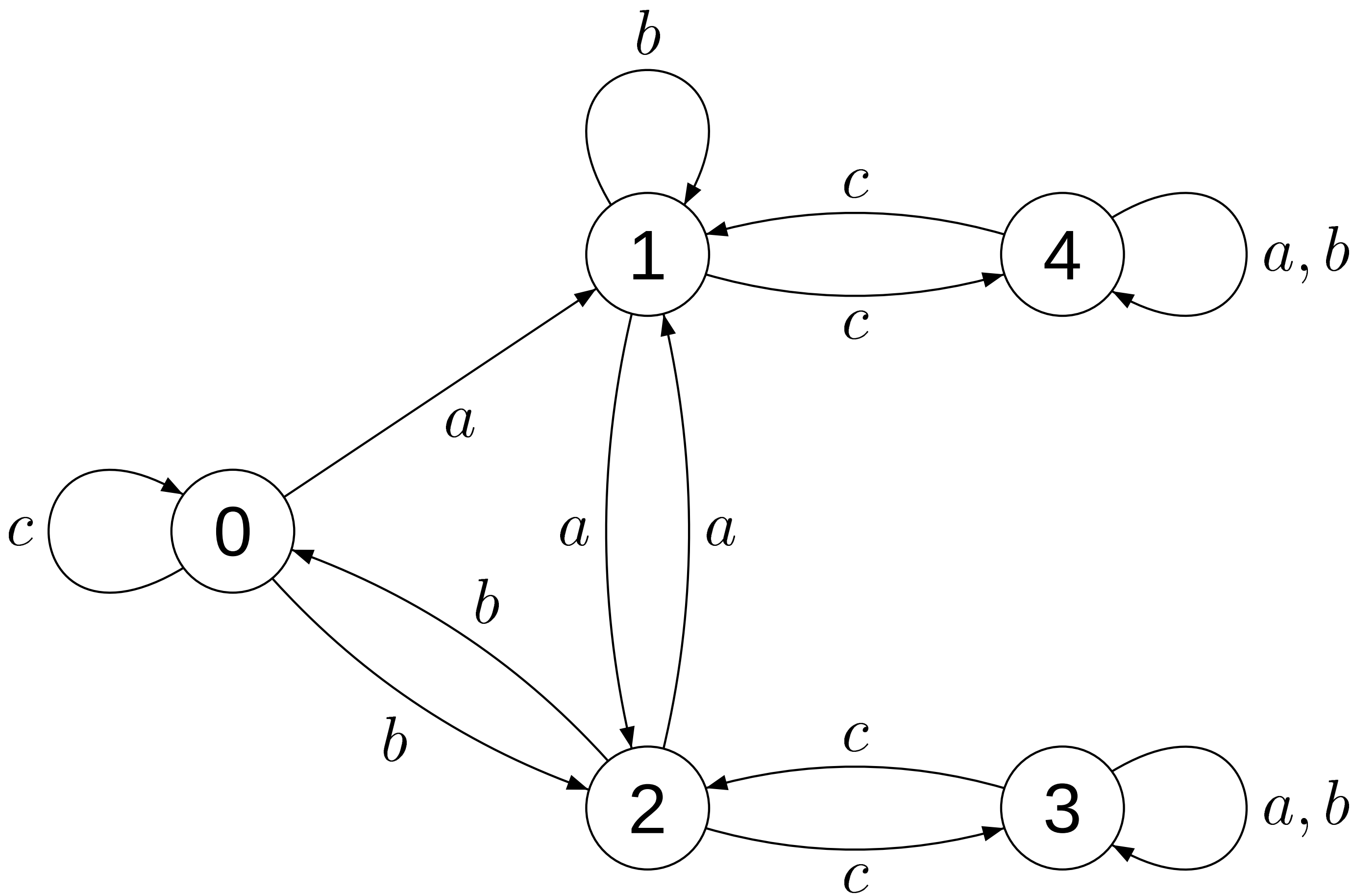

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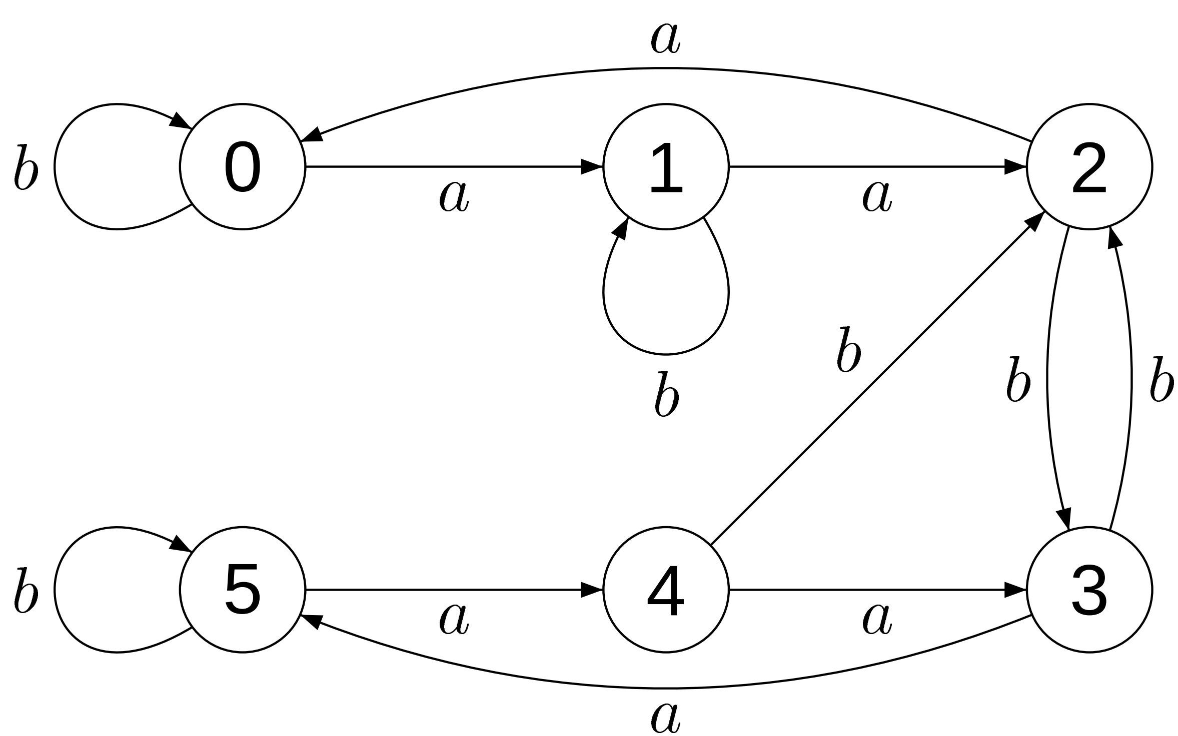

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