P(l)aying for Synchronization

Mikhail Volkov

Ural Federal University, Ekaterinburg, Russia

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Definitions and Terminology

We consider complete deterministic finite automata (DFA) $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ where Q stands for the state set, Σ is the input alphabet, and $\delta : Q \times \Sigma \to Q$ is a (total) transition function.

To simplify notation we often write $q \,.\, w$ for $\delta(q, w)$ and $P \,.\, w$ for $\{\delta(q, w) \mid q \in P\}$.

$\mathscr{A}$ is called synchronizing if there is a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves $\mathscr{A}$ in one particular state no matter at which state in Q it started: $q \,.\, w = q' \,.\, w$ for all $q, q' \in Q$. In short, $|Q \,.\, w| = 1$.

Any w with this property is a reset word for $\mathscr{A}$.

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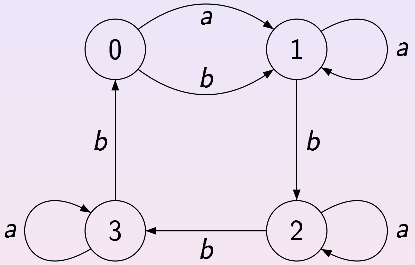

A reset word is *abbbabbba*. In fact, we will see that this is the shortest reset word for this automaton.

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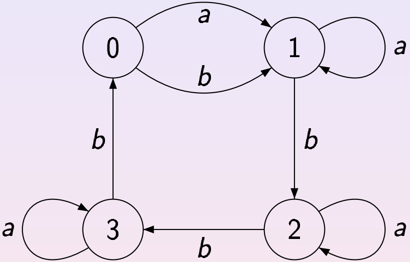

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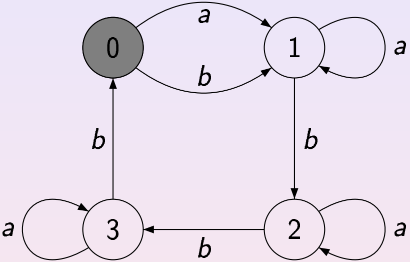

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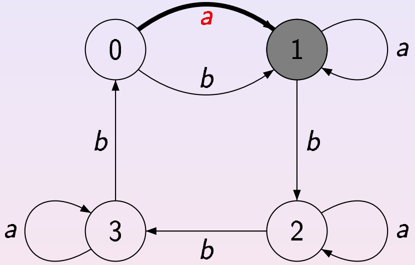

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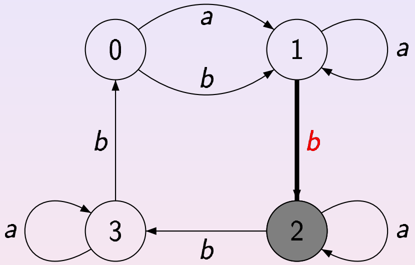

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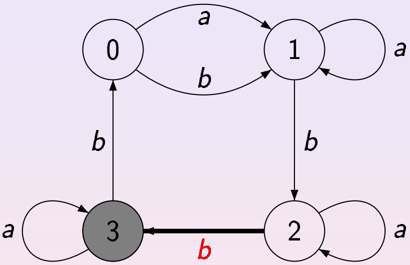

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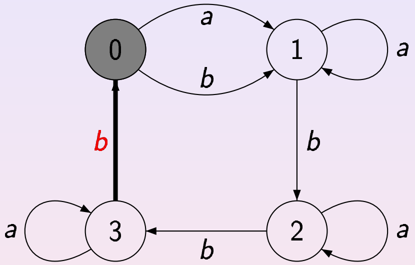

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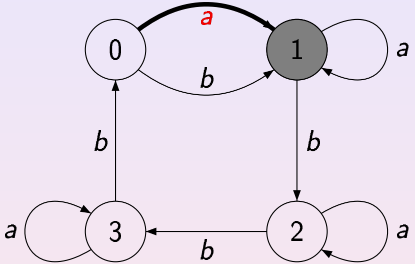

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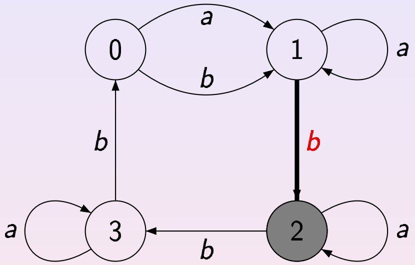

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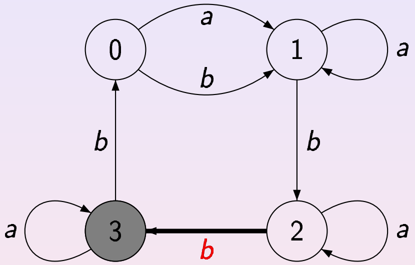

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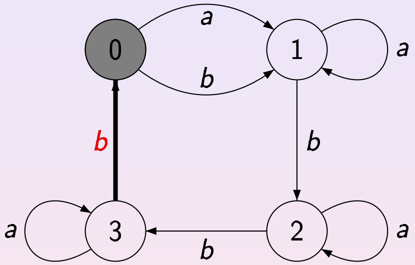

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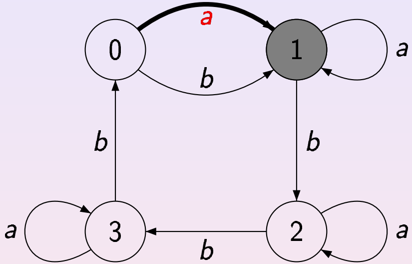

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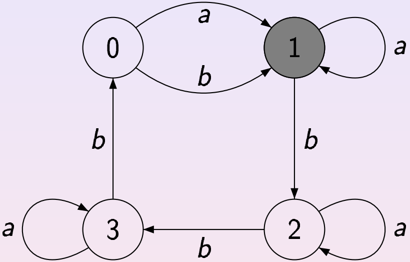

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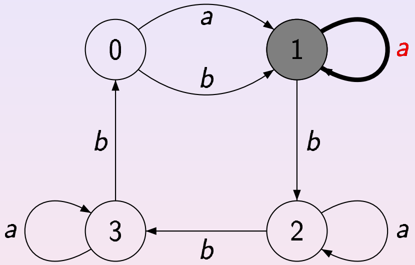

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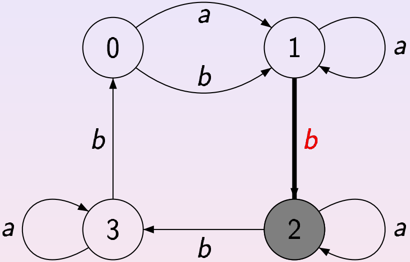

