Regular Semigroups Beyond Regular Varieties

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The problem: is $\mathbf{A} \vee \mathbf{G}$ decidable?

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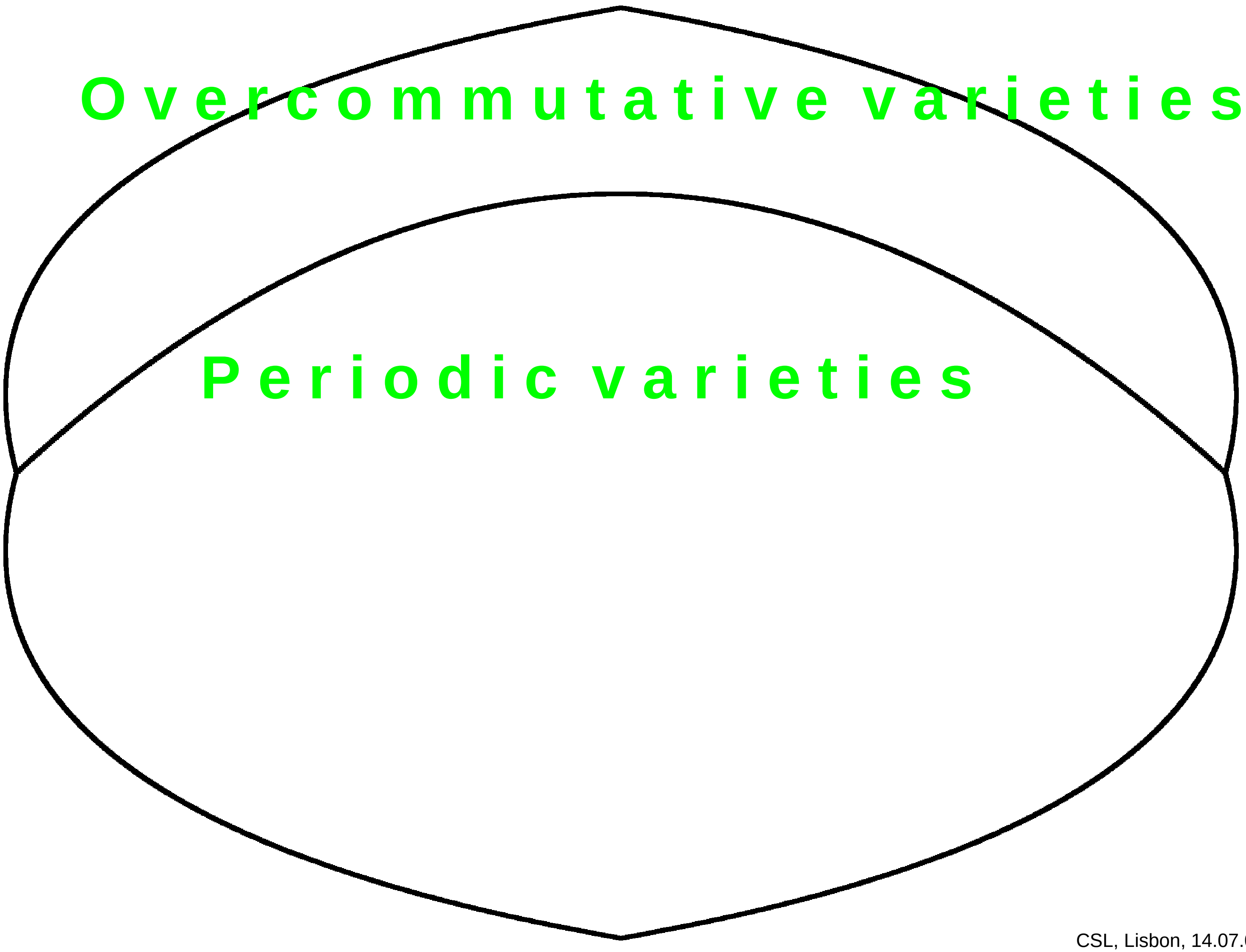
The problem therefore reduces to understanding the relationship between regular and non-regular members of $\mathbf{A} \vee \mathbf{G}$.

