Synchronizing Automata – III

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$|Q\,.\,w| = 1$. Here $Q\,.\,v = \{\delta(q, v) \mid q \in Q\}$.

Any w with this property is said to be a *reset word* for the automaton.

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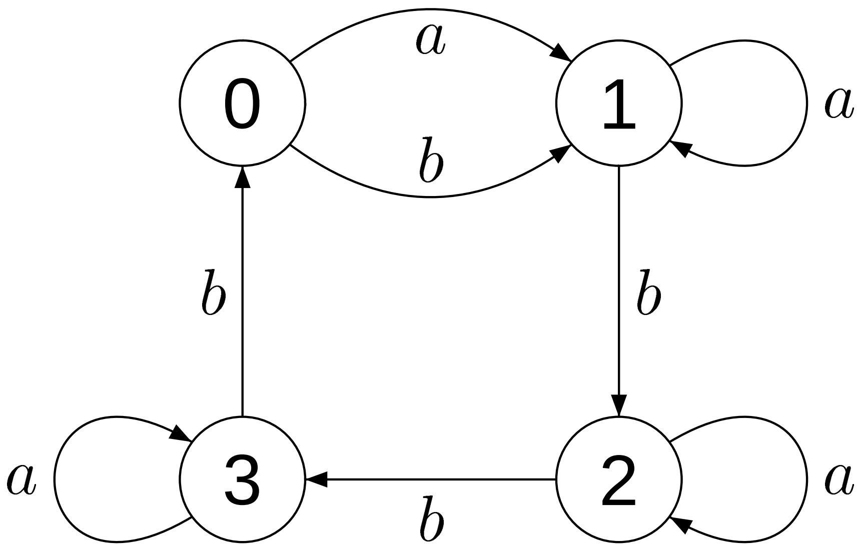

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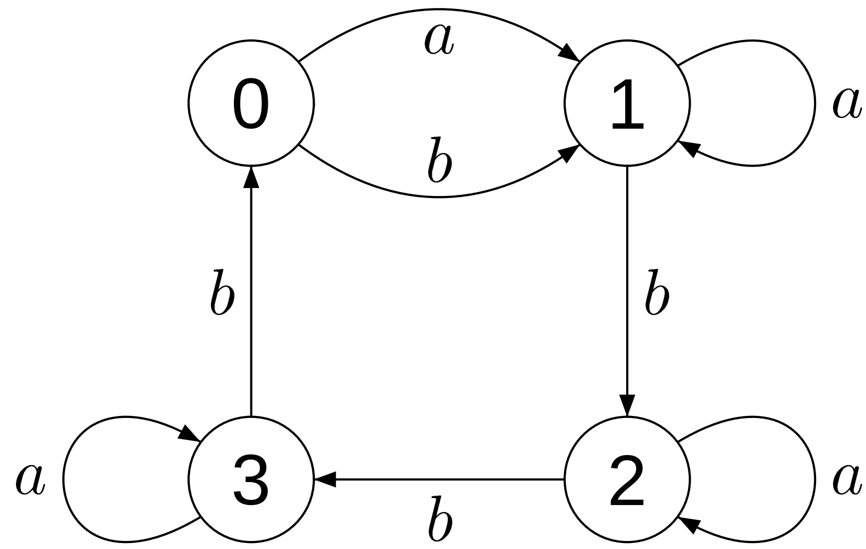

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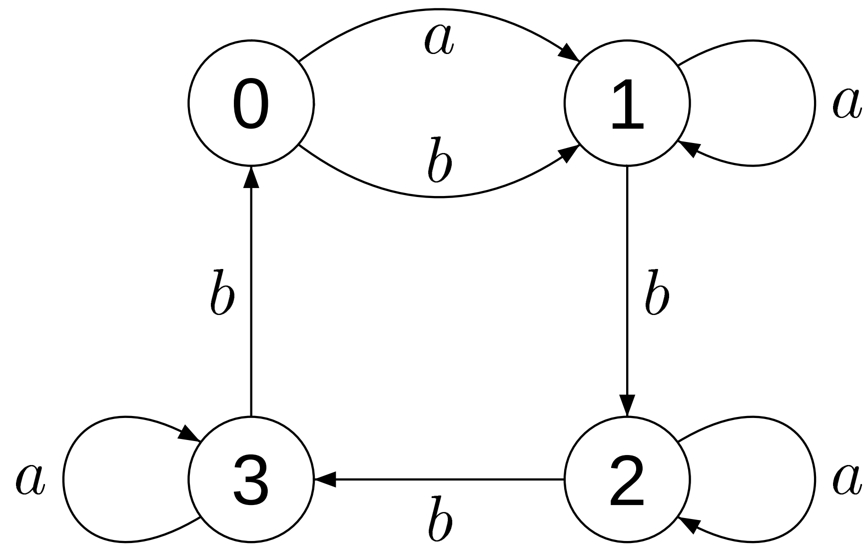

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Observe that this automaton is strongly connected (as a digraph). In fact, almost all our examples have been strongly connected.

Strongly connected digraphs

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Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a synchronizing automaton with n states. Consider the set S of all states to which $\mathscr{A}$ can be synchronized and let $m = |S|$. If $q \in S$, then there exists a reset word $w \in \Sigma^*$ such that $Q.w = \{q\}$. For each $a \in \Sigma$, we have $Q.wa = \{\delta(q, a)\}$ whence wa also is a reset word and $\delta(q, a) \in S$. Thus, restricting the function δ to $S \times \Sigma$, we get a subautomaton $\mathscr{S}$ with the state set S.

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We recall the notion of a congruence and the related notion of the quotient automaton w.r.t. a congruence in the next slide. They will be essentially used in this lecture!

Congruences and Quotient Automata

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called a *congruence* if $(p, q) \in \pi$ implies $\big(\delta(p, a), \delta(q, a)\big) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$.

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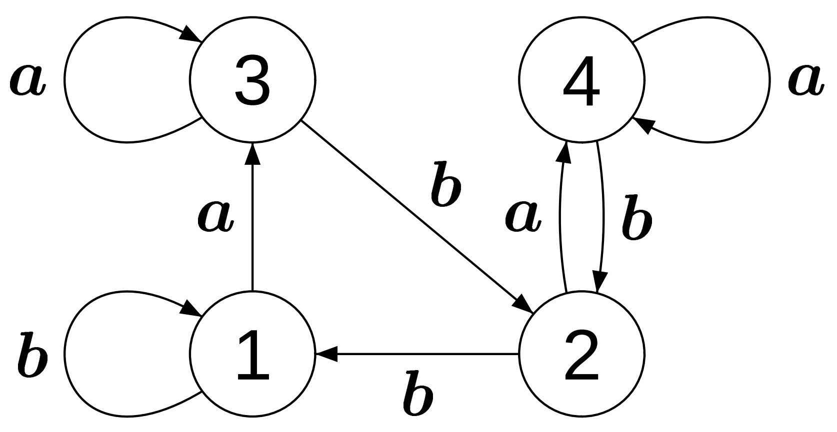

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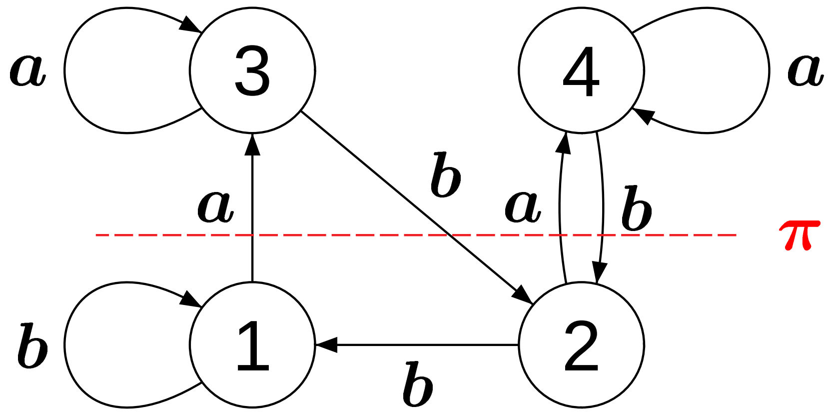

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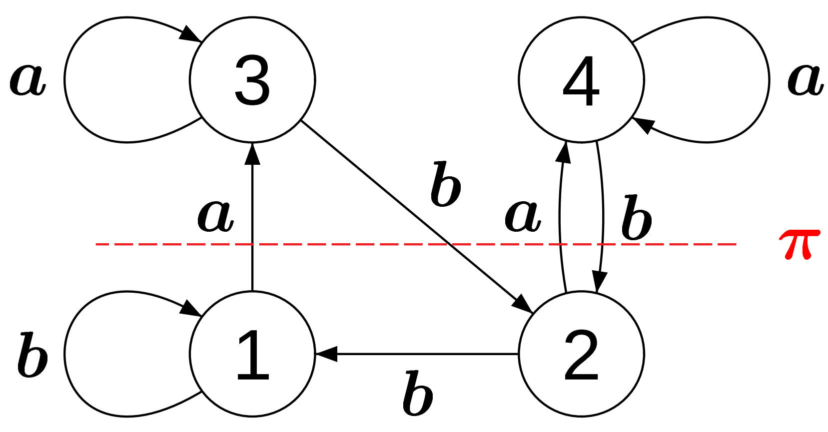
The *quotient* $\mathscr{A}/\pi$ is the DFA $\langle Q/\pi, \Sigma, \delta_\pi \rangle$ where $Q/\pi = \{[q]_\pi \mid q \in Q\}$ and the function δ_π is defined by the rule $\delta_\pi([q]_\pi, a) = [\delta(q, a)]_\pi$.