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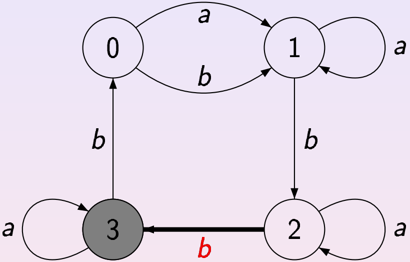

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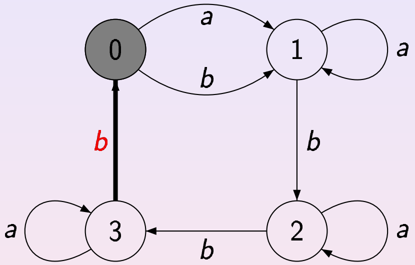

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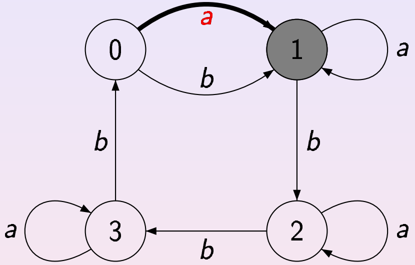

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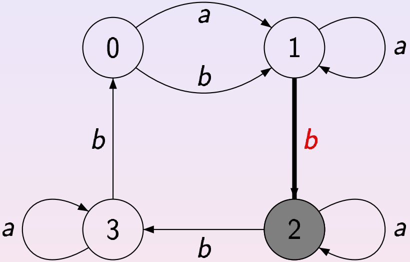

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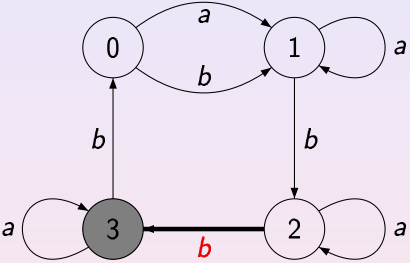

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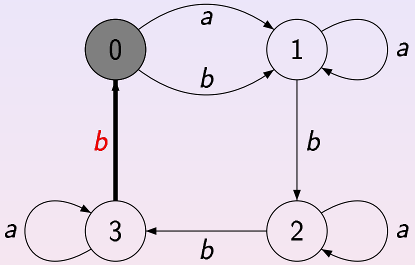

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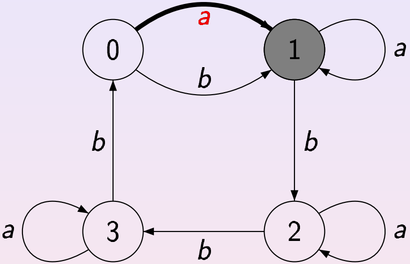

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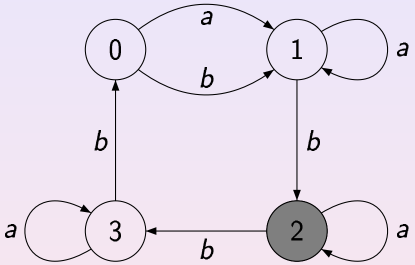

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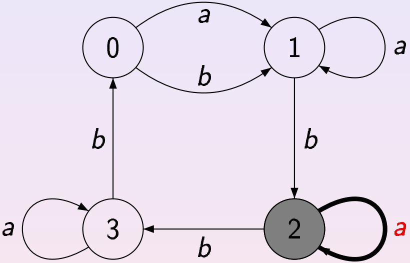

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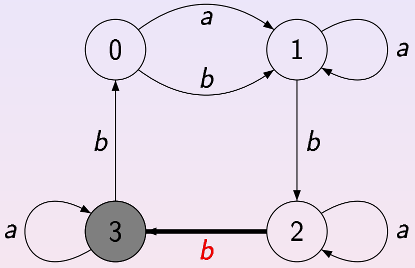

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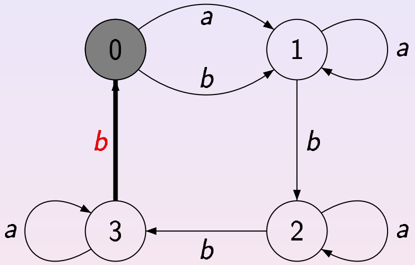

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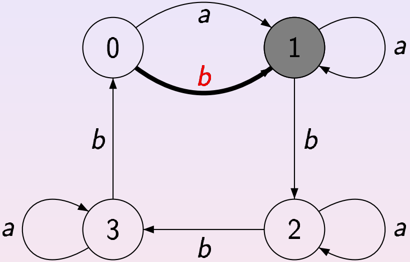

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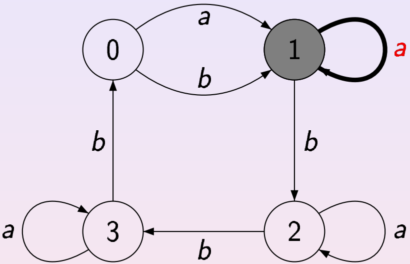

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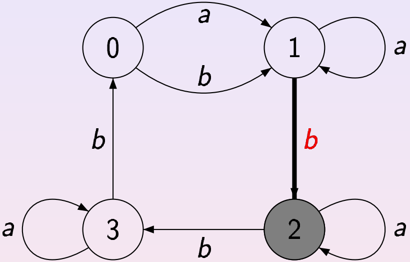

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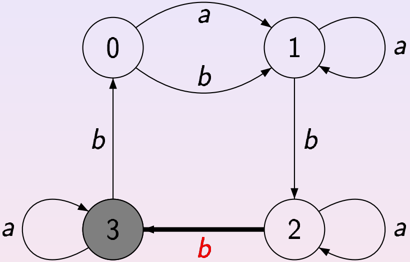

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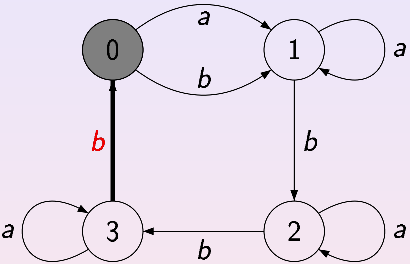

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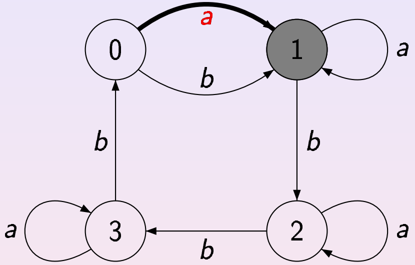