However, in $\mathscr{K}_6$ there is no word w of length $16 = (6-2)^2$ such that $|Q \,.\, w| = 2$.

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The *rank* of a DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is the minimum cardinality of the sets $\boldsymbol{Q} \,.\, \boldsymbol{w}$ where $\boldsymbol{w}$ runs over $\boldsymbol{\Sigma}^*$.

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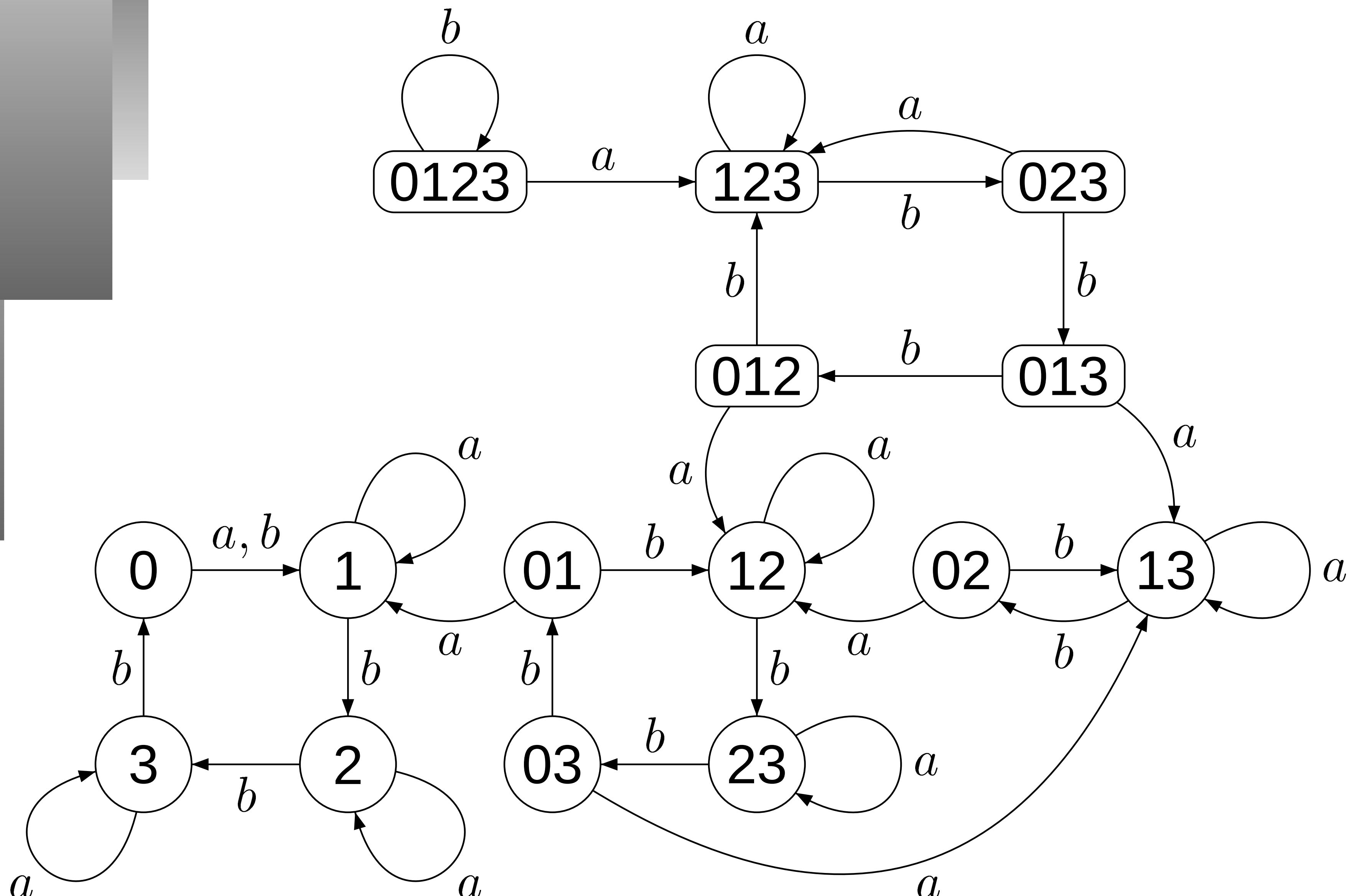
Extensibility



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