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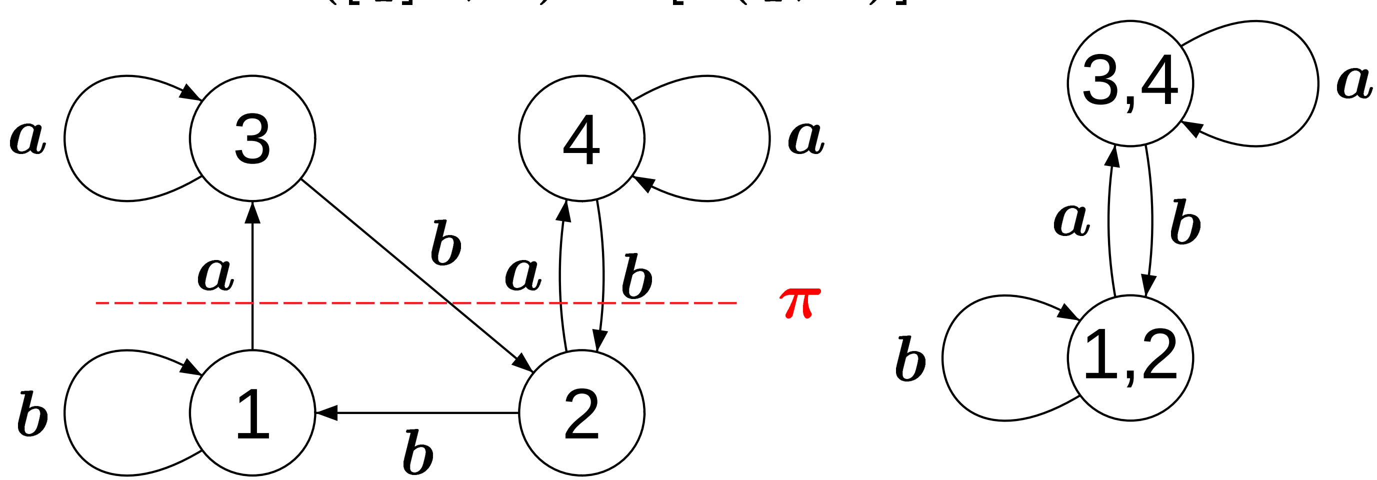

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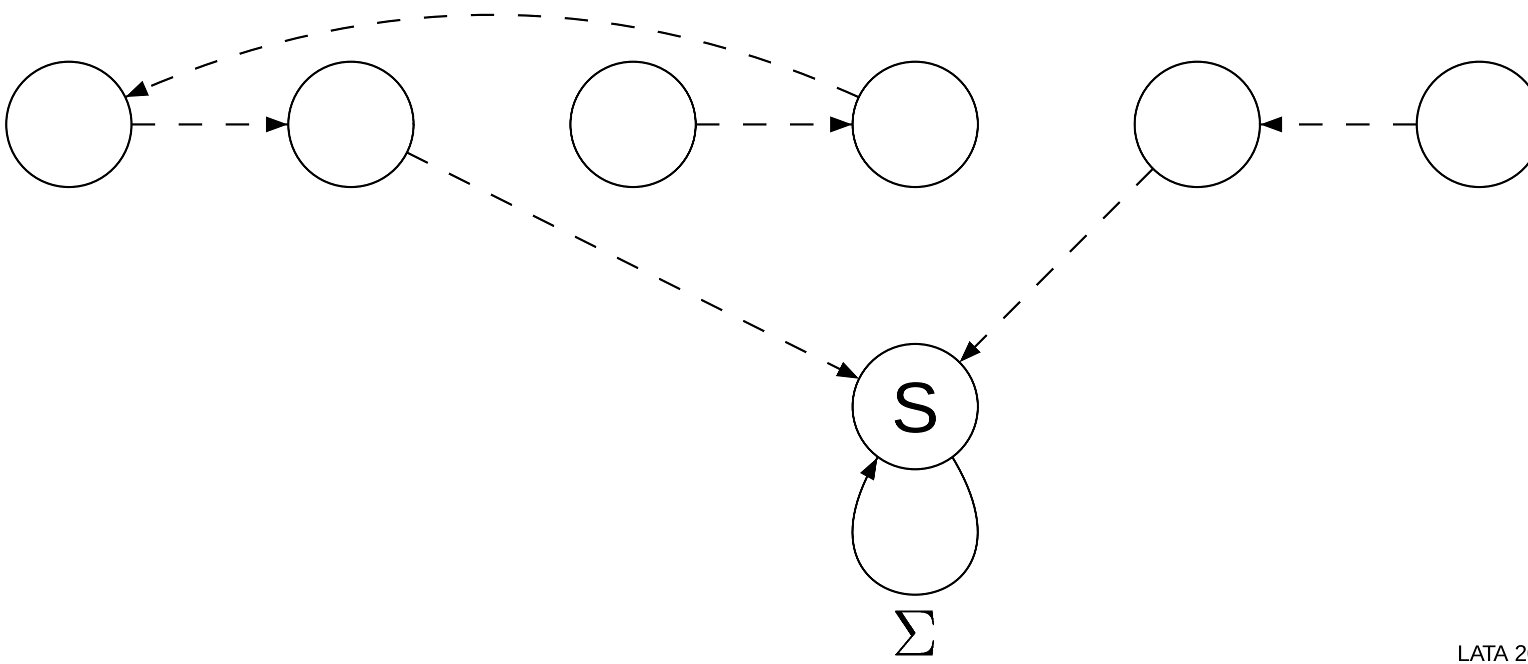

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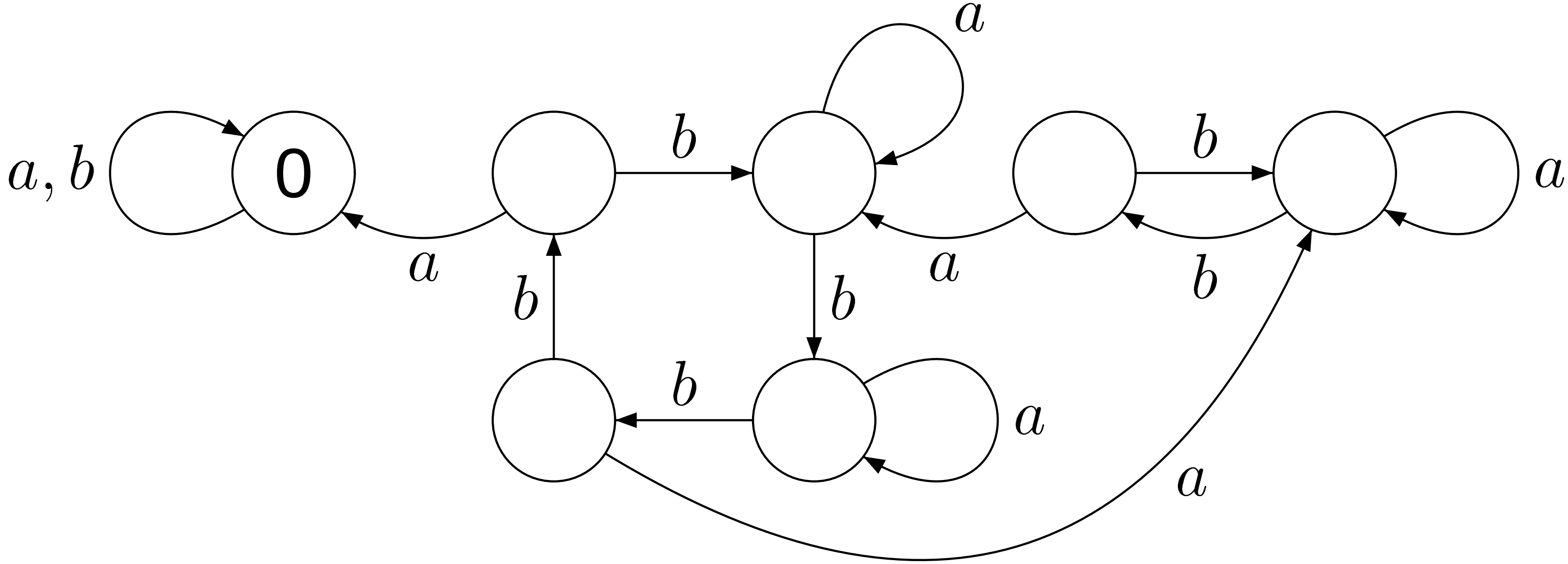

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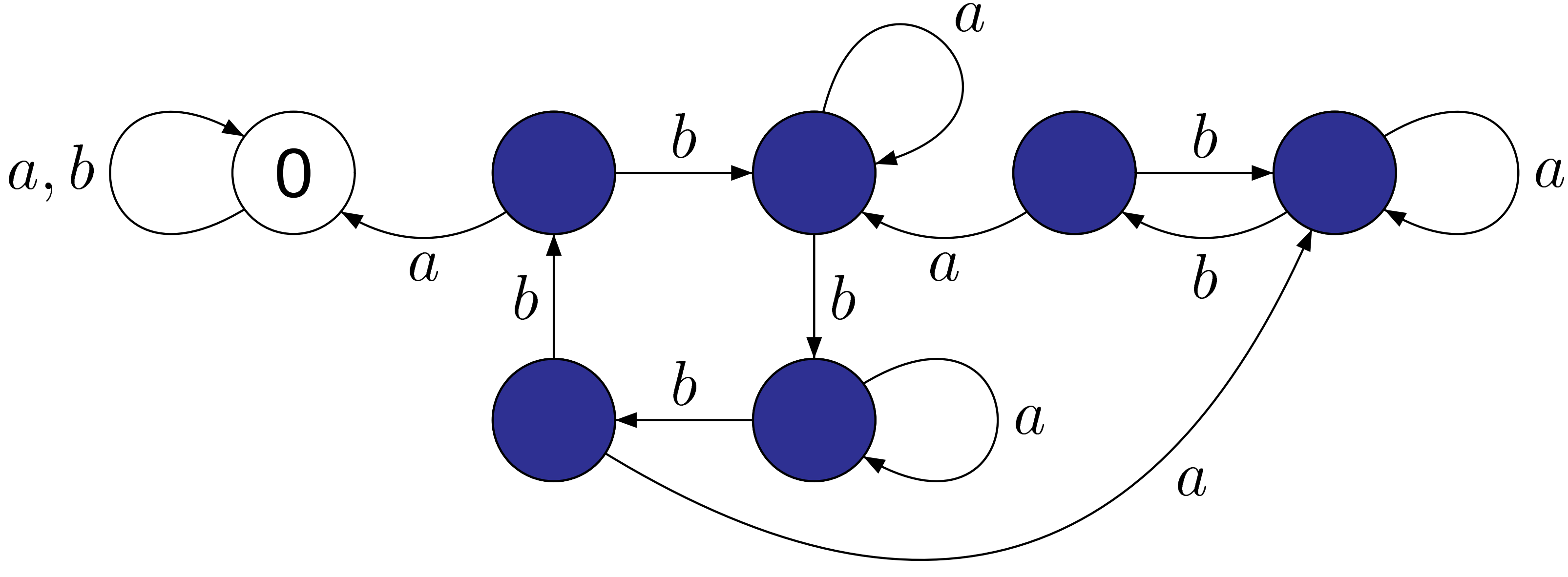