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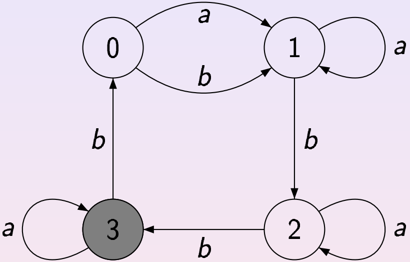

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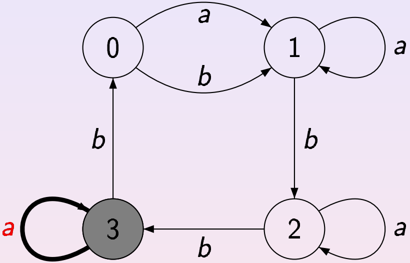

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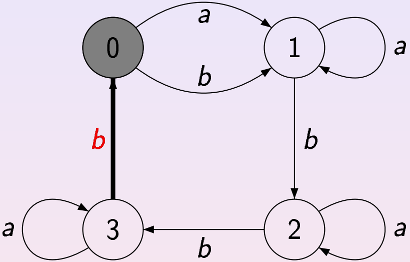

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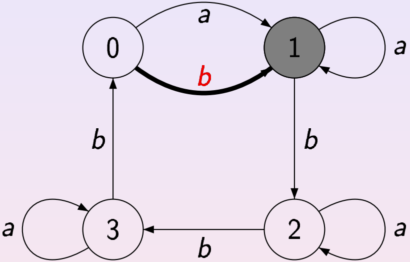

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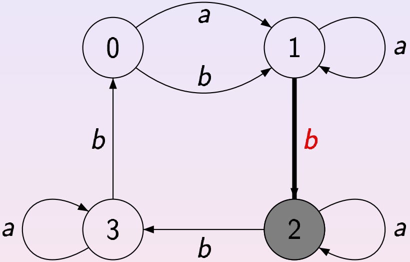

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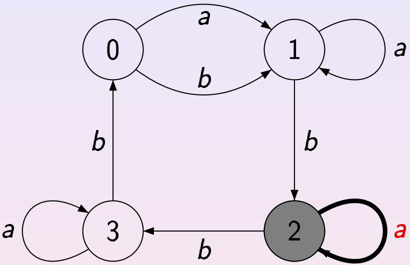

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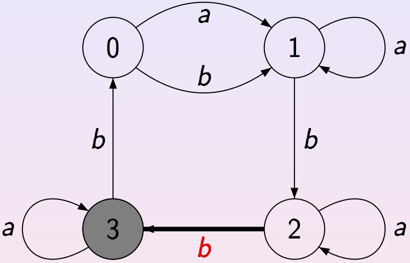

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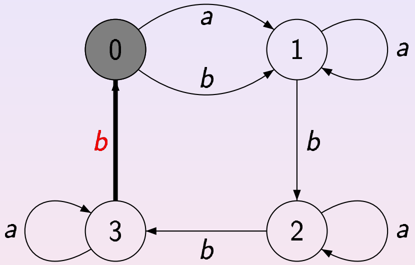

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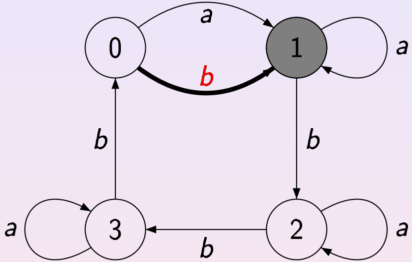

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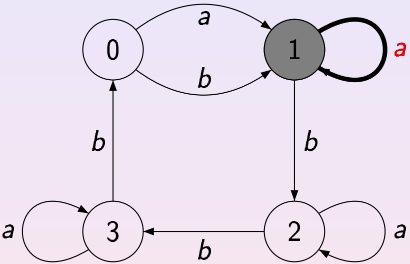

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Our Game

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- The digraph of $\mathscr{A}$ — the game-board.
- The initial position — each state holds a coin.
- Each letter $c \in \Sigma$ defines a move — coins slide along the arrows labelled c and, whenever two coins meet at some state, one of them is removed.
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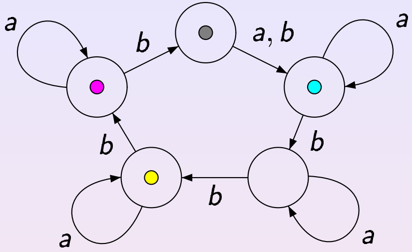

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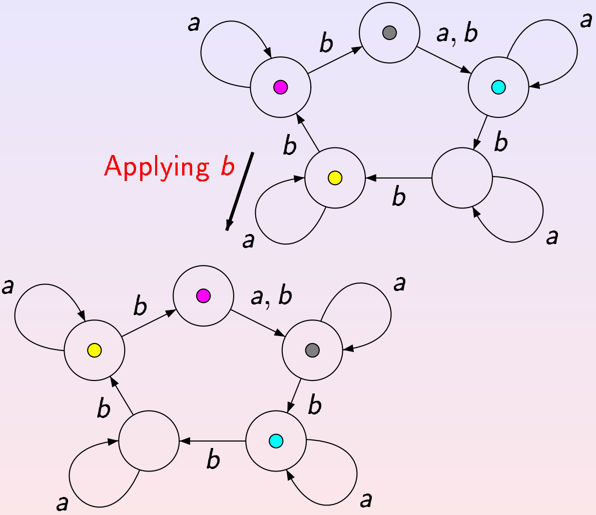

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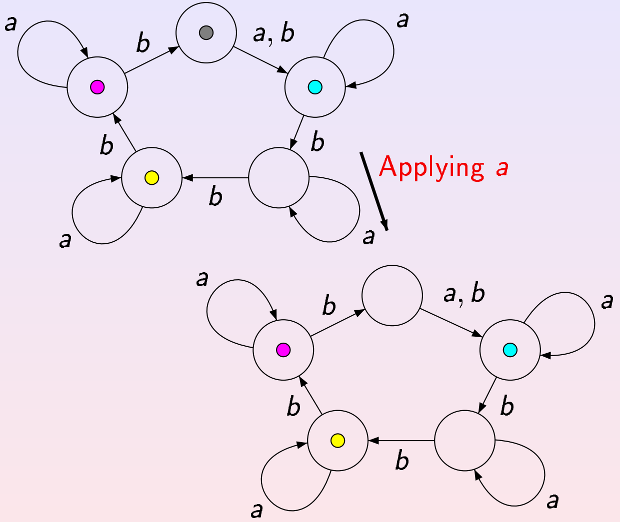