Semigroup Varieties: An Overview



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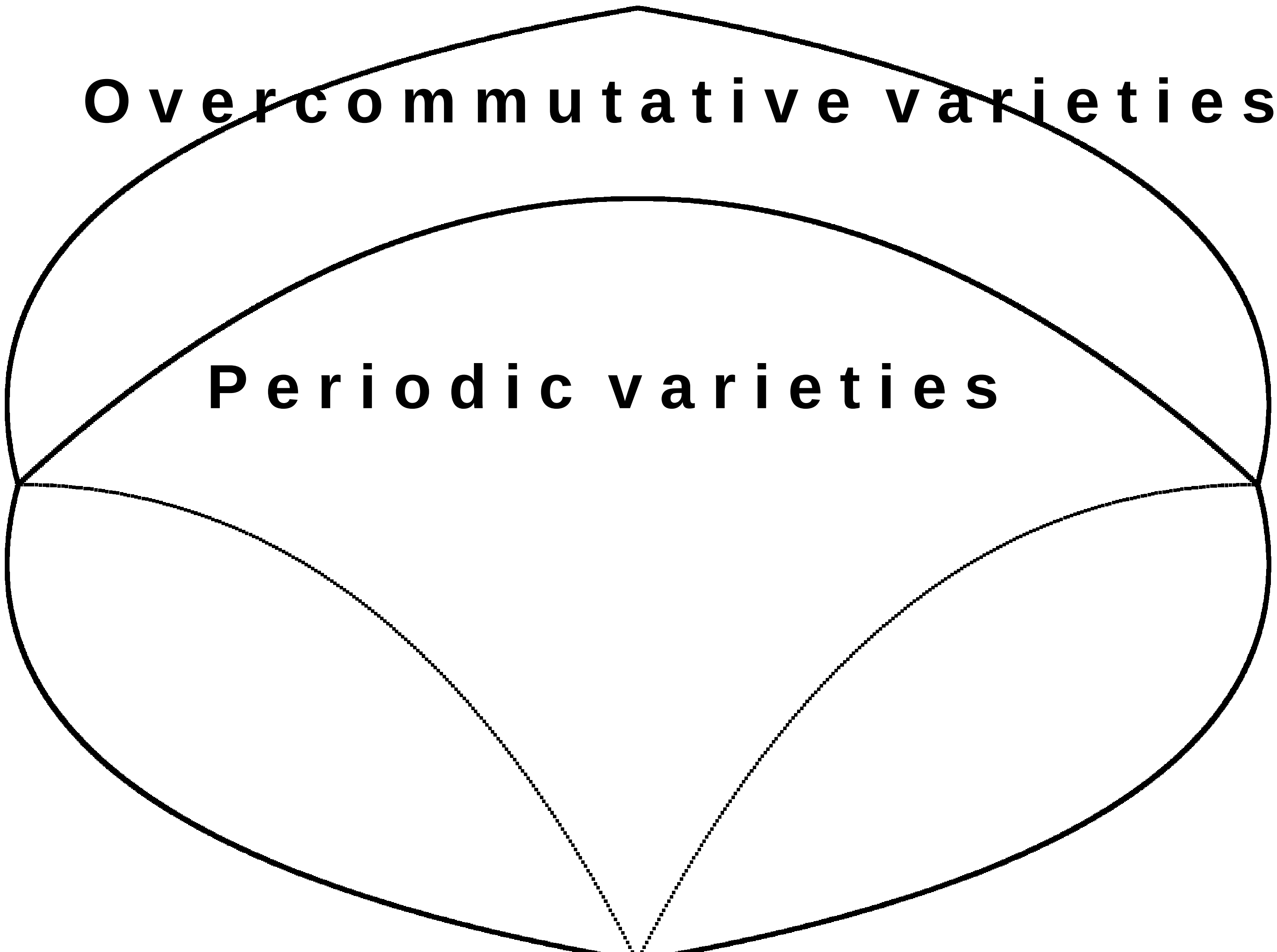

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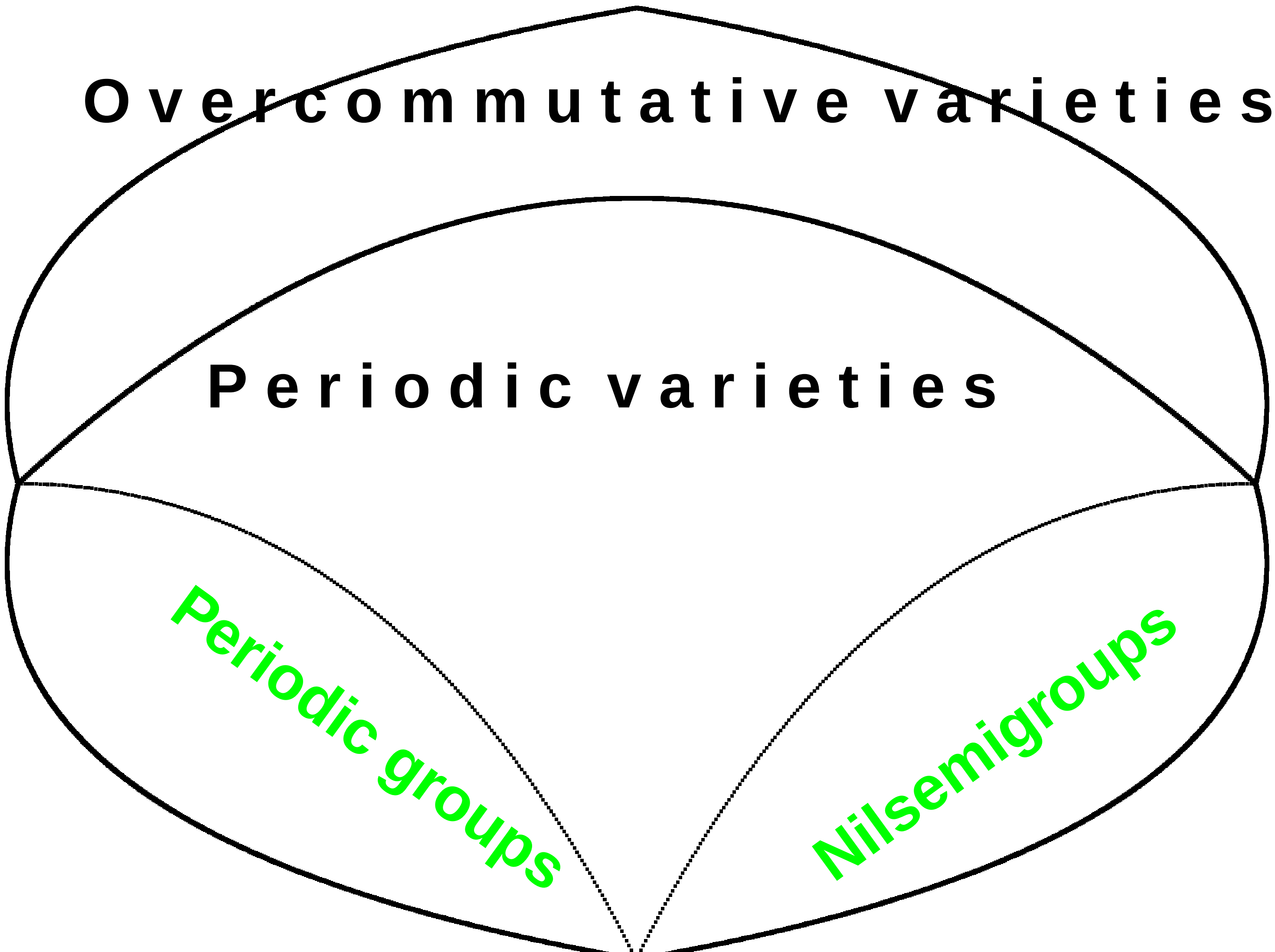