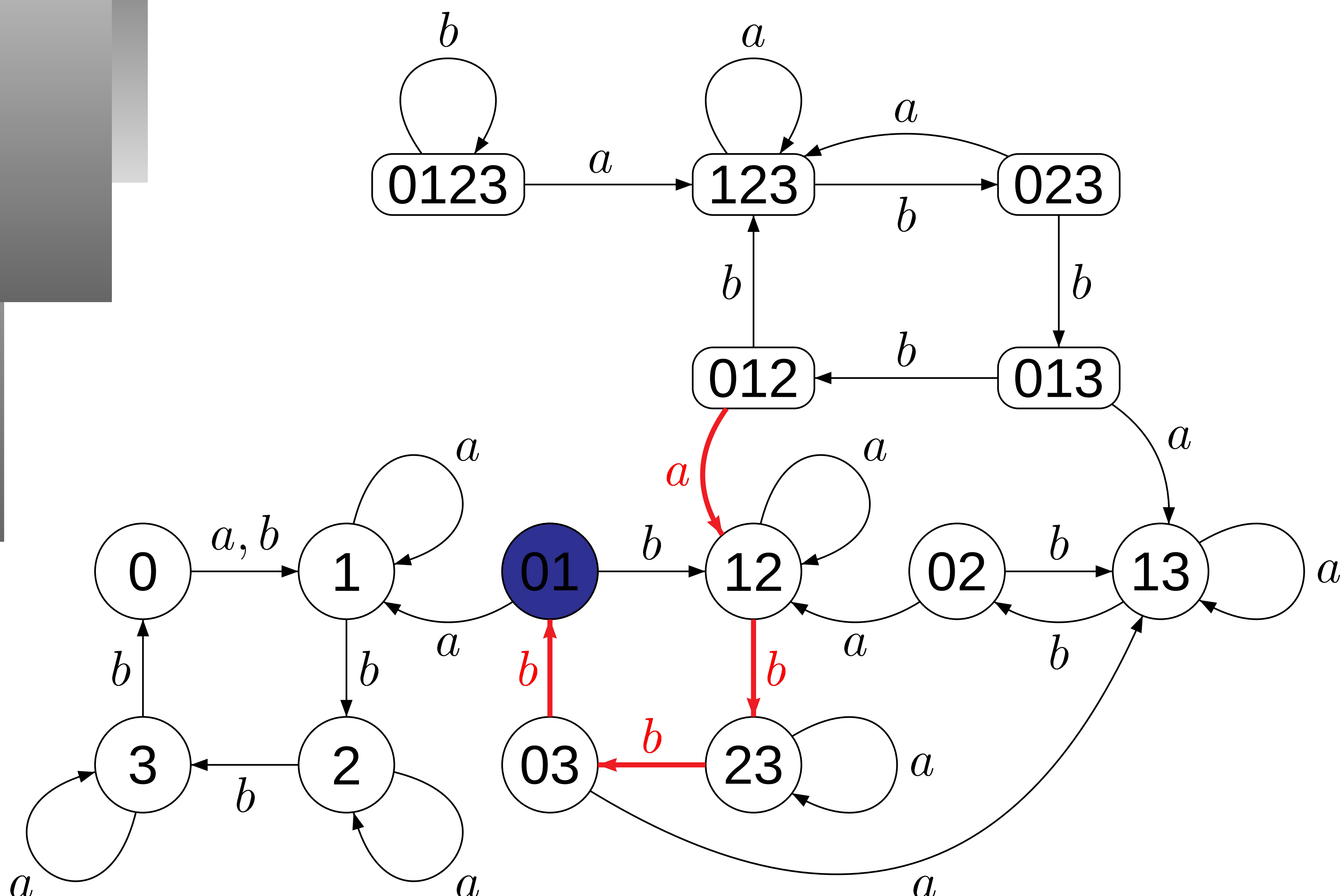

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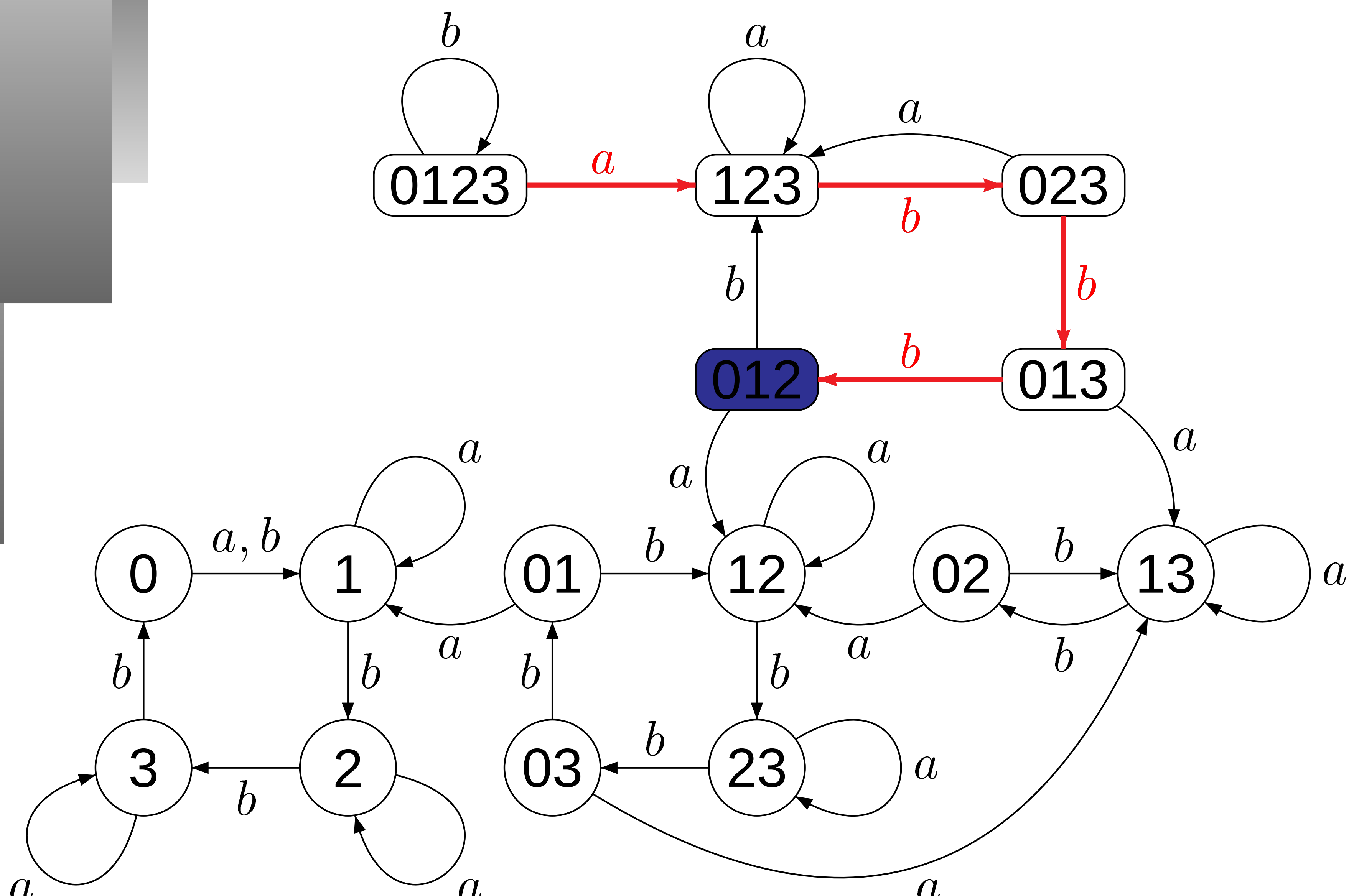