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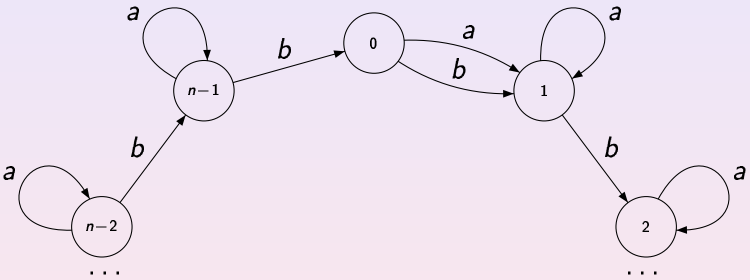
In his 1964 paper Jan Černý constructed a series $\mathscr{C}_n$, $n = 2, 3, \ldots,$ of synchronizing automata over 2 letters:



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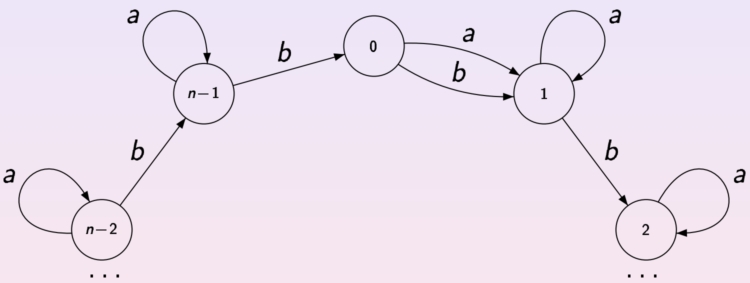

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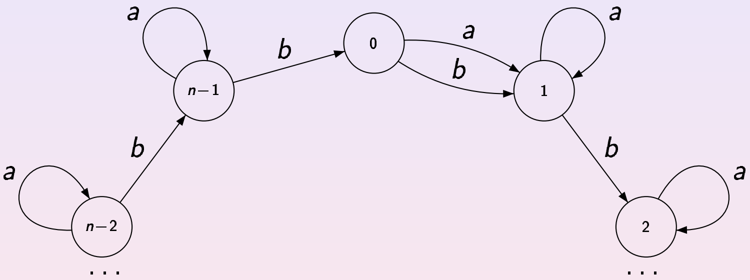

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Extra Conventions

In the game on $\mathscr{C}_n$ we adopt a few extra conventions:

- The coins are pairwise distinct.
- Whenever two coins meet at the state 1, the coin arriving from 0 is removed.
- The only coin that remains at the end of the game is called the golden coin G.

LMRC, February 2nd, 2012

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Let P_0 be an initial distribution of coins, w a reset word. Denote by P_i the position that arises when we apply the prefix of w of length i to the position P_0. We want to define the weight $\mathrm{wg}(P_i)$ of the position such that

(i) $\mathrm{wg}(P_0) \geq n(n-1)$ and $\mathrm{wg}(P_{|w|}) \leq n-1$;

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Then $$|w| = \sum_{i=1}^{|w|} 1 \geq \sum_{i=1}^{|w|} (\mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)) =$$

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where $m_i(C)$ is the number of moves from the state holding C to 0 and $d_i(C)$ is the number of moves from the state holding C to one holding G. (Recall that G stands for the golden coin G which is present in all positions.)

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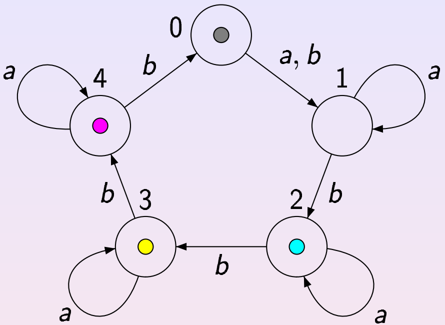

Assume that the yellow coin is the golden one. Then its weight is 2. The weight of the cyan coin is $5 \cdot 1 + 3 = 8$. The weight of the gray coin is $5 \cdot 3 + 0 = 15$. The weight of the magenta coin is $5 \cdot 4 + 1 = 21$, and this is the weight of the position.

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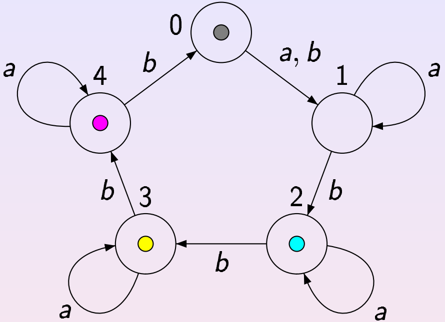

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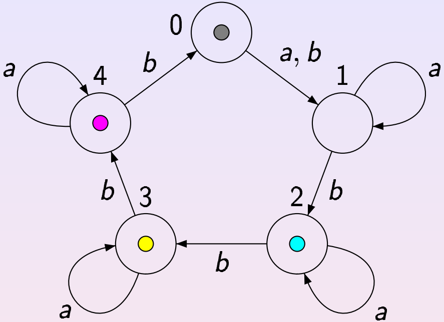

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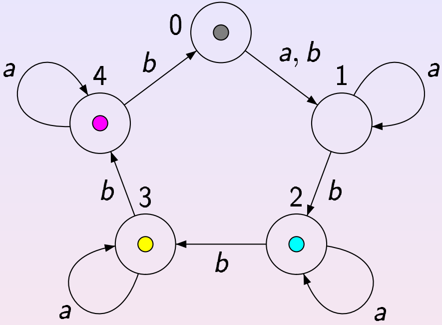

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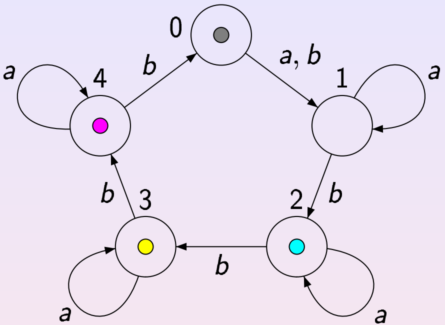

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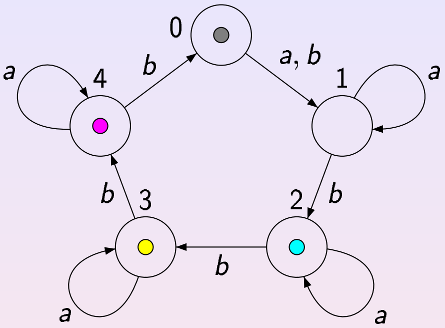

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We have to check that our weight function satisfies the conditions

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In the initial position all states are covered with coins. Consider the coin C that covers the state in one step clockwise after the state covered with the golden coin. Then $d_0(C) = n-1$ whence

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In the final position only the golden coin G remains, whence the weight of $P_{|w|}$ is the weight of G. Clearly, $\mathrm{wg}(G, P_i) = m_i(G) \leq n-1$ for any position P_i.