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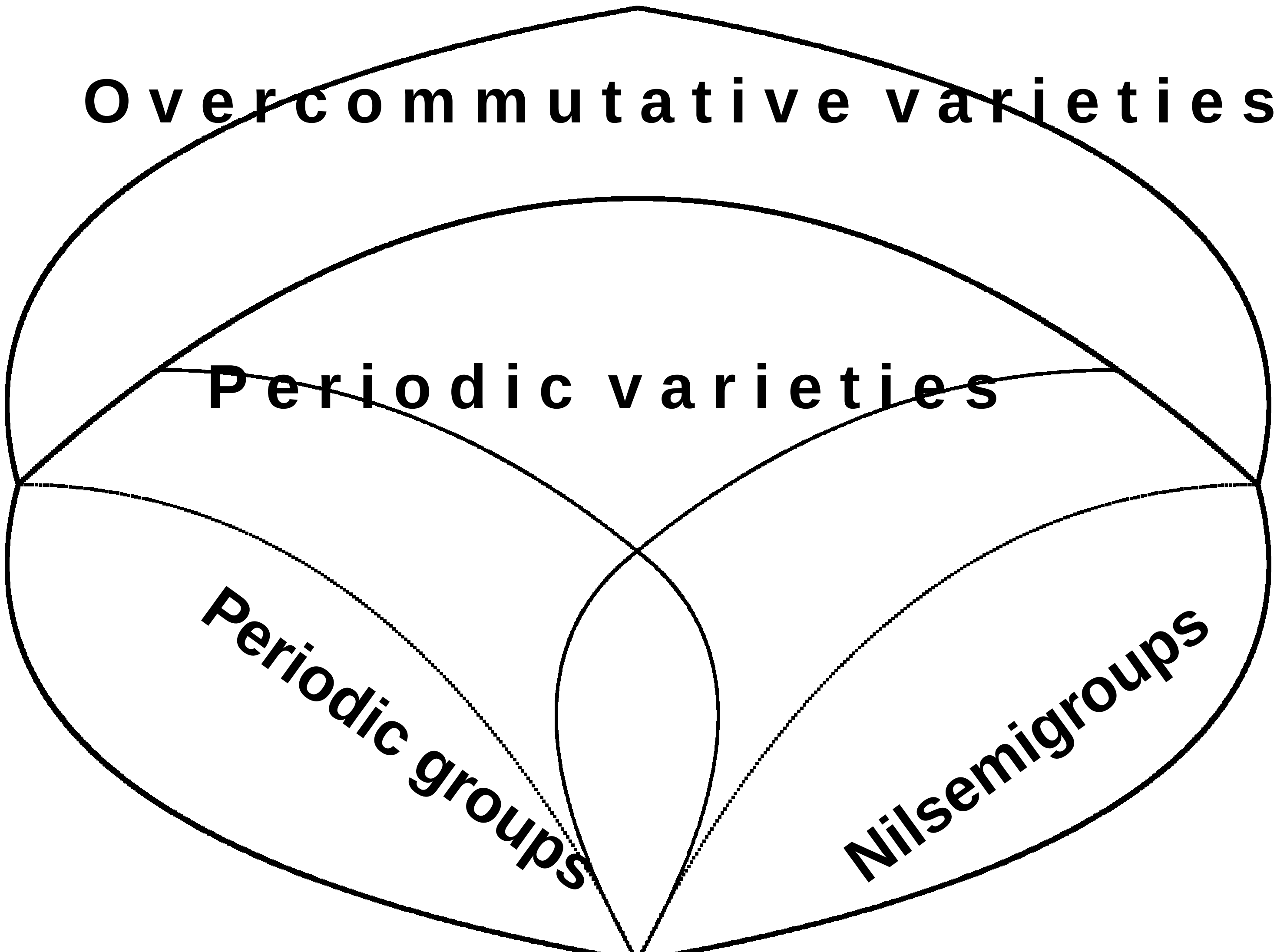