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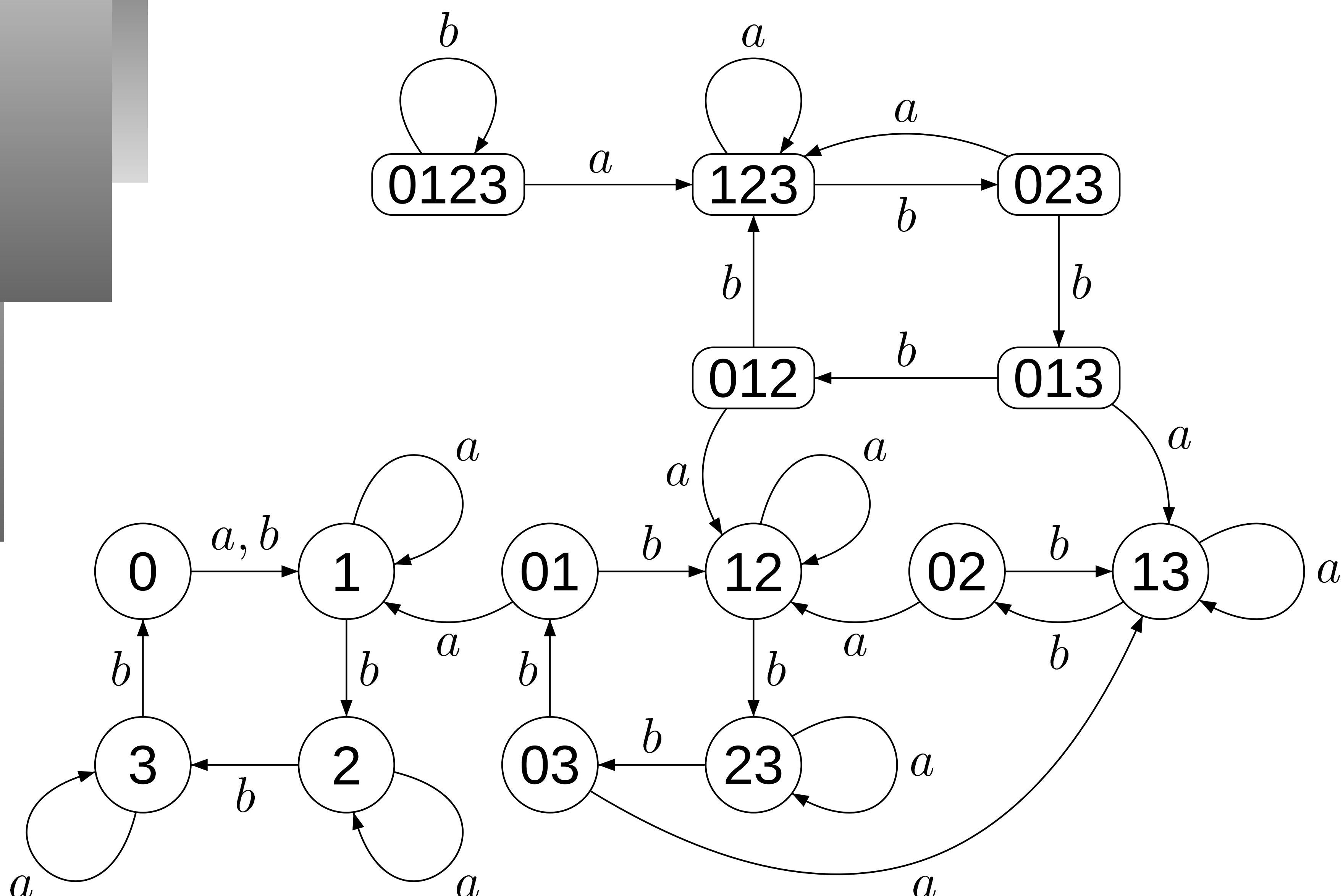

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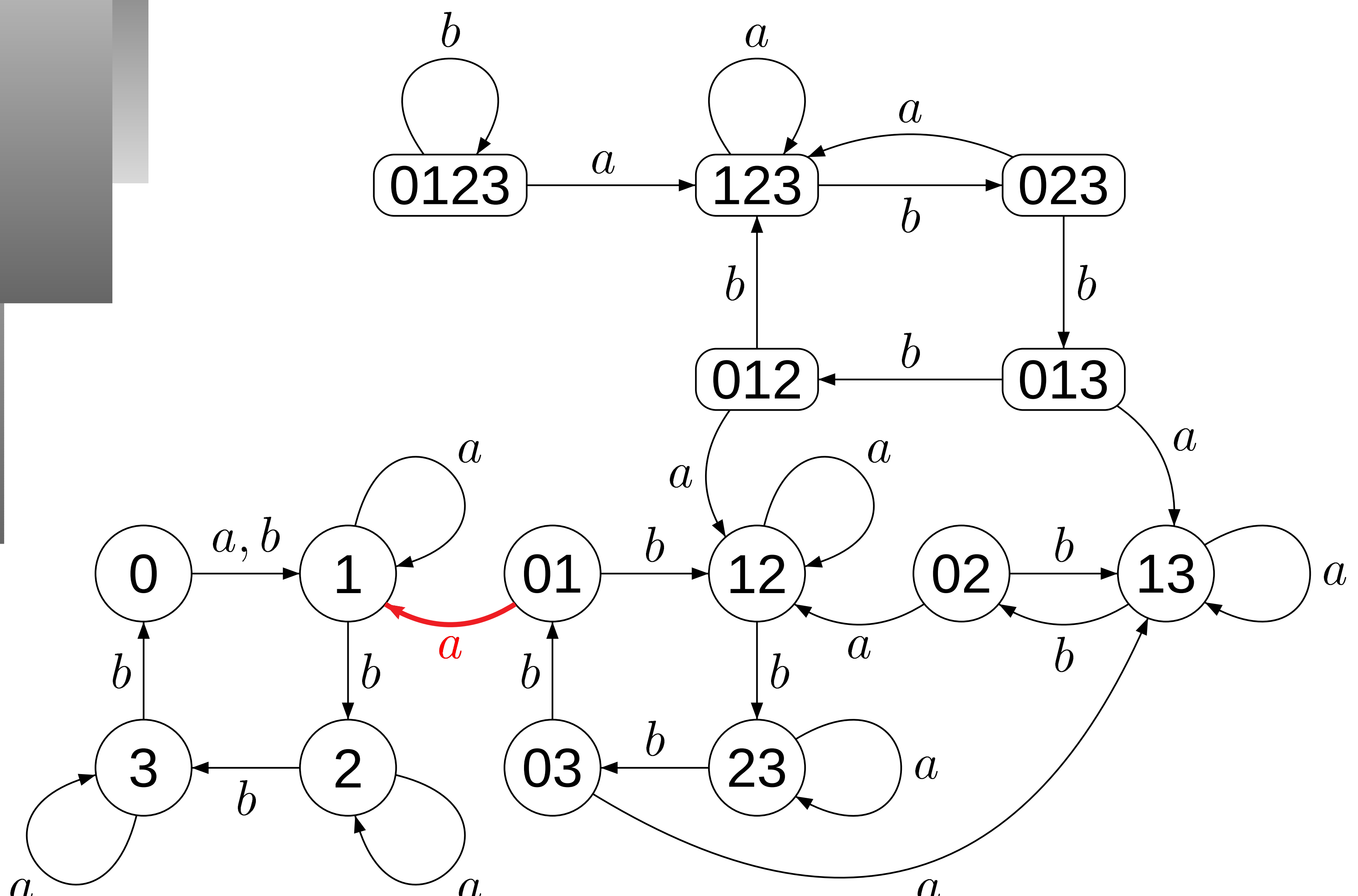