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A Two-Players Game

Assume that there are two players: Alice (Synchronizer) and Bob (Desynchronizer) whose moves alternate. Alice (who plays first) wants to synchronize the given automaton, Bob aims to make her task as hard as possible.

Provided that both players play optimally, the outcome of such a game depends only on the underlying automaton so studying synchronization games is a way to study synchronizing automata. The most natural questions here are the following:

- Clearly, Bob wins on non-synchronizing automata. May he win on a synchronizing automaton?
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The answers to the above questions have been found by Födor Fominykh.

- Bob can win on a synchronizing automaton (for instance, he wins on $\mathscr{C}_n$).
- Given $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, one can decide who wins on $\mathscr{A}$ in $O(|Q|^2 \cdot |\Sigma|)$ time.
- If Alice wins, then she can win in $O(|Q|^3|)$ moves.
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A Two-Players Game: Answers

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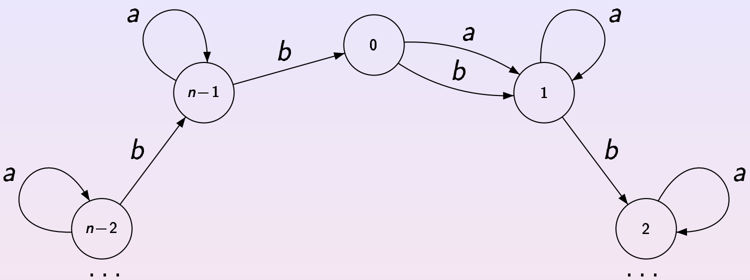

Bob only has to trace the coins that cover the states $n-1$ and 1 in the initial position. He can always say a except two cases: when the chosen coins cover either $n-2$ and 0 or 0 and 2 in which cases Bob says b. This way Bob can always keep the coins two steps apart from each other whence Alice can remove none of them.

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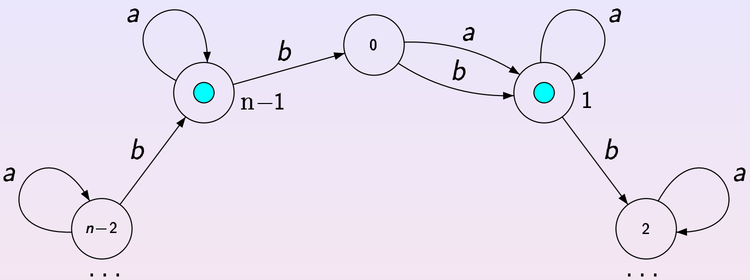

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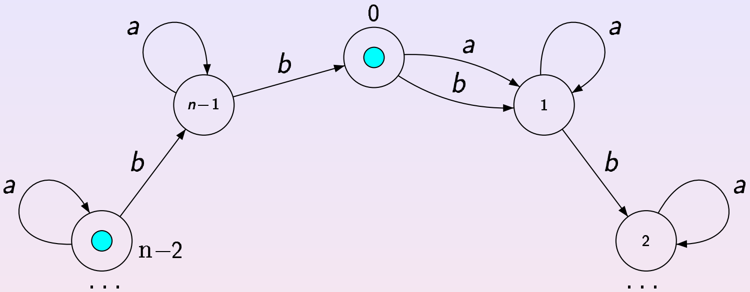

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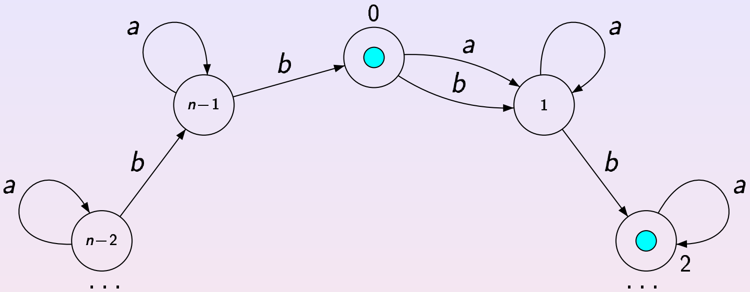

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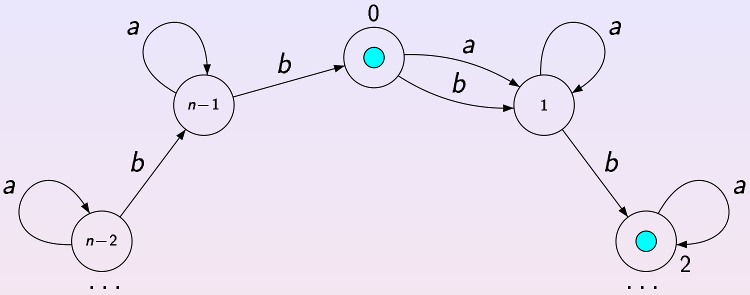

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Proposition. *Alice wins the game on a DFA $\mathscr{A}$ iff she wins in every position with only two coins.*

Then we construct a new automaton whose states are all $O(|Q|^2)$ positions with two coins plus a sink state (corresponding to all positions with one coin) and mark its states in a standard way: A state p is an Alice state if either Alice can reach the sink state from p by a single move or she has a move leading to a position in which every Bob's reply leads to an Alice state. The marking is basically a BFS on the reverse graph, and Alice wins iff all states will eventually be marked as Alice states.

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Upper Bounds for Decision Time and Game Length