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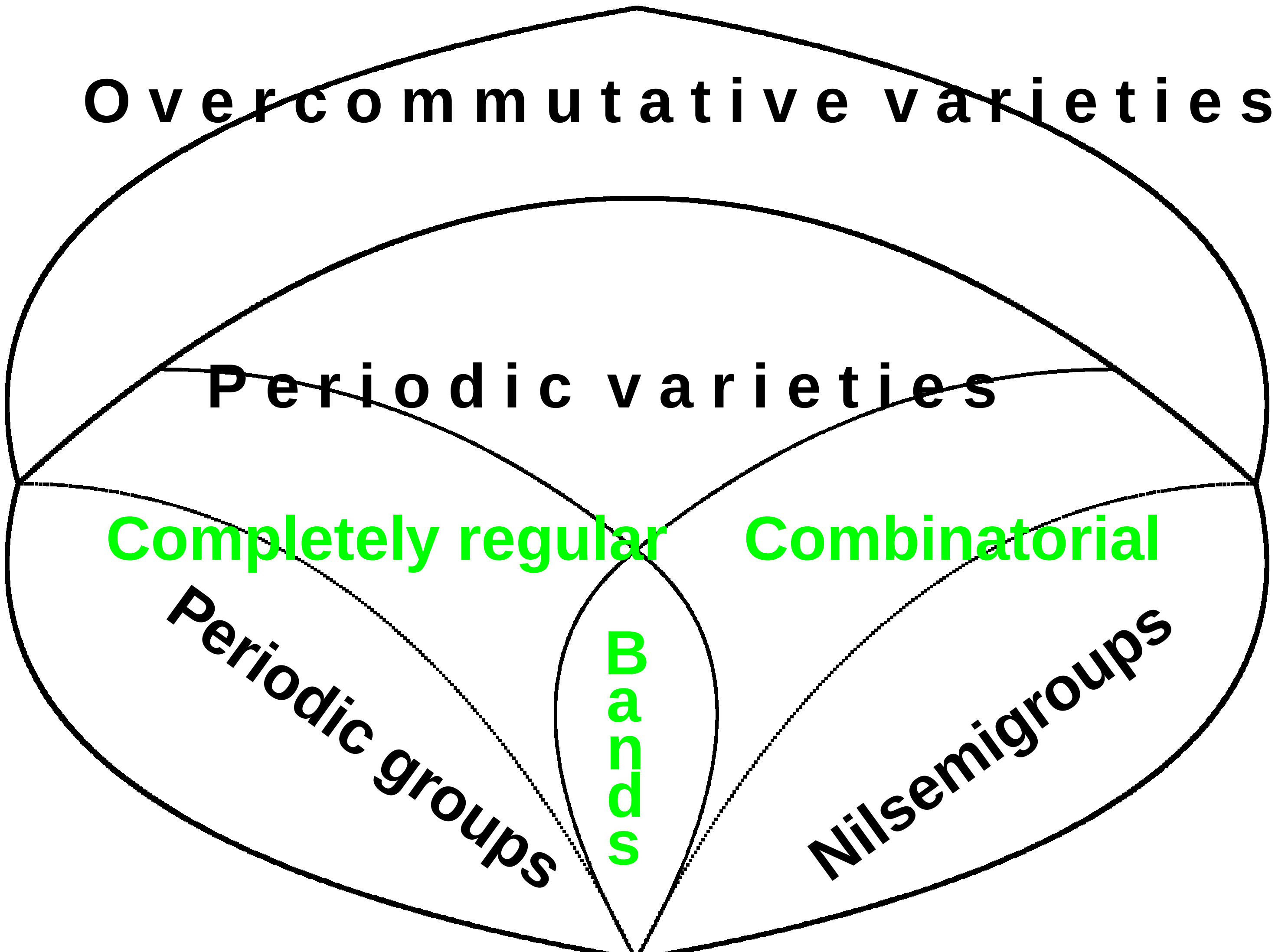