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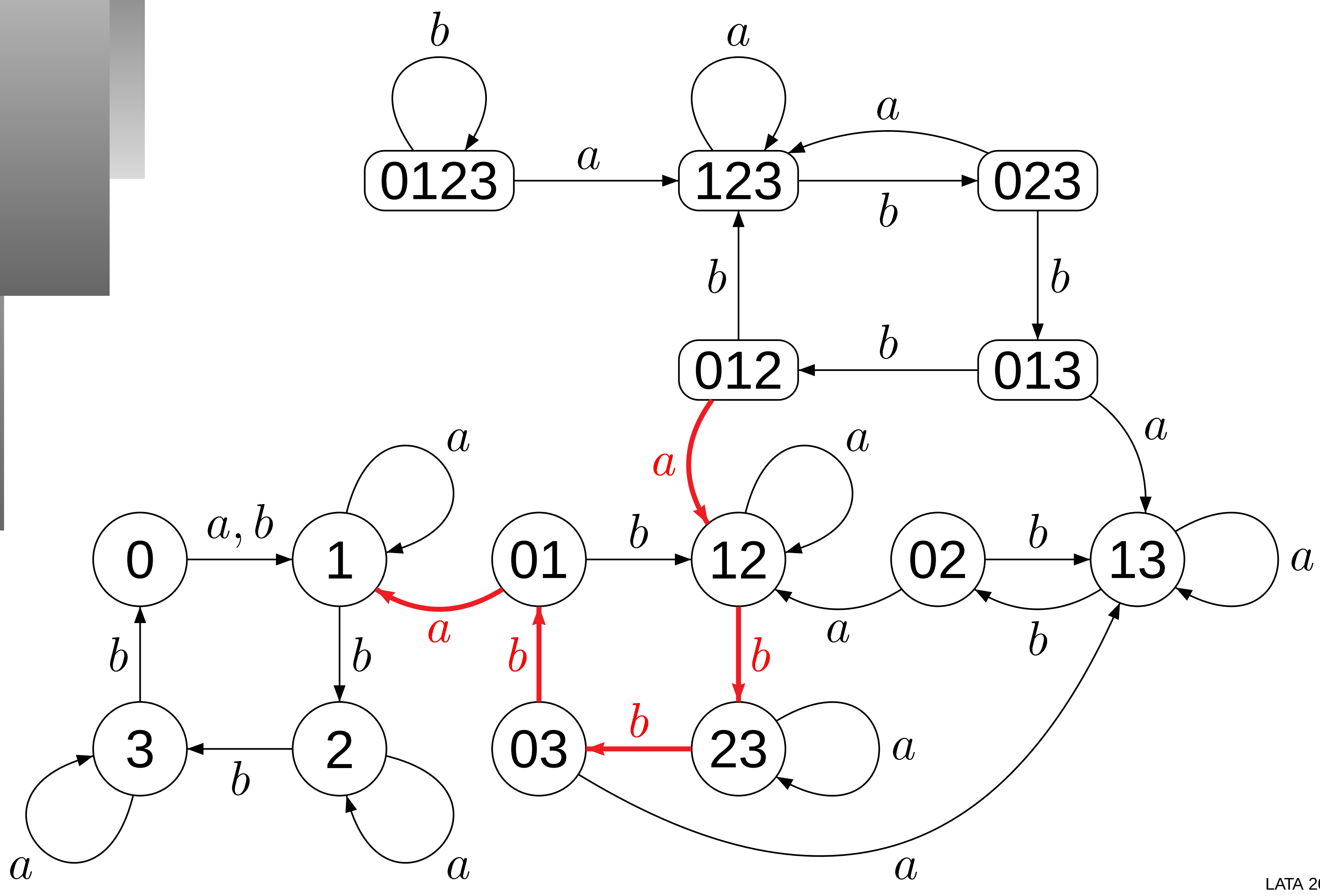

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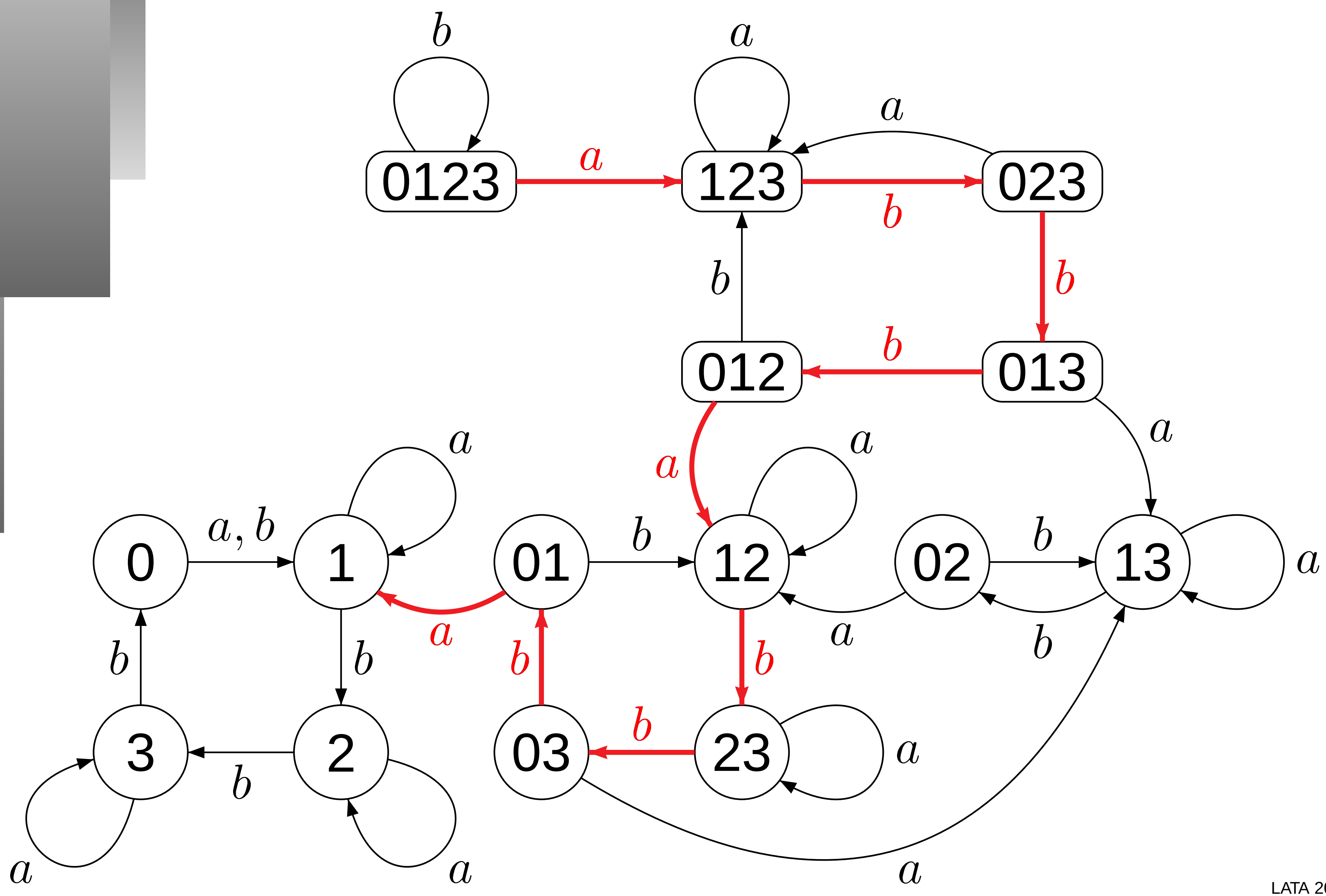

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Thus, the extensibility conjecture fails, and the approach based on it cannot prove the Černý conjecture in general.