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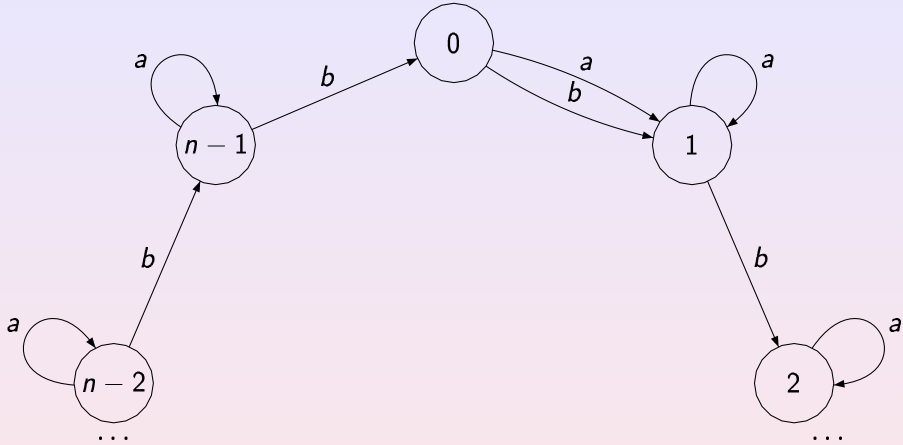
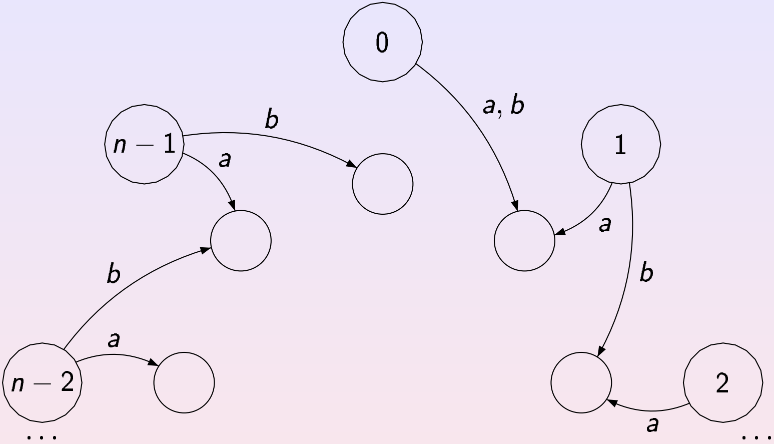


We start with the Černý automaton $\mathscr{C}_n$

Long Games



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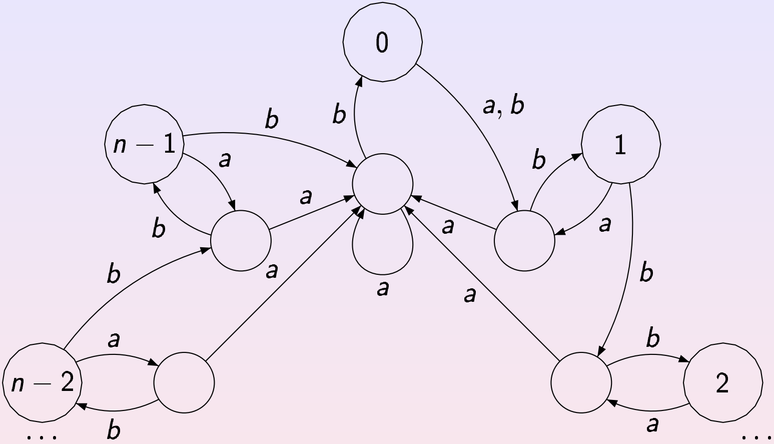

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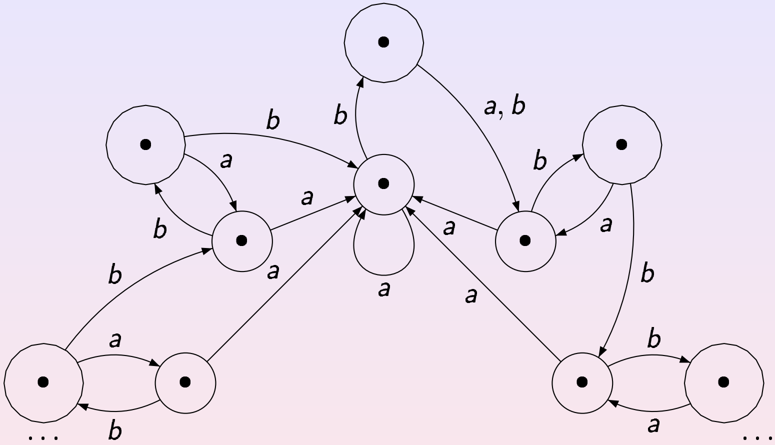
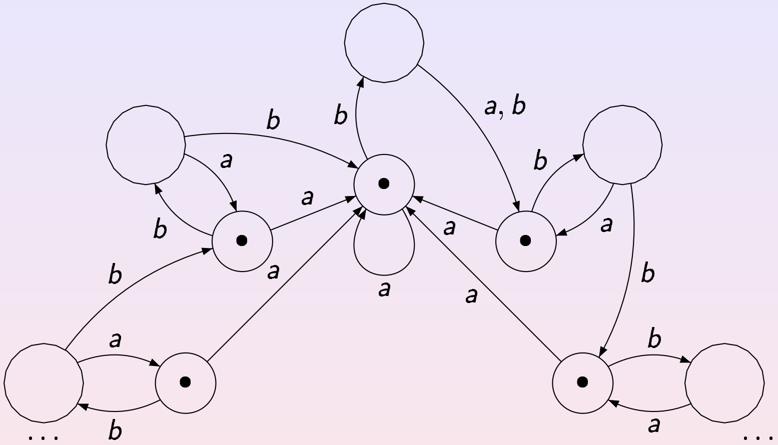


The initial position.

Long Games



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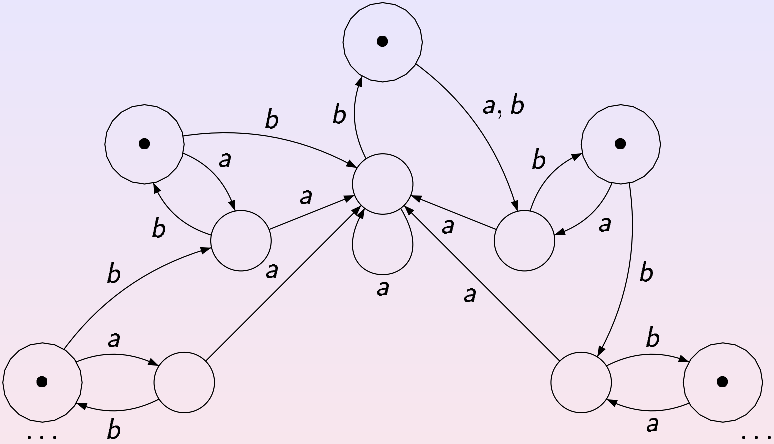

Bob must reply b otherwise he loses immediately. Now the position imitates the initial position in the game on $\mathscr{C}_n$.

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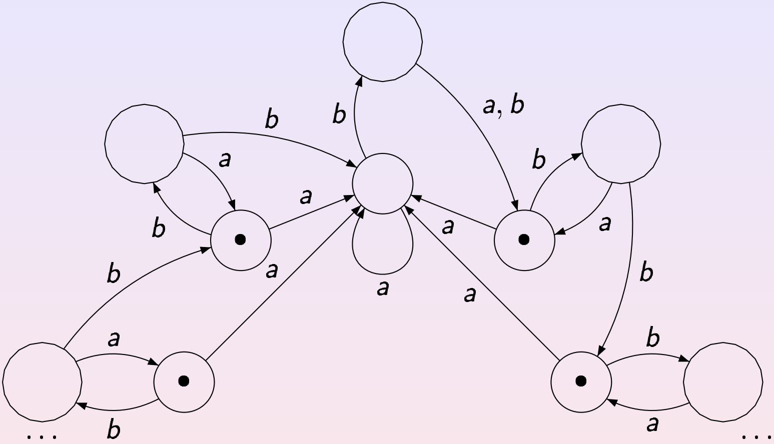

Alice says a again.