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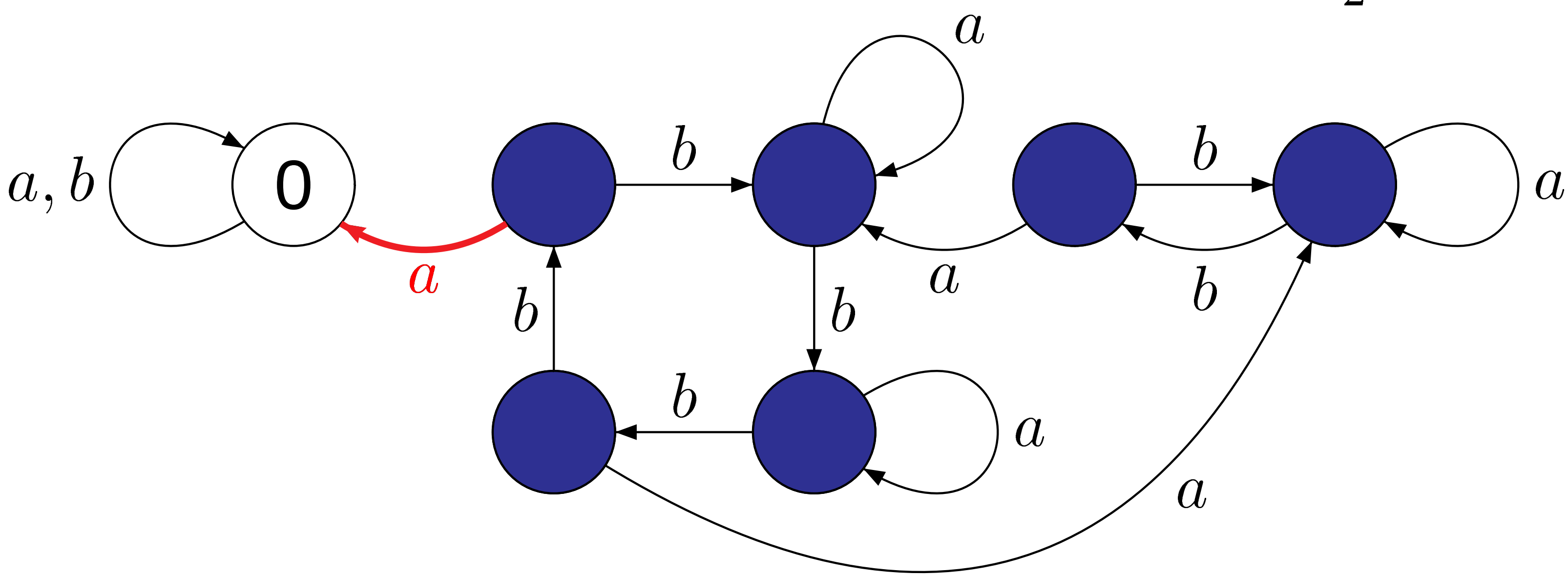

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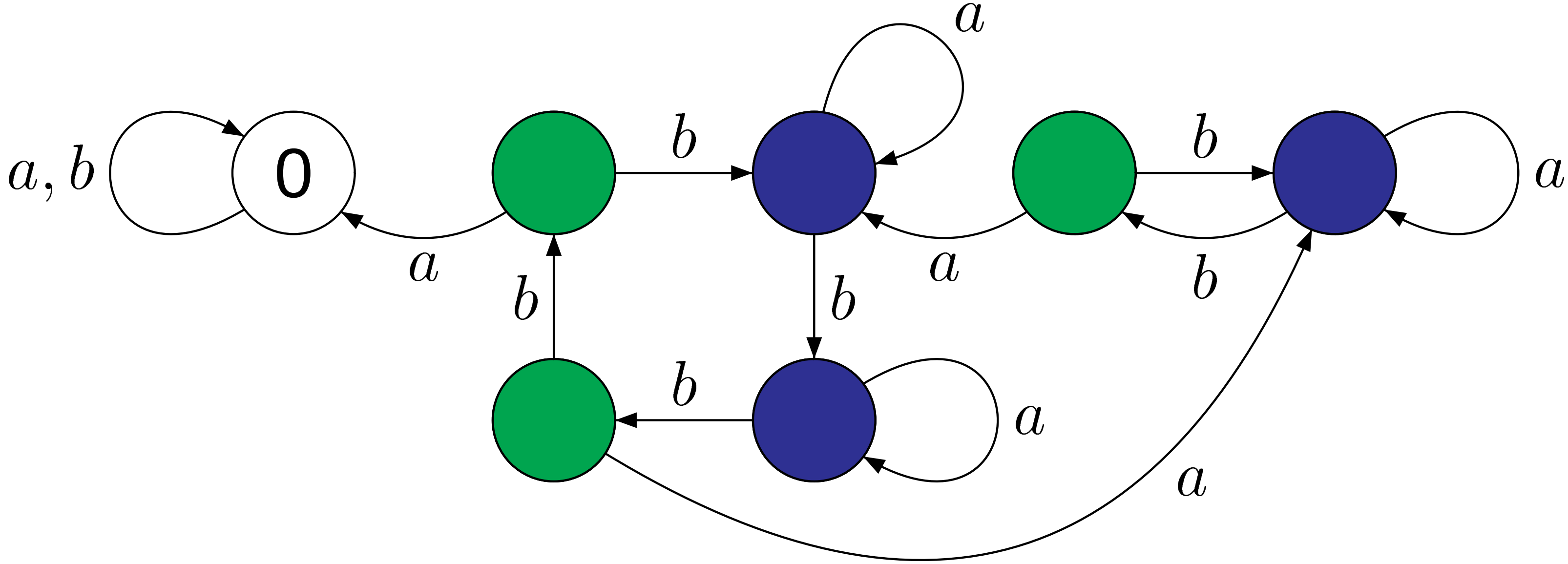

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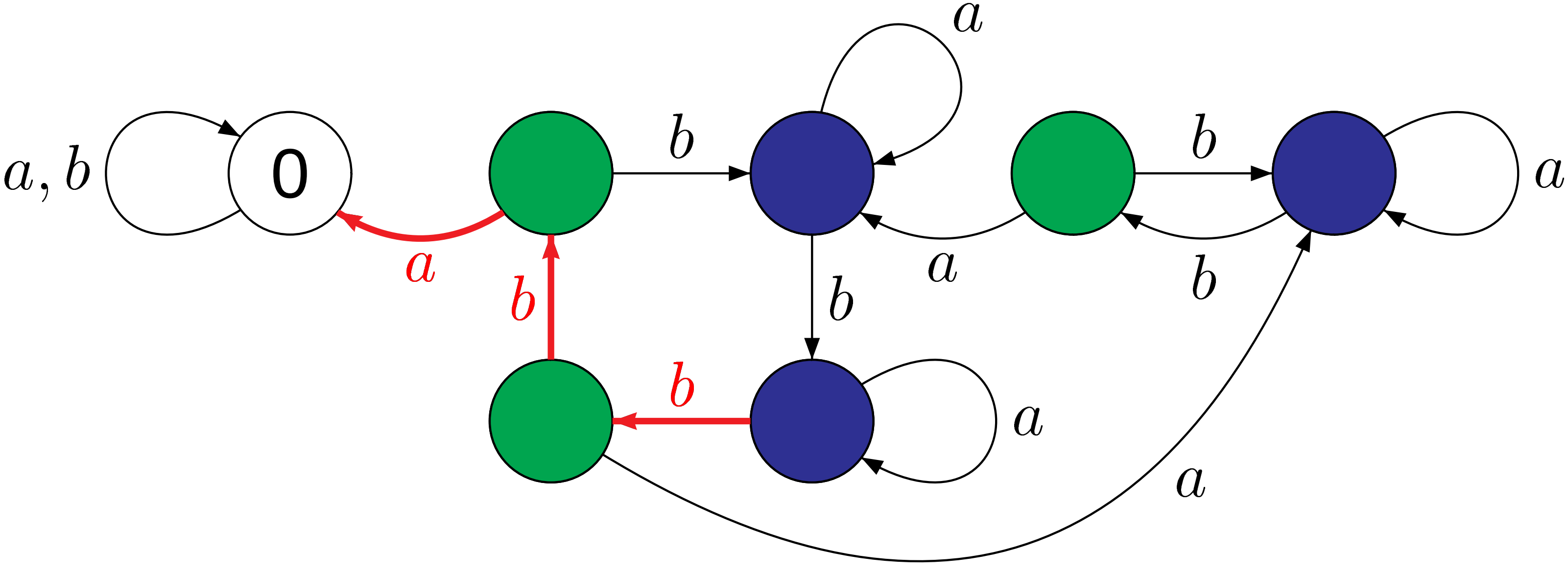

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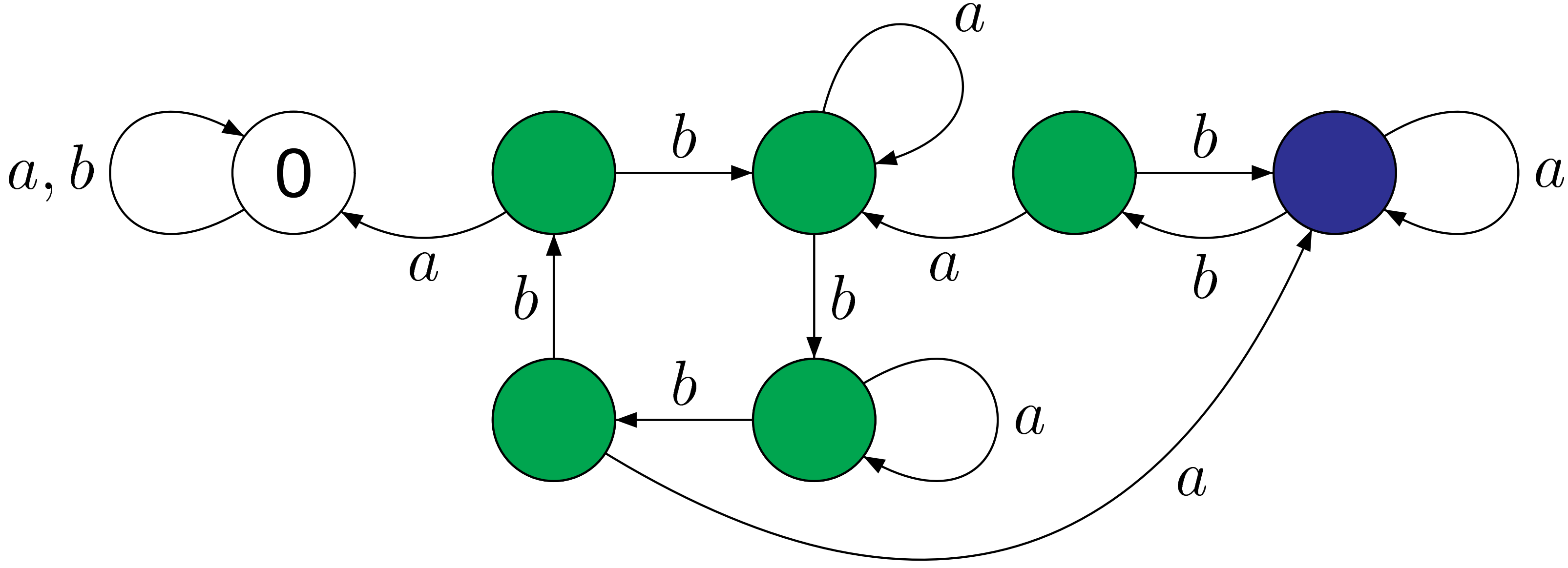