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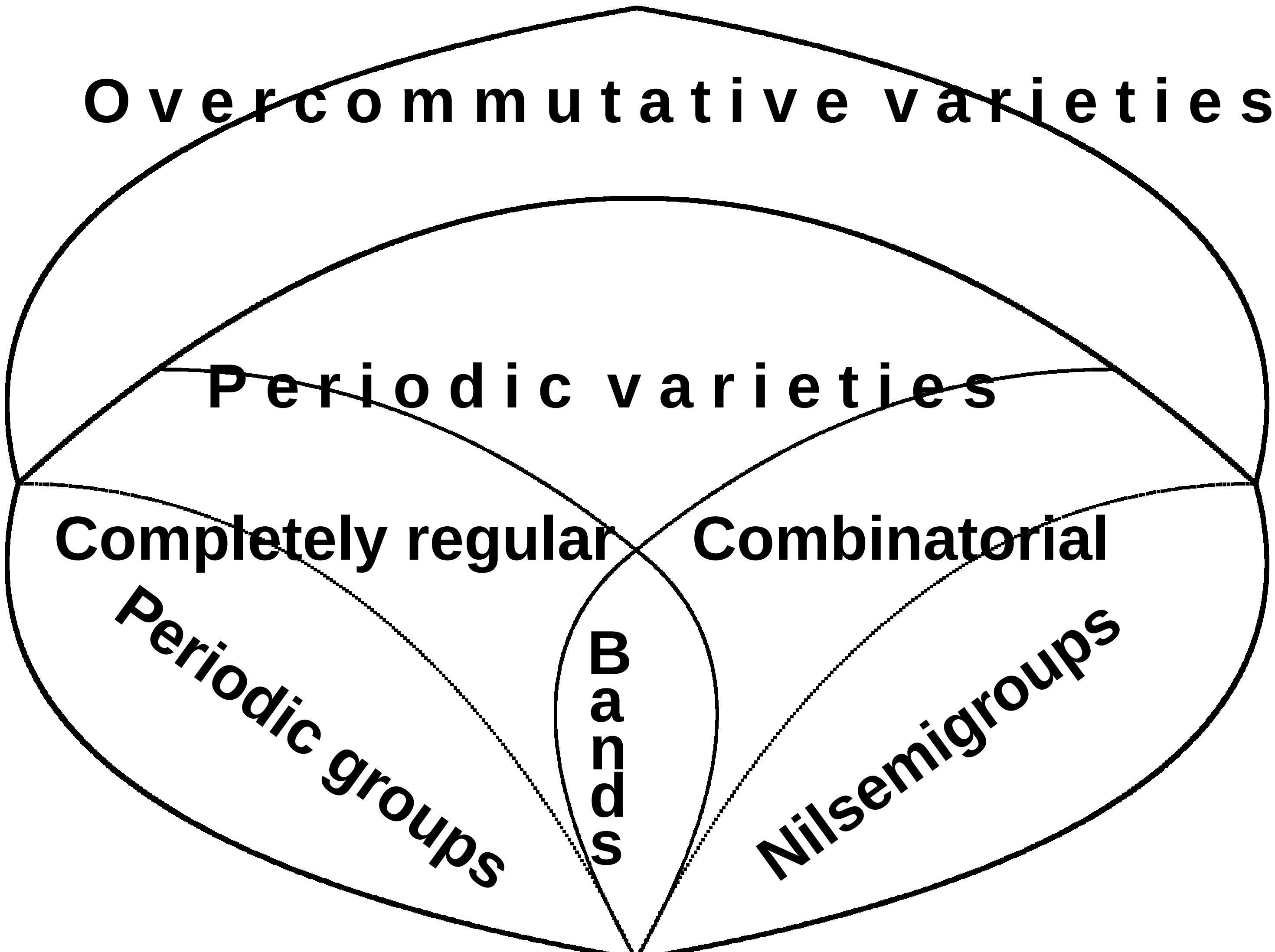