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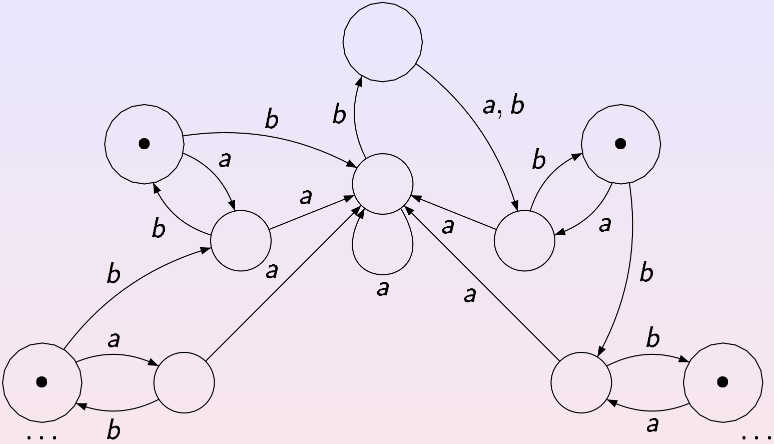

Bob must reply b otherwise he loses immediately. Now the position imitates the one after the first move in the one-player game on $\mathscr{C}_n$.

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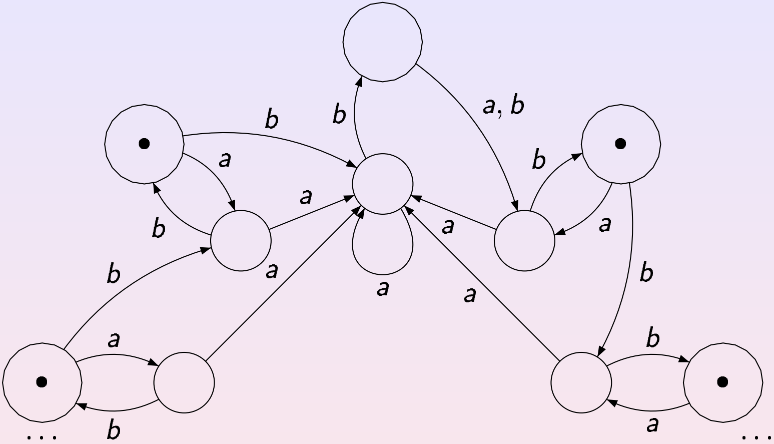

Continuing, we see that Alice wins by spelling out the reset word for $\mathscr{C}_n$ but cannot win faster if Bob replies b on each move.

A Corollary and Further Games

We can register the following rather unexpected corollary:

If Alice has an $O(n^2)$-strategy for each n-state automaton with a reset word of length 2 on which she can win, then there is a quadratic upper bound in the Černý problem.

Other possible synchronization games include:

- Games against the Nature (Nature replies by random moves) — see Andreas Blass, Yuri Gurevich, Lev Nachmanson, Margus Veanes: Play to Test, in: Formal Approaches to Software Testing, 5th International Workshop, FATES 2005 (Lect. Notes Comp. Sci., vol. 3997), Springer, 2006, 32–46.
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LMRC, February 2nd, 2012

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Synchronization Costs

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For $w = a_1 \cdots a_k \in \Sigma^*$ and $q \in Q$, the cost of applying w at q is

$$\gamma(q, w) = \sum_{i=0}^{k-1} \gamma(\delta(q, a_1 \cdots a_i), a_{i+1}).$$

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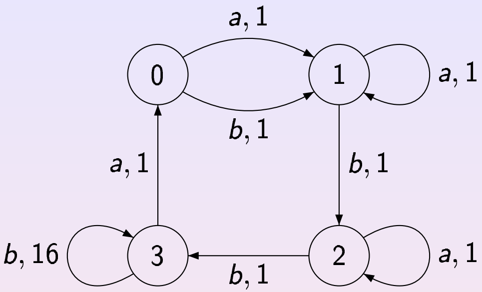

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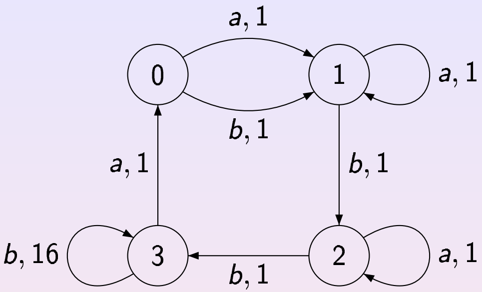

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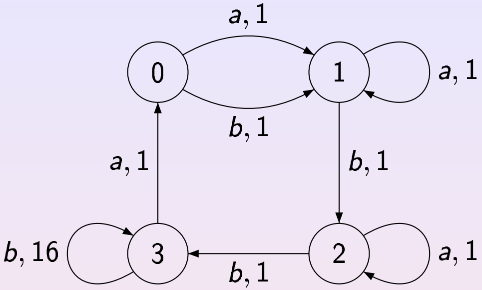

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SYNCHRONIZING ON BUDGET: *Given a synchronizing automaton* $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ *with the cost function* γ *and a positive integer* B, *is it true that* $\mathscr{A}$ *has a reset word with* $\gamma(w) \leq B$*?*

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Clearly, SHORT-RESET-WORD is a special case of SYNCHRONIZING ON BUDGET when $\gamma(q, a) = 1$ for all $q \in Q$ and $a \in \Sigma$. Thus, SYNCHRONIZING ON BUDGET is NP-hard.

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Clearly, Short-Reset-Word is a special case of Synchronizing on Budget when $\gamma(q, a) = 1$ for all $q \in Q$ and $a \in \Sigma$. Thus, Synchronizing on Budget is NP-hard.

Easy Case

If the transition costs and the budget are given in unary, then SYNCHRONIZING ON BUDGET is in NP.

One can non-deterministically guess a word $w \in \Sigma^*$ of length $\ell \leq B$ and then check if w is a reset word for $\mathscr{A}$ with $\gamma(w) \leq B$. This can be done in $\ell B|Q|$ time.

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Now assume that the transition costs $\gamma(q, a)$ and the budget B are given in binary. Then one can show that for some synchronizing automata any reset word satisfying $\gamma(w) \leq B$ is exponentially long in $|Q|$. Therefore the above non-deterministic algorithm is not polynomial in time.

