Interpreting graphs in 0-simple semigroups with reversion

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(joint work with Marcel Jackson, La Trobe University, Australia)

ICGS, May 20, 2009

Definitions and terminology

We consider a graph as a structure $G := \langle V; \sim \rangle$, where V is a set and $\sim \subseteq V \times V$ is a binary relation.

In other words, we consider directed graphs and do not allow multiple edges.

The adjacency matrix P_G of a graph $G = \langle V; \sim \rangle$ is a $V \times V$-matrix given by $P_G(x, y) = 1$ if $x \sim y$ and 0 otherwise.

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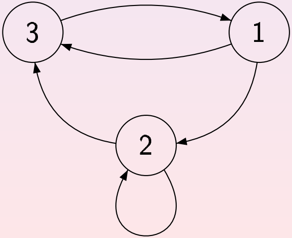

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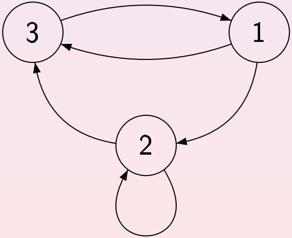

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

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Graph Algebras

Since graphs form a universal language of discrete mathematics, the idea to relate graphs and algebras appear to be natural.

A well known approach: graph algebras

Given a graph $G = \langle V; \sim \rangle$, its graph algebra is a groupoid on the carrier set $V \cup \{\infty\}$ with the multiplication defined by

$$xy = \begin{cases} x & \text{if } \ x, y \in V \ \text{ and } \ x \sim y, \\ \infty & \text{otherwise.} \end{cases}$$

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Given a graph $G = \langle V; \sim \rangle$, its adjacency semigroup is defined on the set $(V \times V) \cup \{\mathbf{0}\}$ and the multiplication rule is

$$(x, y)(z, t) = \begin{cases} (x, t) & \text{if } y \sim z, \\ \mathbf{0} & \text{if } y \nsim z; \end{cases}$$

$$a\mathbf{0} = \mathbf{0}a = \mathbf{0} \quad \text{for all } a \in A(G).$$

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Reversion

Yes, to some extent.

Example: $A(G)$ is 0-simple iff every vertex of G has nonzero indegree and nonzero outdegree.

In general, however, connections between $A(G)$ and G seem to be far too weak in order to be useful.

New idea: to equip $A(G)$ with an additional unary operation (reversion):

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Interpreting Graphs

- $x \sim x$ (reflexivity of G) is equivalent to $A(G) \models XX'X = X$;

Natural graph properties correspond to natural semigroup properties. We encounter identities on the semigroup side. What is the syntactic nature of conditions appearing at the graph side?

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Universal Horn Classes

A universal Horn sentence is a sentence of one of the two forms:

$$(\forall x_1 \forall x_2 \ldots)\left(\left(\mathop{\&}_{1\le i\le n} \Phi_i\right) \to \Phi_0\right),$$

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where the Φ_i are of the form $x_j \sim x_k$ or $x_j = x_k$.

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- Preorders (reflexivity + transitivity)

More about the last example. The following graph Co_3 is 3-colorable but not 2-colorable.

Co_3 is a generator of minimum size for the uH-class of all 3-colorable graphs (Nešetřil and Pultr, 1978).

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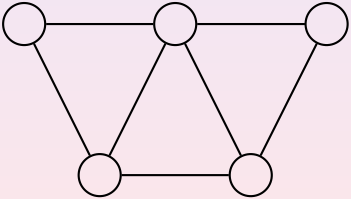

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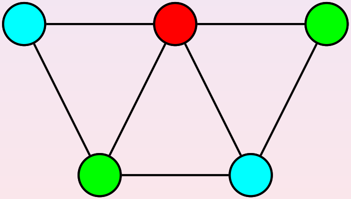

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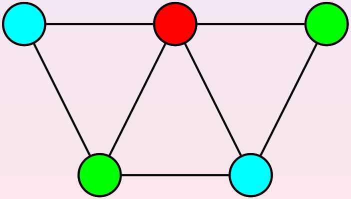

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Embedding

Theorem

The assignment $G \mapsto A(G)$ induces an injective order-preserving map from the lattice of all uH-classes of graphs to the subvariety lattice of the variety $\mathcal{A}$ generated by all adjacency semigroups.

Thus, the equational logic of rather a transparent class of unary semigroups interprets the (universal Horn) theory of graphs.

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Theorem

Let K be any nonempty class of graphs and let G be a graph. G belongs to the uH-class generated by K iff $A(G)$ belongs to the variety generated by the semigroups $A(H)$ with $H \in K$.

The Variety Membership Problem

The theorem in the above strong form has interesting application to the complexity of the variety membership problem for unary semigroups.