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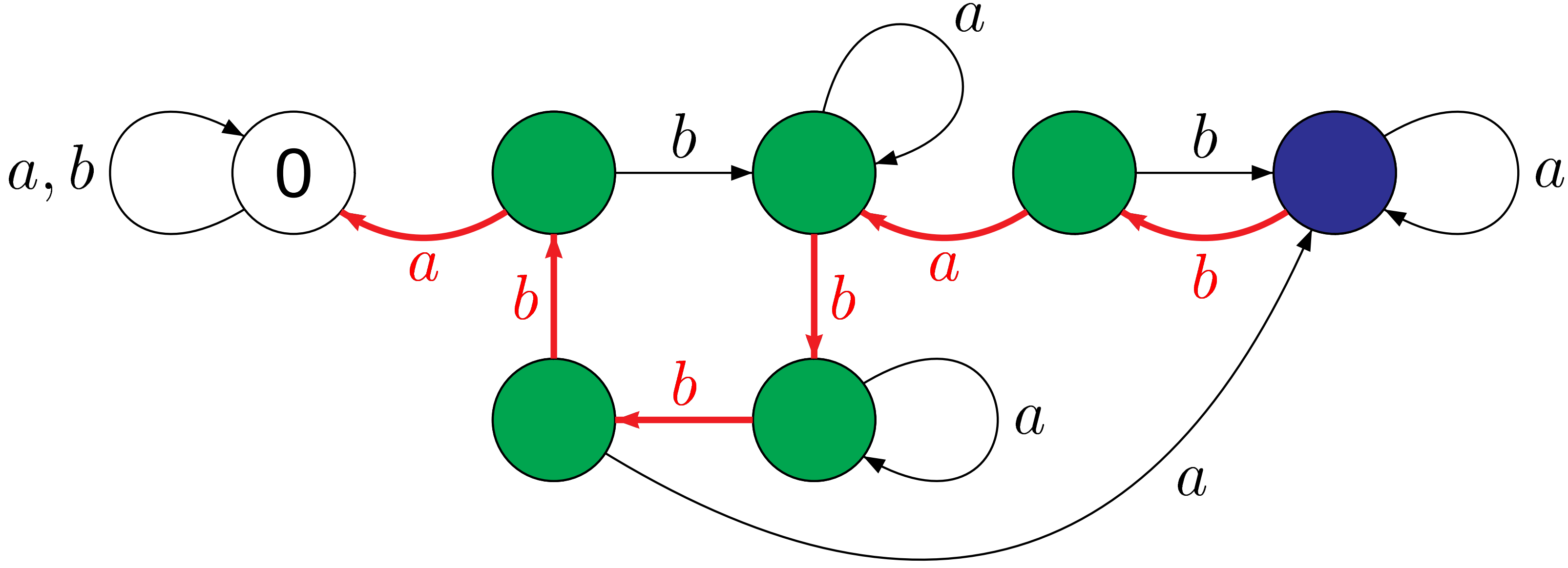

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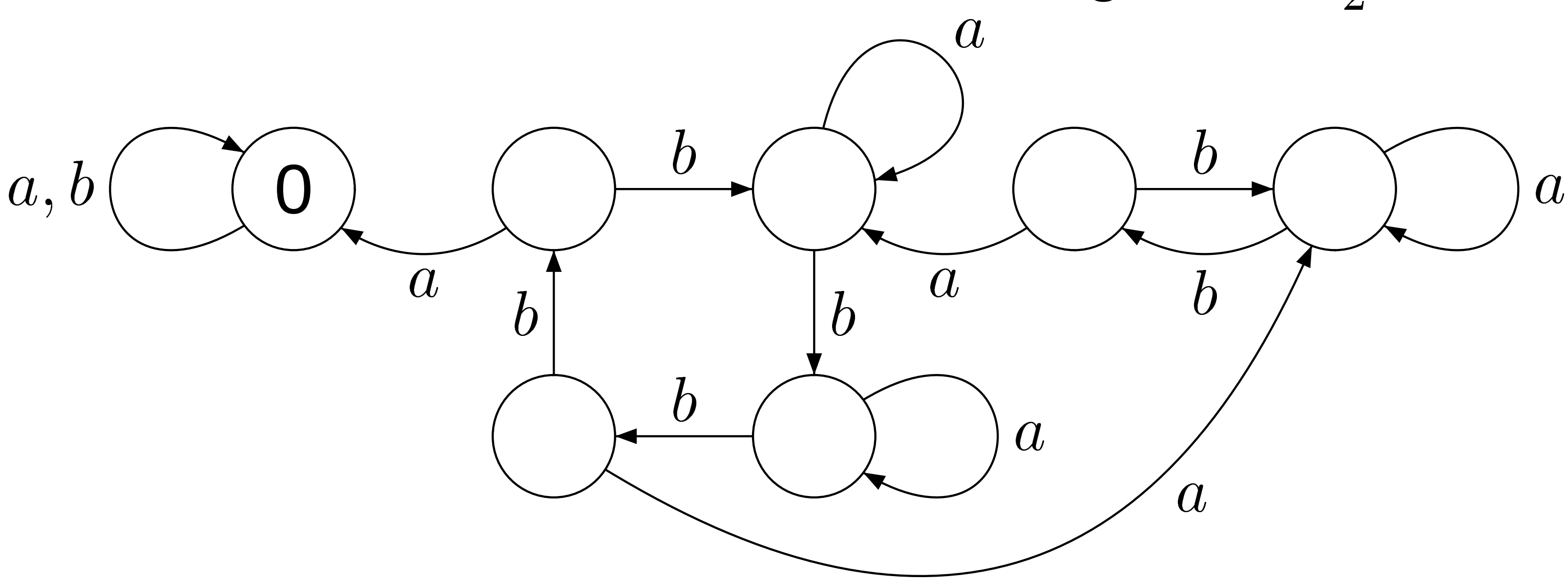

The algorithm makes at most $k-1$ steps and the length of the segment added in the step when t states still holds coins ($k-1 \geq t \geq 1$) is at most $k-t$. The total length is $\leq 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$.

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$$\frac{(n-m+1)(n-m)}{2} + (m-1)^2 \leq (n-1)^2.$$

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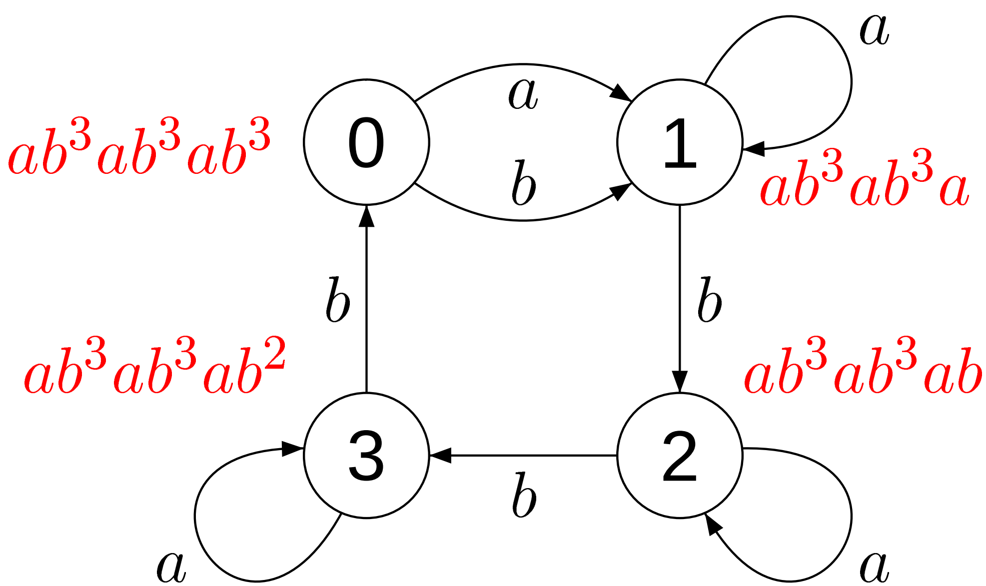

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Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.