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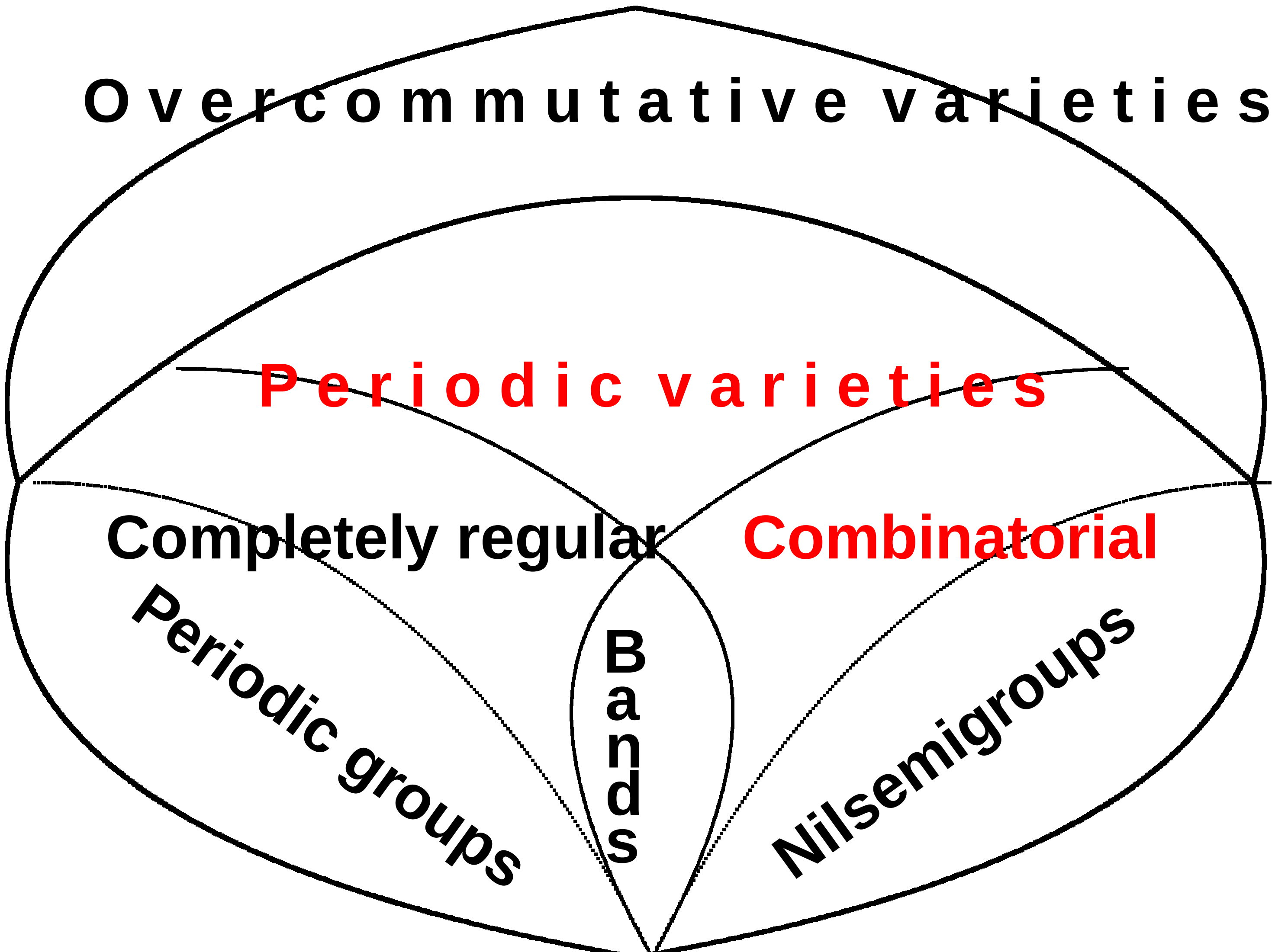

Rees–Sushkevich Varieties

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Goal – to describe the lattice of Rees–Sushkevich varieties modulo the lattice of group varieties with exponent dividing n. In particular, we aim to describe the the lattice of combinatorial Rees–Sushkevich varieties, that is, the subvariety lattice of $\mathbf{RS}_1$.

Rees–Sushkevich Varieties

$$A_2 = \langle a, b \mid aba = a^2 = a,\ bab = b,\ b^2 = 0\rangle$$
$$= \{a, b, ab, ba, 0\}$$

the 5-element idempotent-generated 0-simple semigroup.

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We have succeeded in understanding the subvariety lattice of $\mathbf{A_2} \vee \mathbf{G}_n \subseteq \mathbf{RS}_n$.

In this talk I restrict to the subvariety lattice of $\mathbf{A_2}$.

Why Rees–Sushkevich Varieties?

$$B_2 = \langle c, d \mid cdc = c,\ dcd = d,\ c^2 = d^2 = 0\rangle$$
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the 5-element Brandt semigroup

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- is inherently non-finitely based (Sapir)

The Main Frame

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Fact. If $\mathbf{V}$ is one of these 13 exact subvarieties, then there exists the largest subvariety $\overline{\mathbf{V}}$ of $\mathbf{A}_2$ that does not contain $\mathbf{V}$.