Let A be a finite algebra. $\textsc{Var-Memb}(A)$ is the decision problem whose instance is a finite algebra B of the same similarity type and whose question is: does B belong to the variety generated by A?

$\textsc{Var-Memb}(A)$ is decidable. An easy consequence of the HSP-theorem: $B \in \operatorname{var} A$ iff B is a homomorphic image of the free $|B|$-generated algebra of $\operatorname{var} A$ and the free algebra has at most $|A|^{(|A|^{|B|})}$ elements. But the resulting algorithm requires doubly exponential time (as a function of $|B|$). Can we do better?

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Now take the adjacency matrix of Co_3 and construct the 26-element adjacency semigroup $A(Co_3)$. The reversion operation on $A(Co_3)$ is an involution since the graph Co_3 is symmetric.

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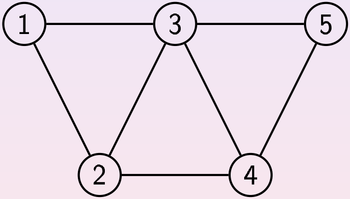

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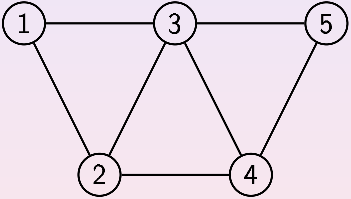

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Some other natural problems about the adjacency semigroup $A(Co_3)$ also have high complexity. For instance, checking identities in $A(Co_3)$ is co-NP-complete.

Further Applications

If K is a class of graphs without a finite basis of uH-sentences, then var $A(K)$ is without a finite basis of identities. If K is a class of graphs whose uH-class has infinitely many (uncountably) many sub-uH-classes, then the variety var $A(K)$ has infinitely many (uncountably many) subvarieties.

For instance, consider the graph S_2 and the graph K_2:

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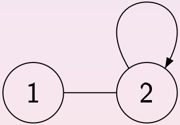

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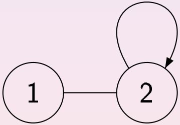

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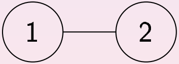

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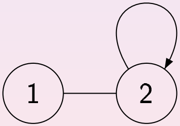

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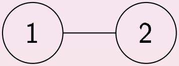

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Reflexive Case

Recall that a graph G is reflexive (has a loop at each vertex) iff the adjacency semigroup $A(G)$ satisfies $XX'X = X$. When restricted to reflexive graphs, our map becomes a "nearly" isomorphism.

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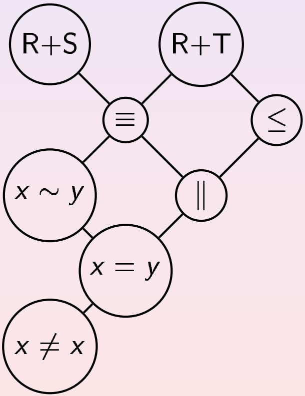

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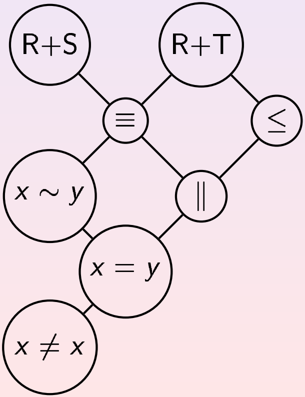

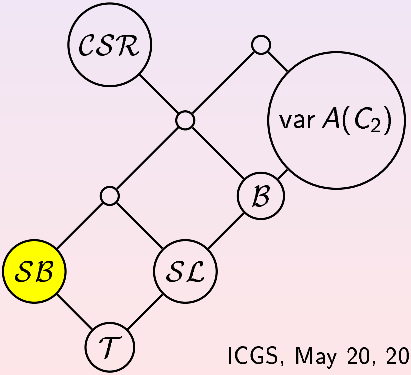

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Here C_2 is the 2-element chain. Compare with S_2 introduced earlier.

$A(C_2)$ and $A(S_2)$ have the same underlying 5-element semigroup (usually denoted by A_2). However, var $A(C_2)$ has only 4 subvarieties while var $A(S_2)$ has uncountably many.