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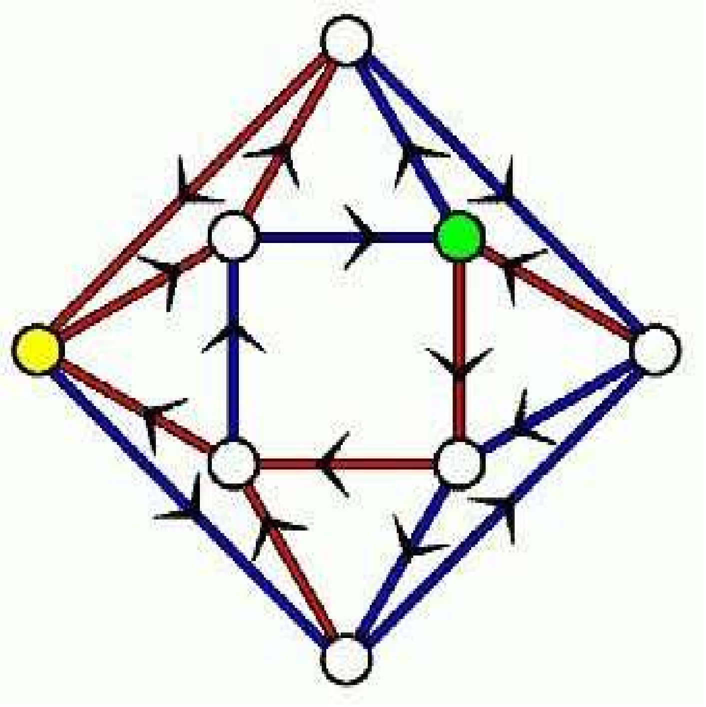

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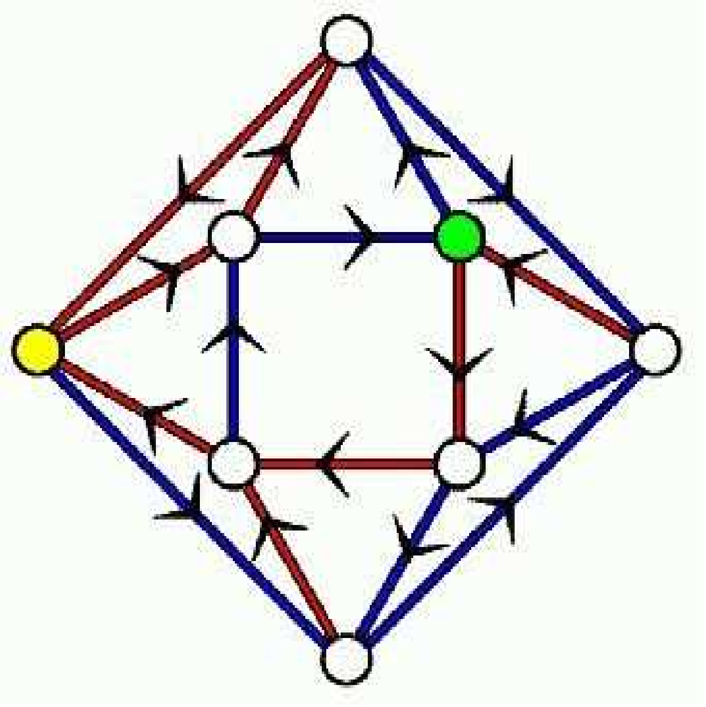

Then for each node there is a sequence of colors that brings one to the chosen node from anywhere.

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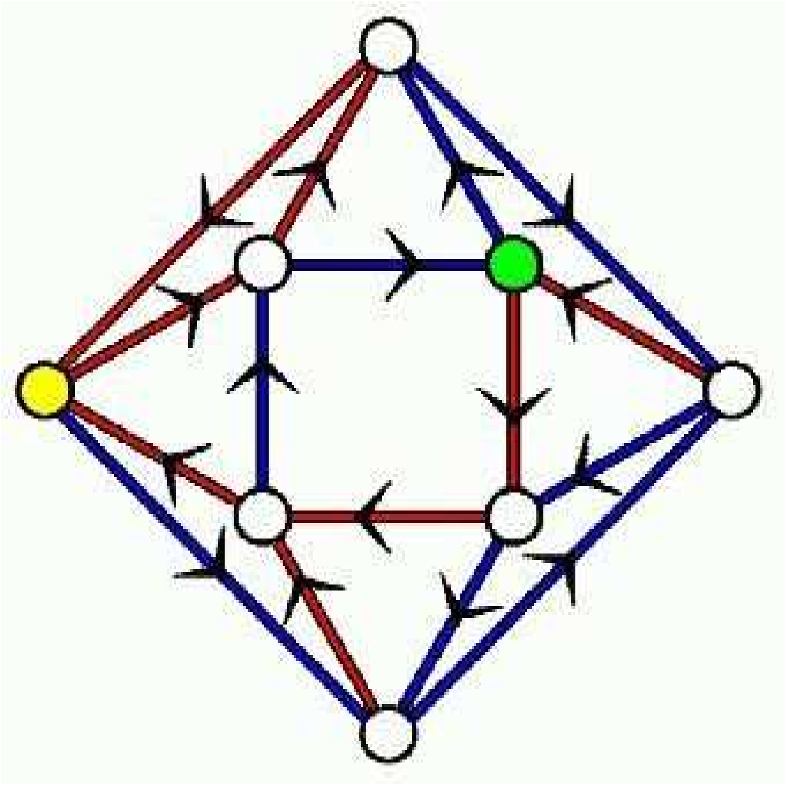

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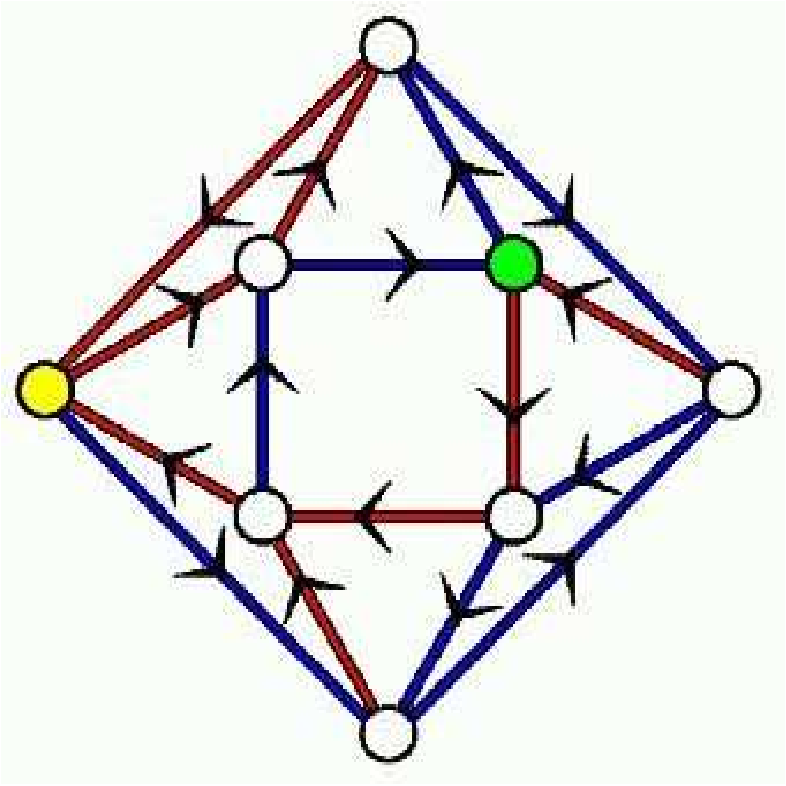

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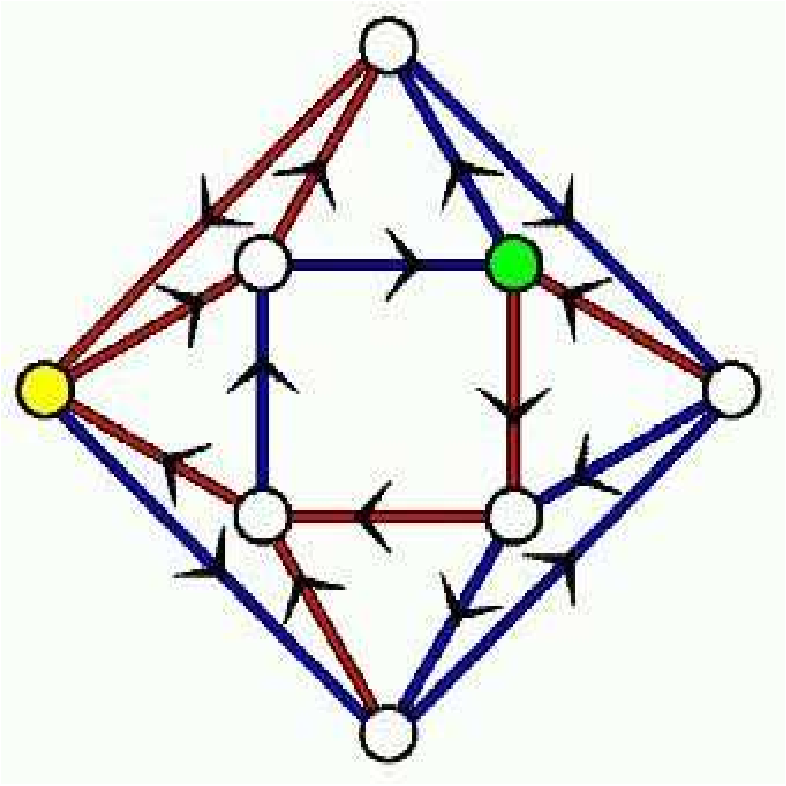

For the green node:
blue-blue-red-blue-blue-red-blue-blue-red.

For the yellow node:
blue-red-red-blue-red-red-blue-red-red.

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An obvious necessary condition:
all vertices should have the same out-degree.
In what follows we refer to this as to the constant out-degree condition.

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$$V_i = \{v \in V \mid \exists \text{ path from } v_0 \text{ to } v \text{ of length } i \pmod{k}\}.$$

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Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

Primitivity

Let $v \in V_i \cap V_j$ where $i \neq j$. This means that in Γ there are two paths from v_0 to v: of length $\ell \equiv i \pmod{k}$ and of length $m \equiv j \pmod{k}$.