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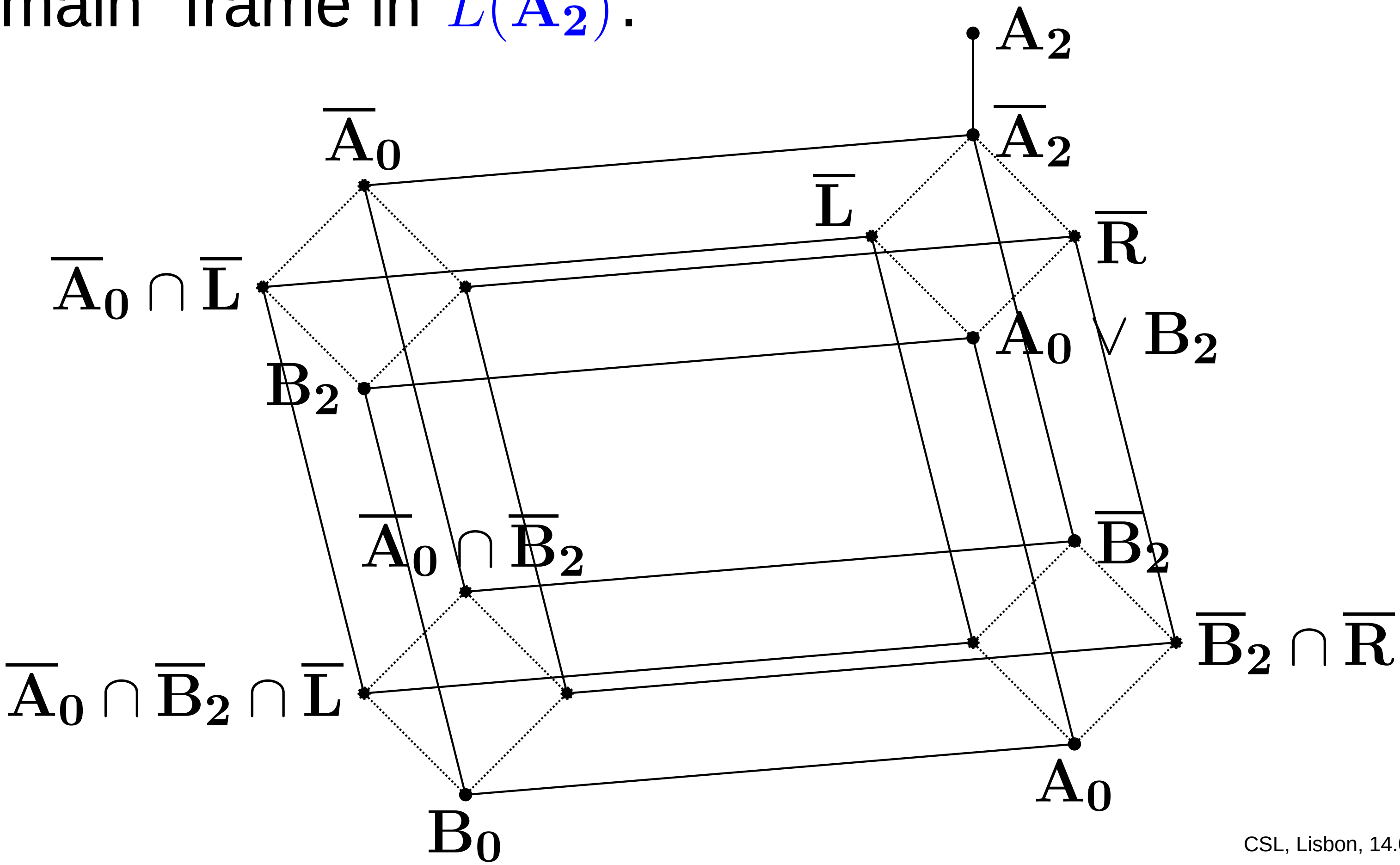

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(in other words, $\overline{\mathbf{B_2}} = \mathbf{A_2} \cap \mathbf{DS}$)

Identity Bases

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The first two results are due to Trahtman,

the other two are by Edmond Lee

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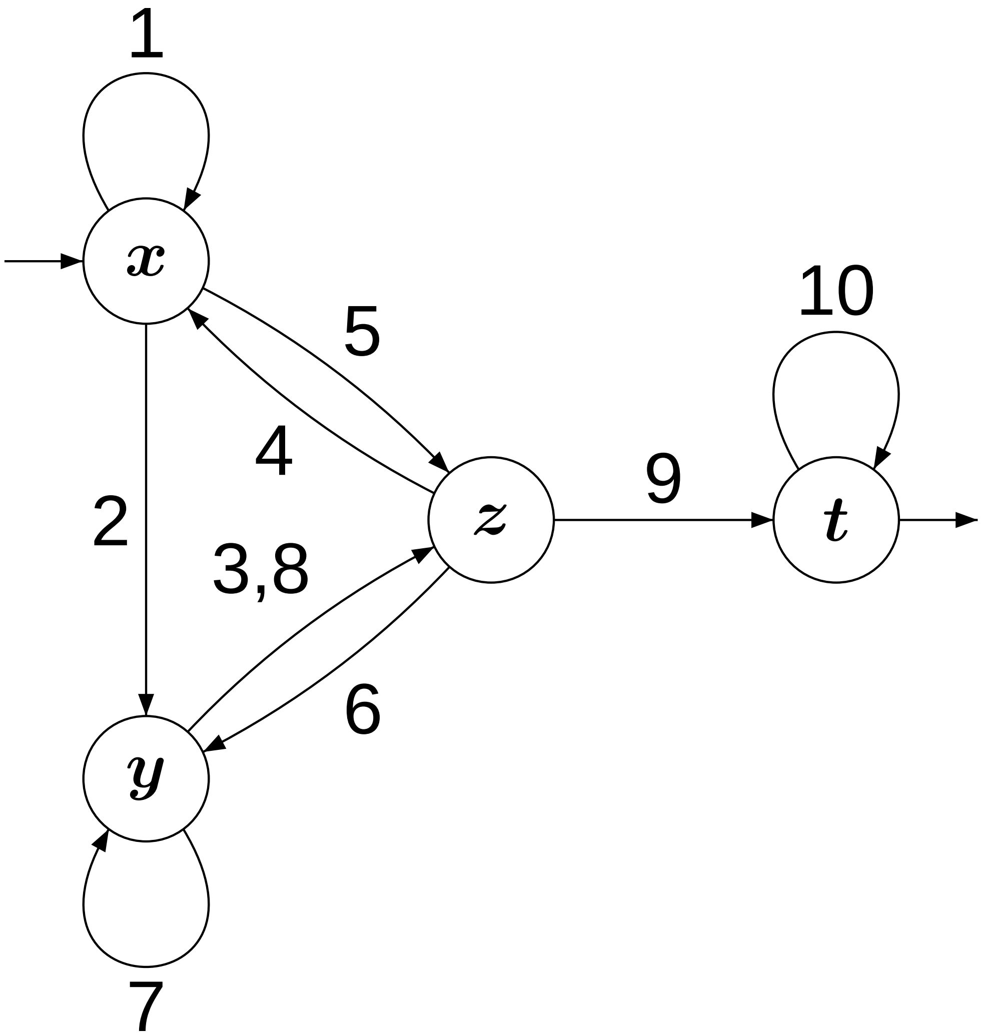
Then w can be thought of as a walk through the graph $G(w)$ that starts at the initial vertex, ends at the final vertex and traverses each edge of $G(w)$ (some of the edges can be traversed more than once).