However, it can be implemented in polynomial space if one guesses the word w letter by letter. One guesses the first letter of w (say, a), apply a at every state $q \in Q$ and save two arrays: $\{\delta(q, a)\}$ and $\{\gamma(q, a)\}$. Then one guesses the second letter of w and updates both arrays, etc.

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We show that Synchronizing on Budget is PSPACE-complete by a reduction from Careful Synchronization.

Recall that an incomplete deterministic automaton $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ is said to be carefully synchronizing if there exists $w = a_1 \cdots a_\ell$ with $a_1, \ldots, a_\ell \in \Sigma$ such that:

1) $\delta(q, a_1)$ is defined for all $q \in Q$,
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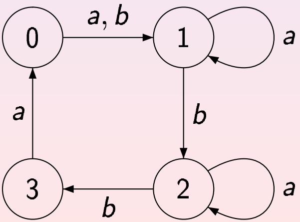

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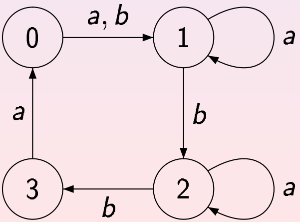

A careful reset word is a^2baba^2.

Theorem (Martyugin, 2010)

Checking if a given incomplete DFA is carefully synchronizing is PSPACE-complete.

There is also an obvious upper bound $2^n - n - 1$ on the minimum length of the shortest careful reset word for carefully synchronizing automata with n states.

It comes from the power automaton $\mathcal{P}'(\mathscr{A})$ of a given incomplete DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$:

- states are the non-empty subsets of Q,
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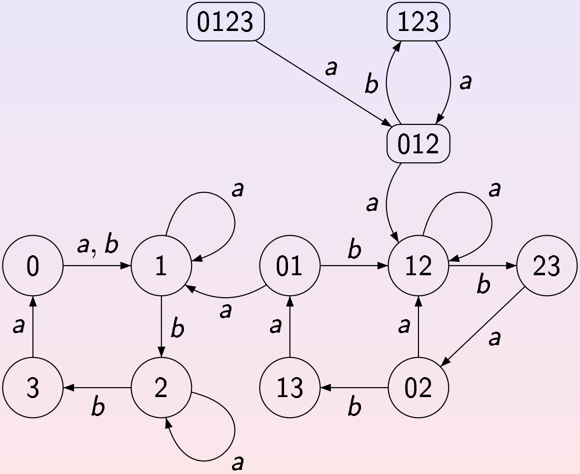

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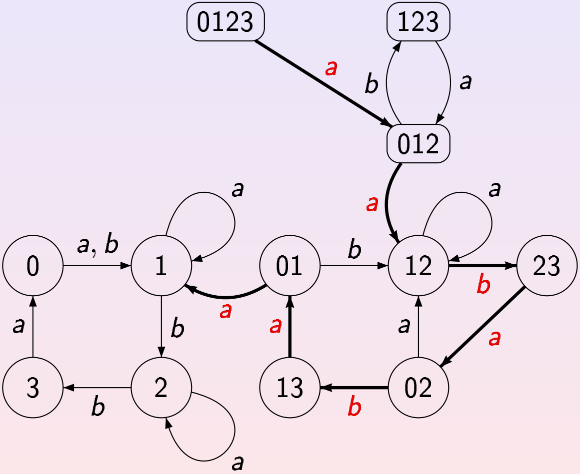

A Reduction

Now, given an incomplete DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, we construct an instance $(\mathscr{A}', B)$ of Synchronizing on Budget as follows:

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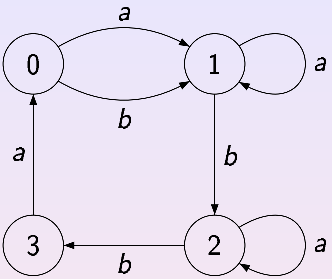

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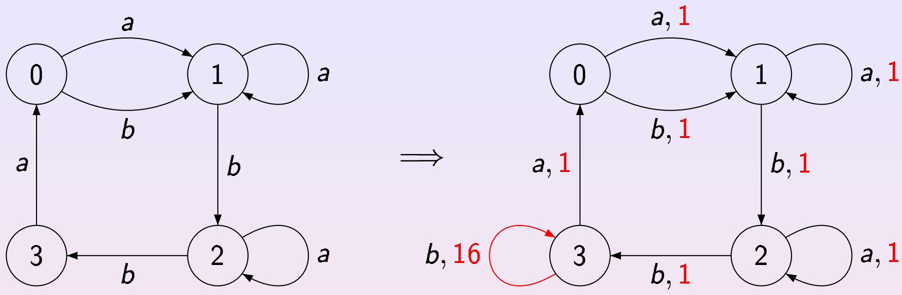

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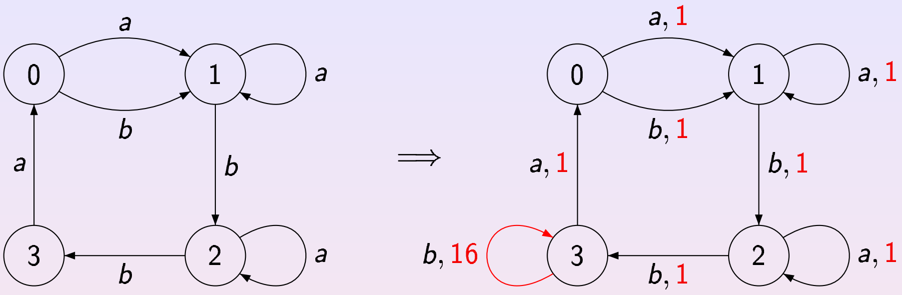

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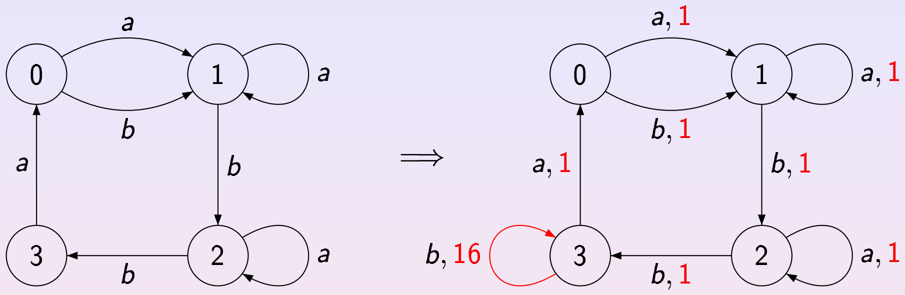

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