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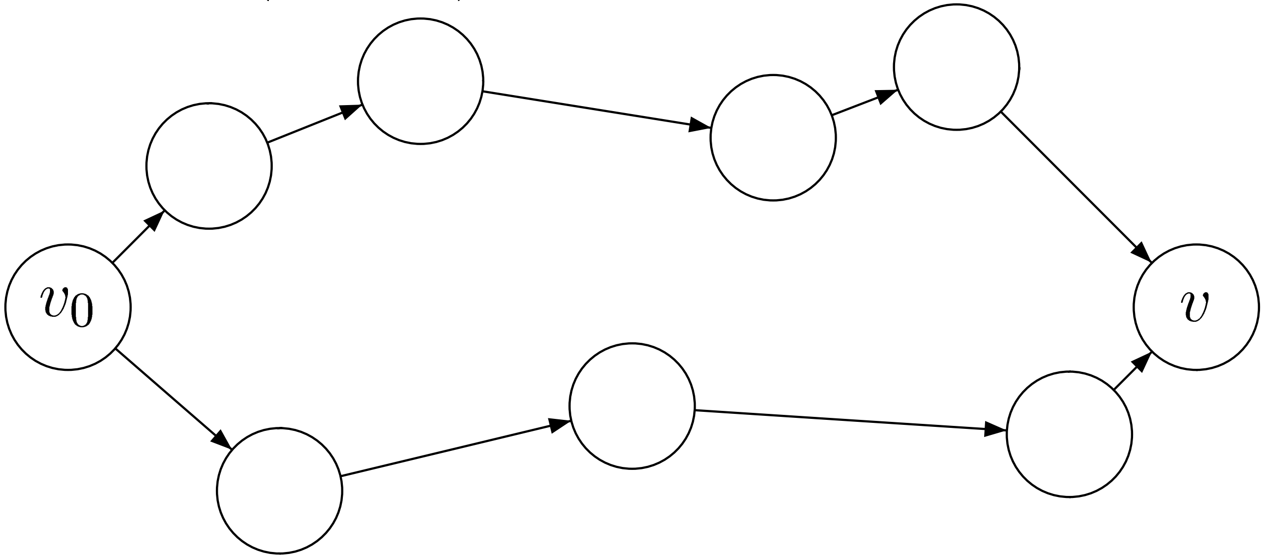

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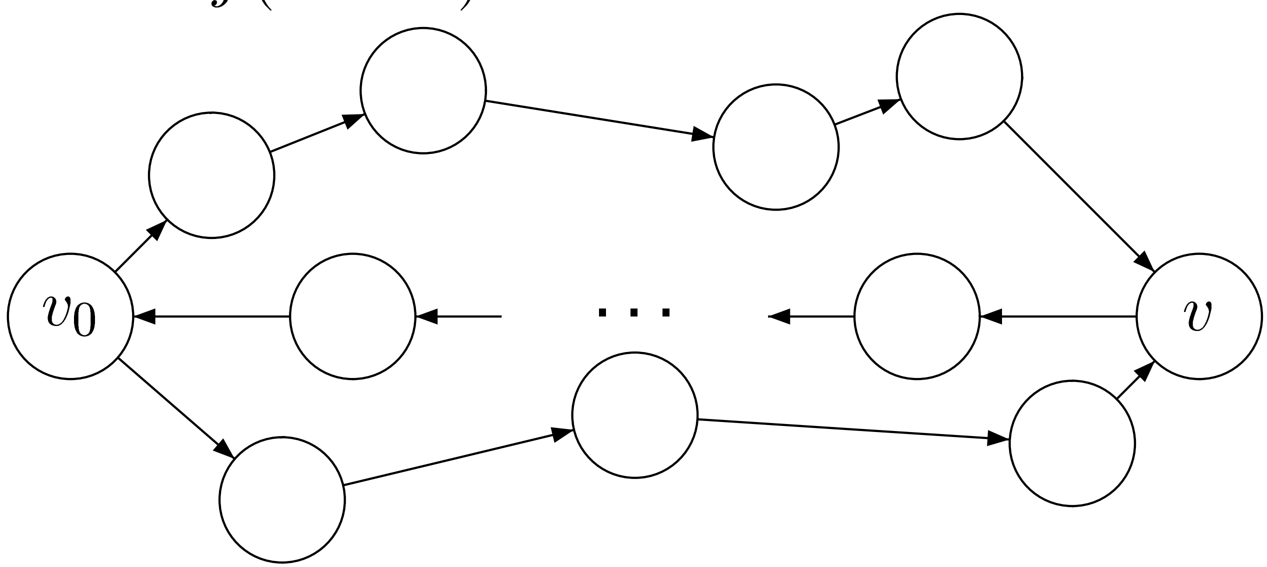

There is also a path v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

Primitivity

Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

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Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Primitivity

Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate in the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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The original motivation for the Road Coloring Conjecture comes from symbolic dynamics, see Marie-Pierre Béal and Dominiues Perrin’s chapter “Symbolic Dynamics and Finite Automata” in Handbook of Formal Languages, Vol.I.

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Stability

Trahtman's proof heavily depends on a neat idea of stability which is due to Karel Culik II, Juhani Karhumäki and Jarkko Kari (A note on synchronized automata and Road Coloring Problem, Int. J. Found. Comput. Sci., 13 (2002) 459–471).

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$\sim$ is called the *stability relation* and any pair (q, q') such that $q \sim q'$ is called *stable*. It is immediate that $\sim$ is a congruence of the automaton $\mathscr{A}$. Also observe that $\mathscr{A}$ is synchronizing iff all pairs are stable.

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We say that a coloring of a digraph with constant out-degree is *stable* if the resulting automaton contains at least one stable pair (q, q') with $q \neq q'$.

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The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/\sim$ admits a synchronizing recoloring by the induction assumption.

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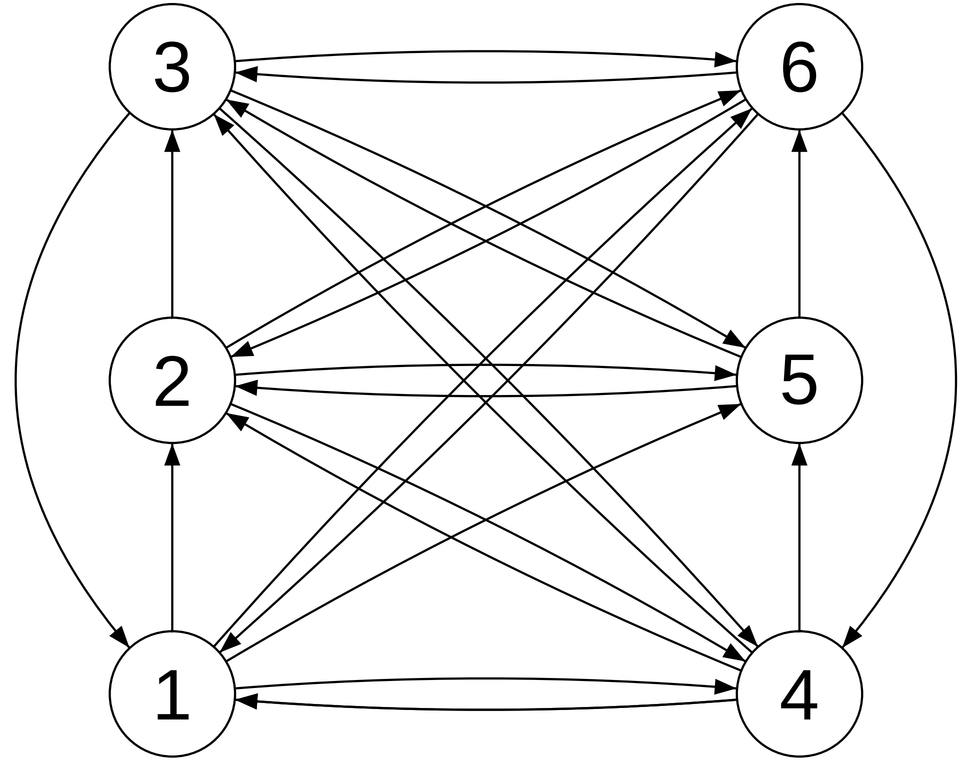

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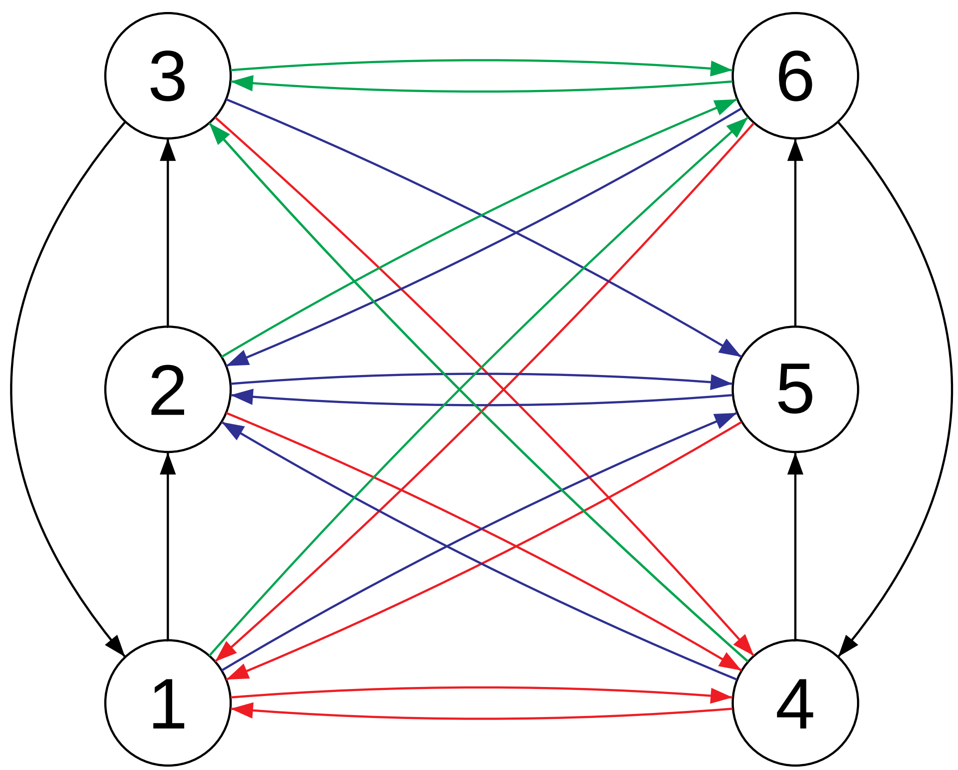

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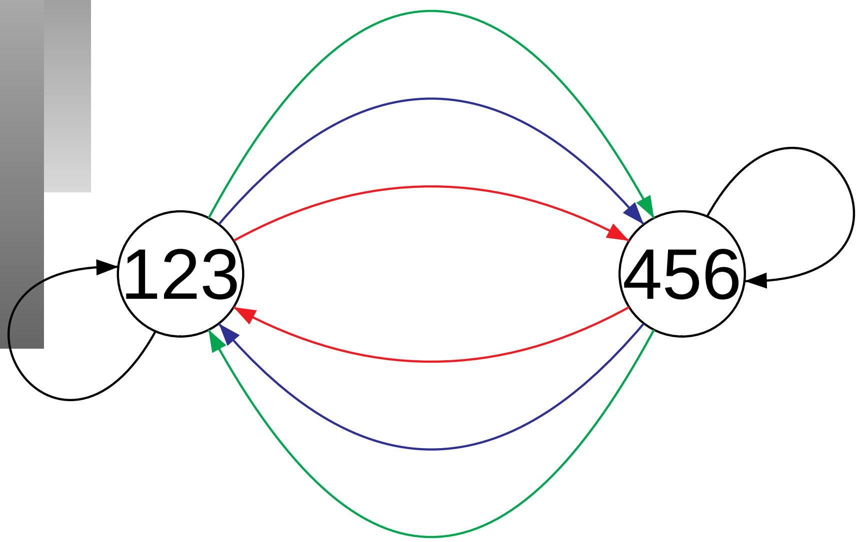