The graph of the word $x^2yzxzy^2zt^2$:



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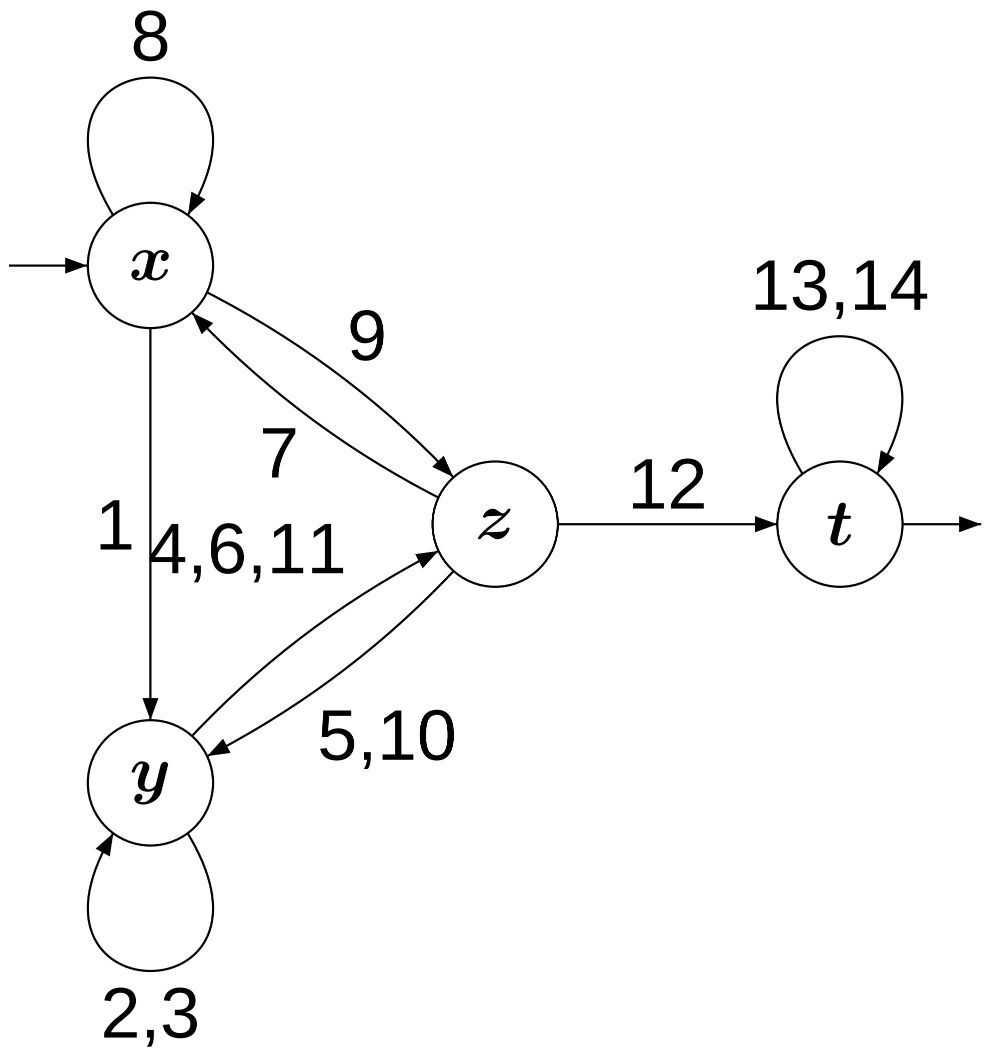
Another walk through the graph, $\boldsymbol{xy^3zyzx^2zyzt^3}$:



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Fact (Kublanovsky) For any semigroup $S \in \mathbf{A}_2$ and distinct regular elements $s, s' \in S$ there exists a completely 0-simple semigroup K and a surjective homomorphism $\varphi : S \to K$ such that $s\varphi \neq s'\varphi$.

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If this is strict, take the distinguishing identity $u = v$ with the least number of letters involved and show that the graphs $G(u)$ and $G(v)$ must be strongly connected.