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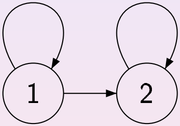

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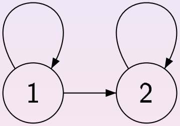

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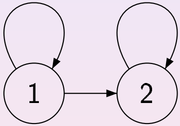

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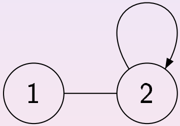

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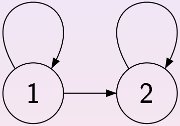

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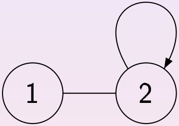

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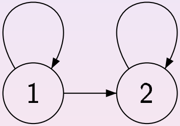

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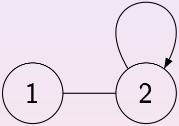

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$$X'' = X, \tag{2}$$

$$X(YZ)' = (Y(XZ')')', \tag{3}$$

$$(XY)'Z = ((X'Z)'Y)', \tag{4}$$

$$(XX')' = XX', \tag{5}$$

$$X'YXZX = (XZX)'YXZX, \tag{6}$$

$$XYXZX' = XYXZ(XYX)', \tag{7}$$

$$XX'X = X. \tag{8}$$

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In particular, a subvariety of the variety generated by adjacency semigroups satisfying $XX'X = X$ is finitely based (finitely generated as a variety) iff it corresponds to the inserted element or to a finitely axiomatized (finitely generated, respectively) uH-class of reflexive graphs.

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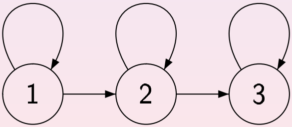

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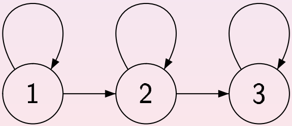

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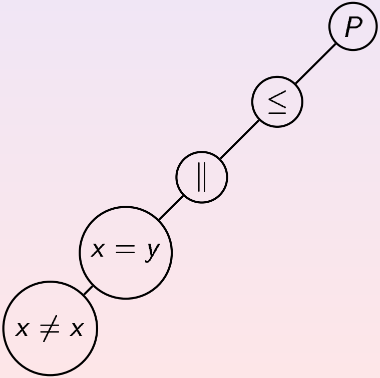

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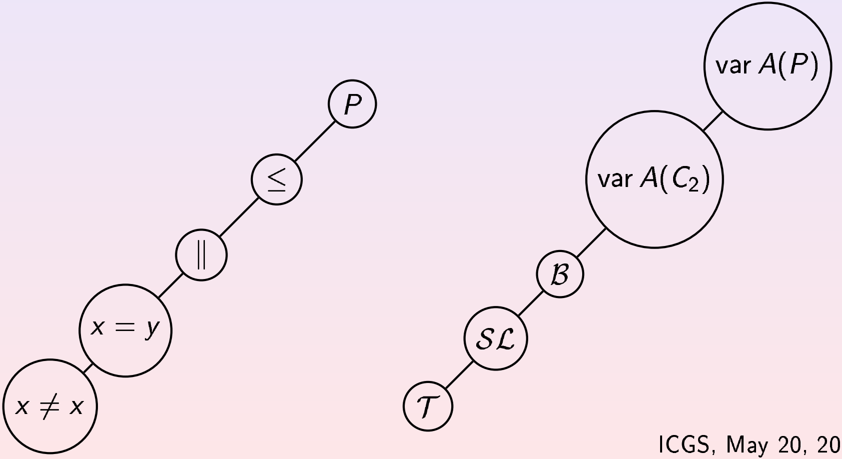

Finite Basis Problem for Adjacency Semigroups

Problem

Is the variety $\mathcal{A}$ generated by all adjacency semigroups finitely based?

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Do the identities (1)–(7)

$$x^3 = x^2,\ xyx = xyxyx,\ xyxzx = xzxyx, \tag{1}$$

$$x'' = x, \tag{2}$$

$$x(yz)' = (y(xz')')', \tag{3}$$

$$(xy)'z = ((x'z)'y)', \tag{4}$$

$$(xx')' = xx', \tag{5}$$

$$x'yxzx = (xzx)'yxzx, \tag{6}$$

$$xyxzx' = xyxz(xyx)'. \tag{7}$$

form a basis for $\mathcal{A}$?

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Lattice of uH-Classes of Reflexive Graphs

Problem

Is the lattice of uH-classes of reflexive graphs/the subvariety lattice of $\mathcal{A}_{\mathrm{ref}}$ countable?

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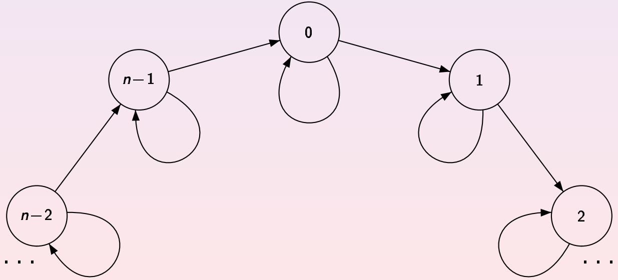

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