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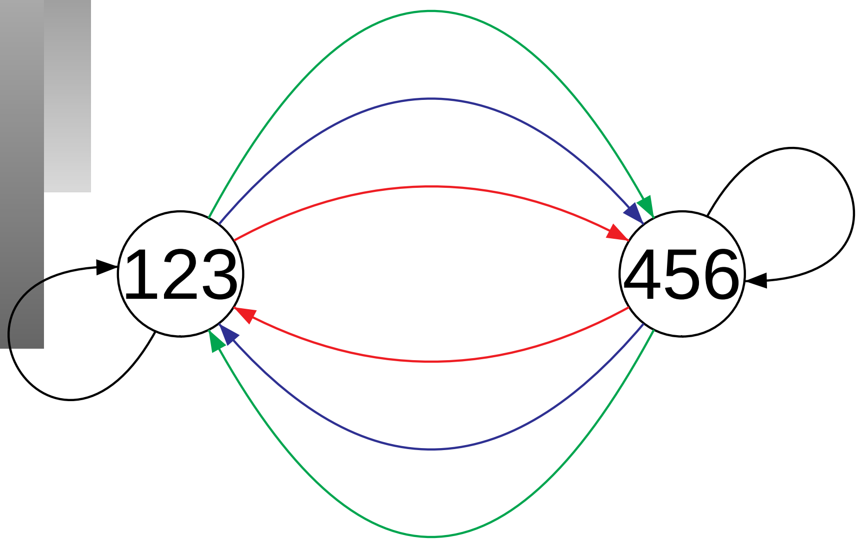

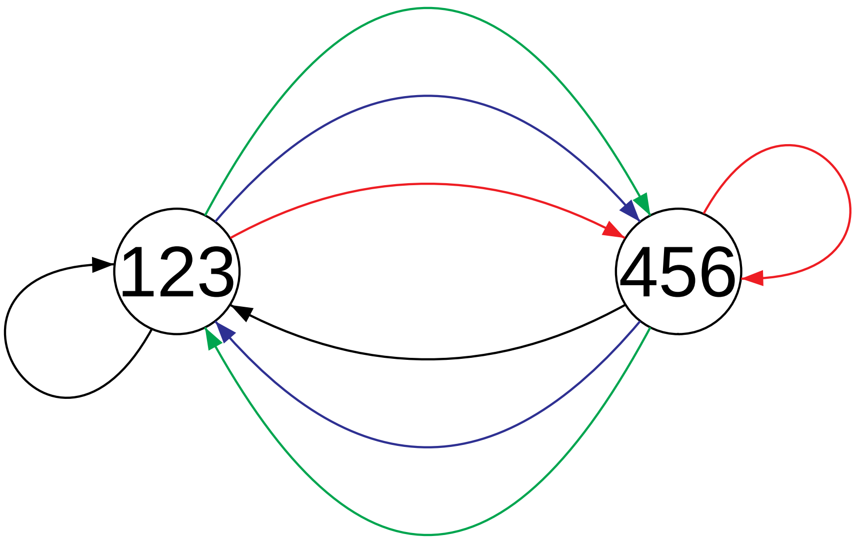

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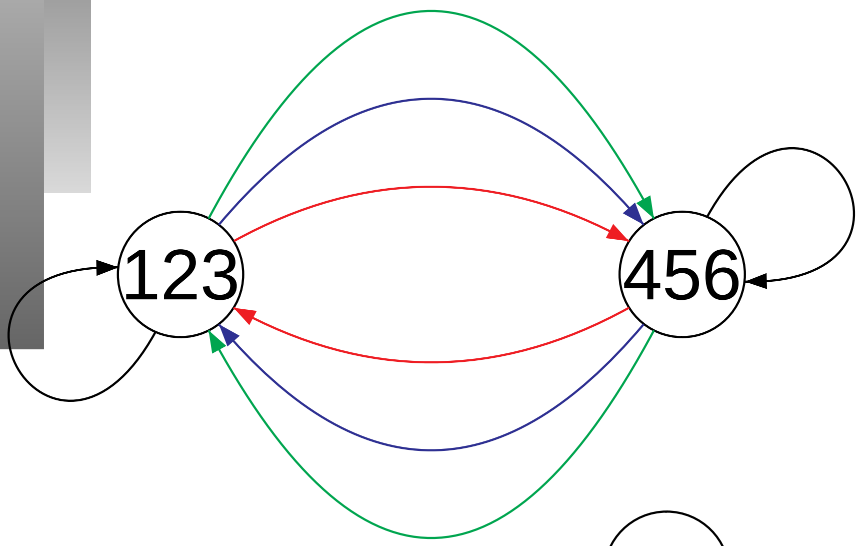

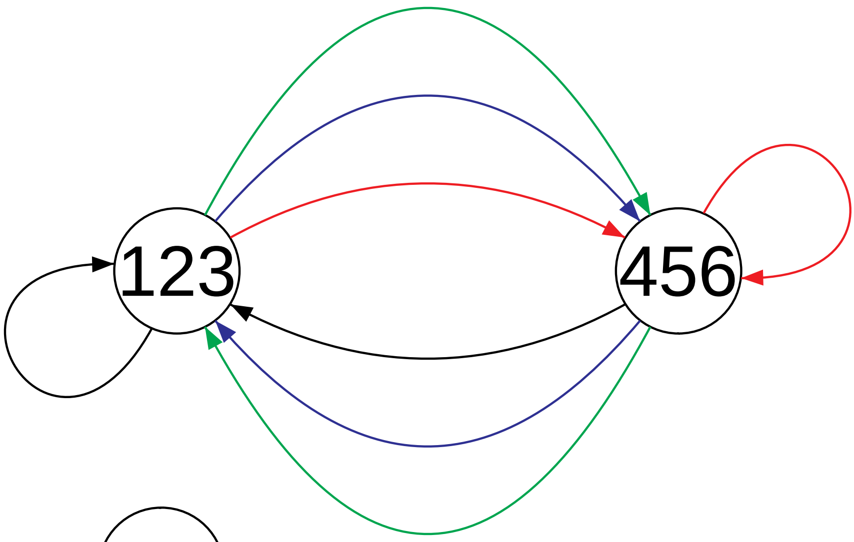

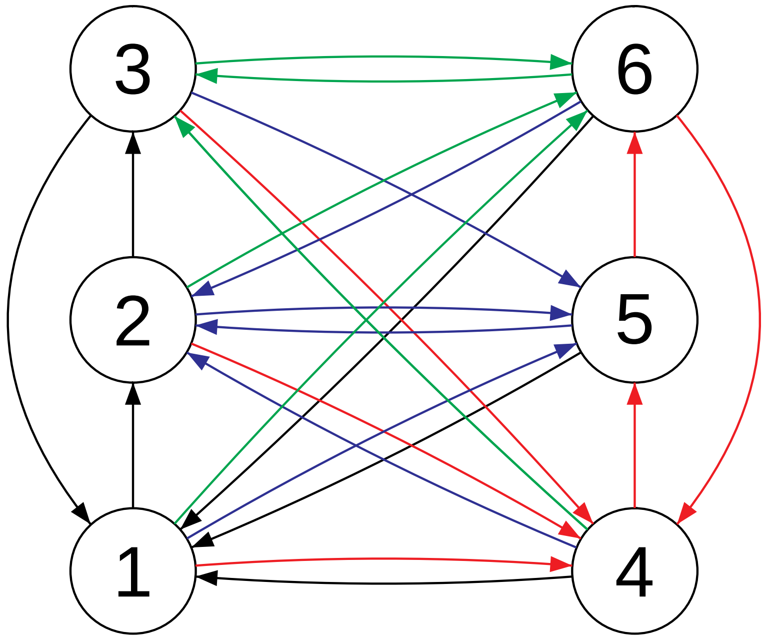

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Trahtman has managed to prove exactly what was needed to use Proposition CKK: every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Thus, Road Coloring Conjecture holds true.

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The proof is not difficult but still a bit too technical for a presentation at the end of our conference.

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- Synchronizing random automata: what is the expectation of the minimum length of reset words for the random automaton with n states? What is the probability distribution of this random variable?

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- The hybrid Černý/Road Coloring problem. Let $\boldsymbol{\Gamma}$ be a strongly connected primitive digraph with constant out-degree and $\boldsymbol{n}$ vertices. What is the minimum length of reset words for synchronizing colorings of $\boldsymbol{\Gamma}$?

For instance, the Černý automata admit synchronizing recolorings with pretty short reset words.

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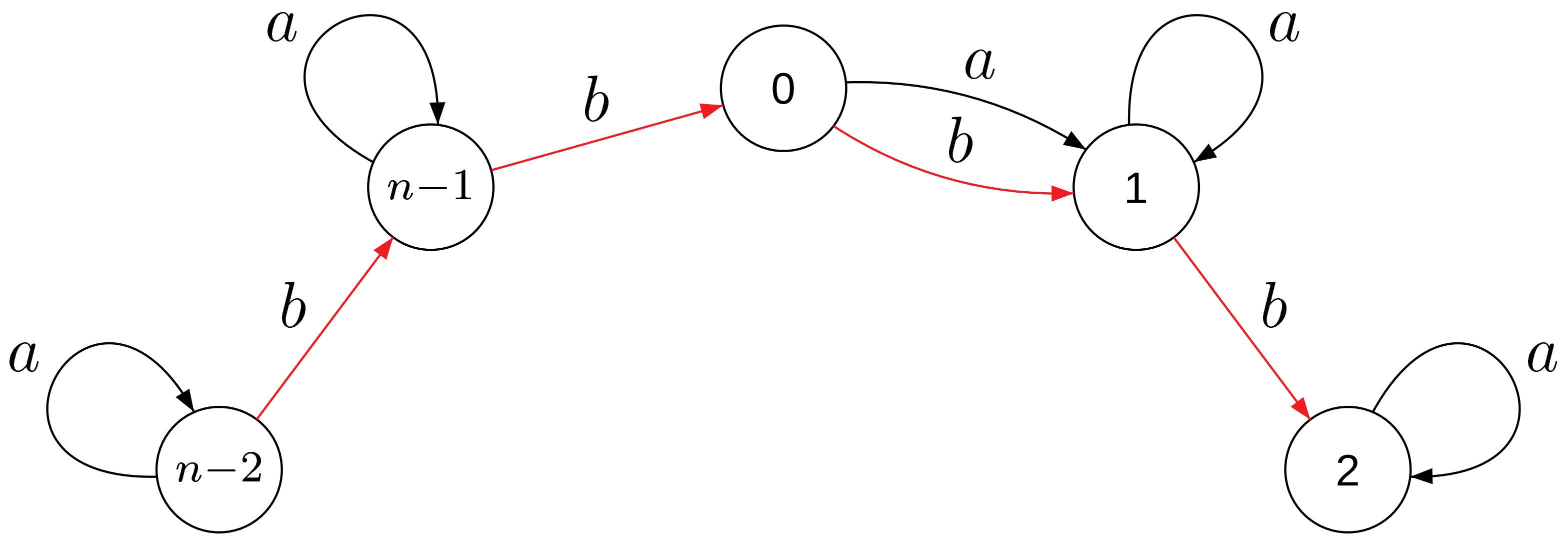

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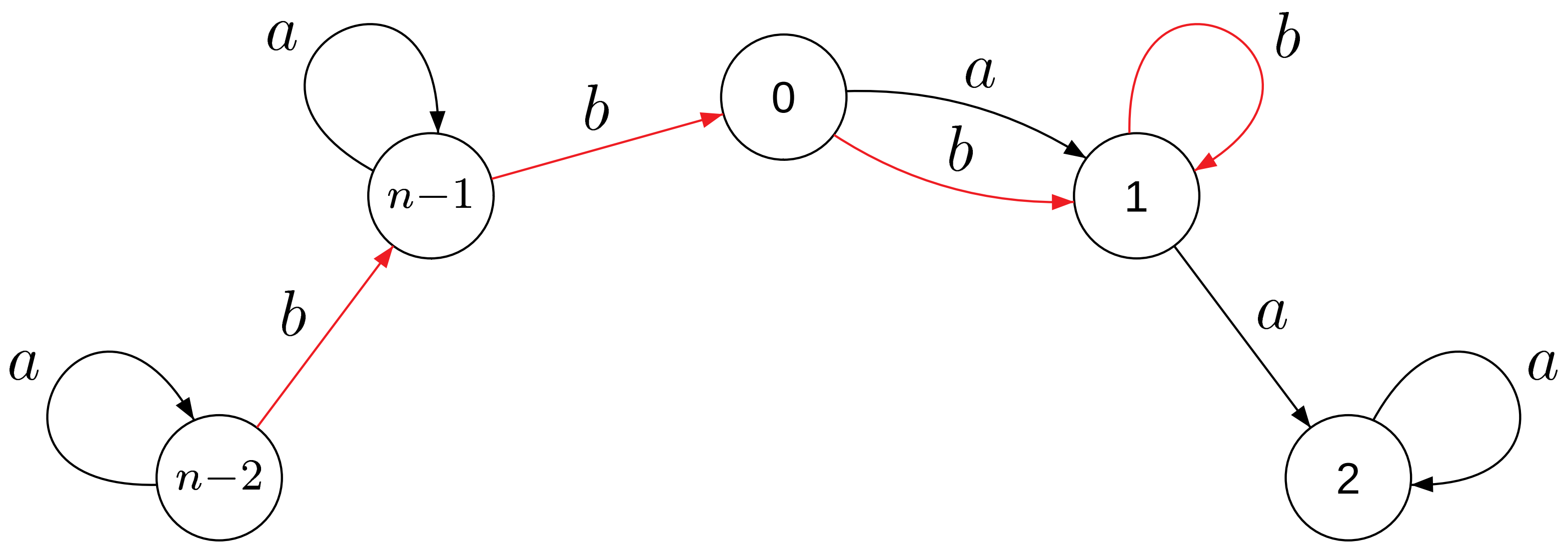

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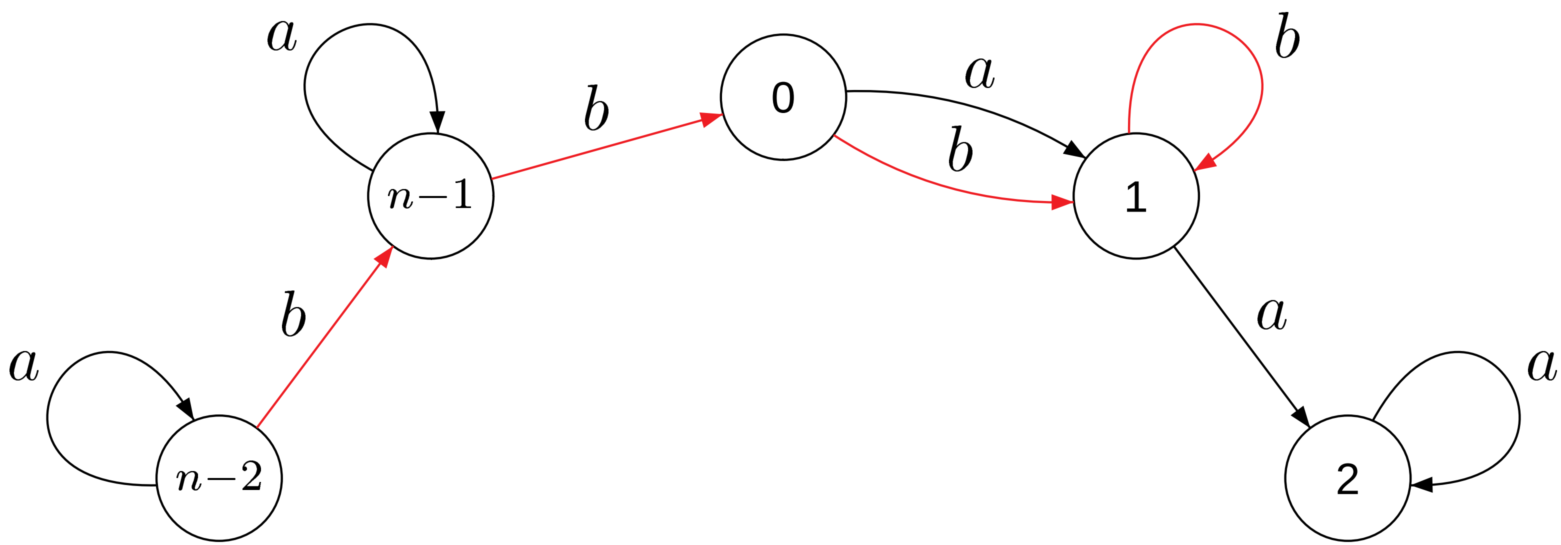

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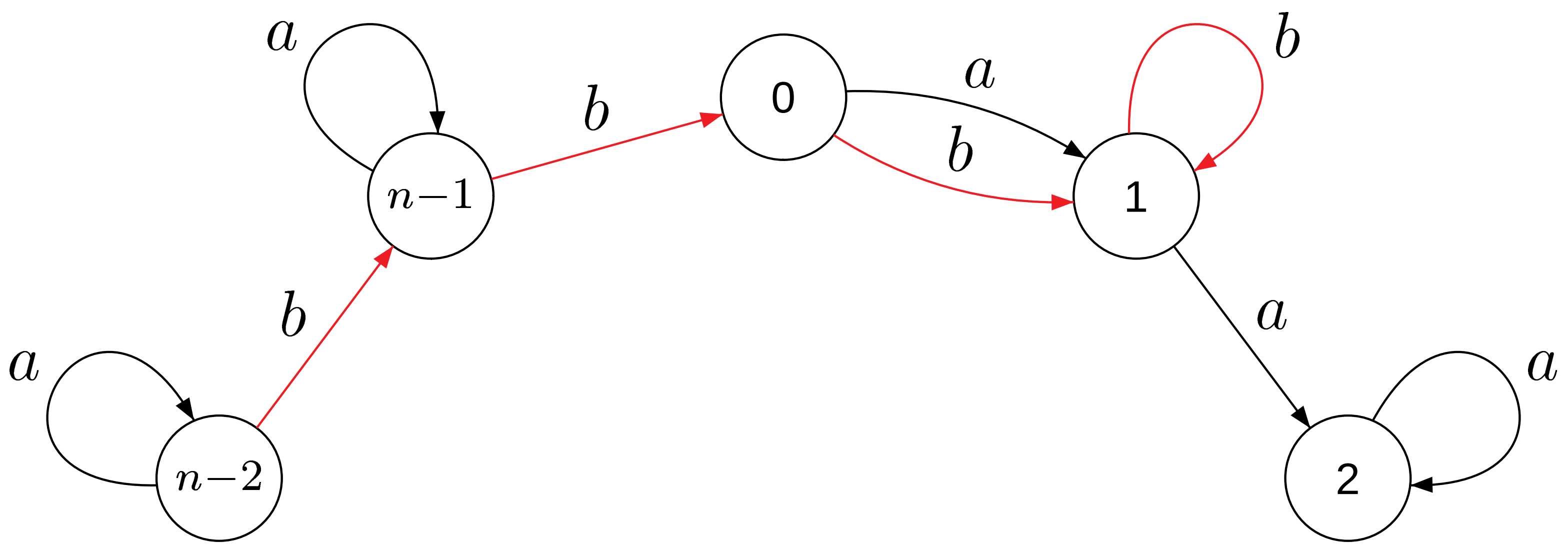

The recolored automaton is reset by the word b^{n-1}.

• Careful Road Coloring Problem. From the viewpoint of transportation network the constant out-degree condition does not seem to be natural. We rather want to find a synchronizing coloring for arbitrary strongly connected primitive digraph Γ, the number of colors being the maximal out-degree of Γ.

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But in the absence of the constant out-degree condition, the resulting automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is incomplete. We need a suitable modification of the notion of a synchronizing automaton for this case.

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In transport network terms this means that following the instruction w is always possible and brings one to the node which is independent of the initial node.

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The Careful Road Coloring Problem asks under which conditions strongly connected digraphs admit carefully synchronizing colorings. Is it true that every primitive strongly connected digraph has such a coloring? (The Careful Road Coloring Conjecture)

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