The Finite Basis Problem for Finite Semigroups Revisited

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Basic definitions

We consider finite semigroups sometimes equipped with extra unary operation.

A semigroup identity is a pair of semigroup words (u, v) usually written as a formal equality $u \bumpeq v$.

A semigroup S satisfies an identity $u \bumpeq v$ (or: $u \bumpeq v$ holds in S) if every evaluation of letters involved in the words u and v at some elements of S produces equal values in S.

Example: the identities $xy \bumpeq yx$ and $x \bumpeq x^2$ hold in the semigroup $\langle\{0, 1\}; \cdot\rangle$ while the identity $x \bumpeq y$ does not.

A semigroup S is finitely based if all identities holding in S can be deduced from some finite set of such identities (called an identity basis for S).

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Indeed, if an identity $u \bumpeq v$ holds in $\langle\{0,1\};\cdot\rangle$, then u and v have the same letters. But the laws $xy \bumpeq yx$ and $x \bumpeq x^2$ allow one to reduce any word to the product of its letter, each taken once, in some fixed order.

$$xyxyzytzx \xrightarrow{xy \bumpeq yx} x^3y^3z^2t \xrightarrow{x \bumpeq x^2} xyzt$$

Hence, whenever two words involve the same letters, they can be reduced to the same product, and thus, to each other.

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Perkins's Example

If a semigroup is not finitely based, it is said to be nonfinitely based.
A finite semigroup can be nonfinitely based (Perkins, 1969). Perkins's example is the 6-element monoid B_2^1 (the Brandt monoid) formed by the following 2×2-matrices:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}.$$

The example published exactly 40 years ago is extremely transparent and natural. It also turns out to be minimal with respect to size of semigroup. (There are two other 6-element examples.)

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Main Problem

Semigroups are the only "classical" algebras for which finite non-finitely based objects exist: finite groups, associative or Lie rings, lattices are finitely based.

Problem: Which finite semigroups are finitely based and which are not?

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Tarski's Problem

Is there an algorithm that when given an effective description of a finite semigroup S decides whether S is finitely based or not?

Previous Survey

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M.V. Volkov, The finite basis problem for finite semigroups, *Sci. Math. Jap.*, Vol. 53, 171–199, 2001.

The area has been progressed a lot over the last decade. An updated version of the above survey will become available tonight through my webpage: `http://csseminar.kadm.usu.ru/volkov`

The present talk is rather intended to display the evolution of ideas that underlie the area.

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- Inherently nonfinitely based semigroups: a finite semigroup is said to be inherently nonfinitely based if it is not contained in any locally finite finitely based variety. Hence S is nonfinitely based (and even inherently nonfinitely based) if $\operatorname{Var} S$ contains an inherently nonfinitely based semigroup.

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- Critical semigroups: a series of semigroups S_n, $n = 1, 2, \ldots,$ such that each S_n does not belong to the variety $\mathrm{Var}\, S$ while all n-generated subsemigroups of S_n belong to $\mathrm{Var}\, S$.

List of Methods contd

- Interpretation: one interprets within the equational theory of S another theory which is known (or can be easily proved) to have no finite axiomatization.

A new method (that can be thought of as a variation of the interpretation method) has been recently found by Jackson and McKenzie: M. Jackson, R. McKenzie, Interpreting graph colorability in finite semigroups, *Int. J. Algebra Comput.* Vol.16, 119–140, 2006.

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- Complexity analysis: if the membership problem for the variety $\mathrm{Var}\, S$ is computationally hard, then S must be nonfinitely based.

FBB vs VMP

The Variety Membership Problem (VMP) for a finite semigroup S is a combinatorial decision problem whose input is a finite semigroup T and whose question is whether T belongs to the variety $\mathrm{Var}\,S$.

VMP and related problems for pseudovarieties play a central role in the modern theory of finite semigroups.

VMP is always decidable: by the HSP-theorem $T \in \mathrm{Var}\,S$ iff T is a homomorphic image of the free $|T|$-generated semigroup of $\mathrm{Var}\,S$ and the free semigroup has at most $|S|^{(|S|^{|T|})}$ elements.

The complexity of VMP is not known (for general algebras it can be extremely high as shown by Kozik).

If S has a finite identity basis Σ, say, then in order to check whether or not T belongs to $\mathrm{Var}\,S$ it suffices to verify whether or not all identities in Σ hold in T, and this is a polynomial (in $|T|$) procedure.

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FBB vs VMP contd

This is an additional strong motivation for studying the FBP for finite semigroups.

On the other hand, the connection between FBB vs VMP allows one to employ complexity-theoretical tools for producing new classes of finite semigroups without finite identity basis.

Recall that the problem 3-Color is NP-complete (Levin, 1973). The following graph Co_3 is 3-colorable but not 2-colorable.

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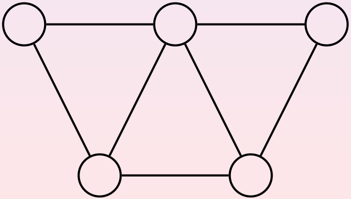

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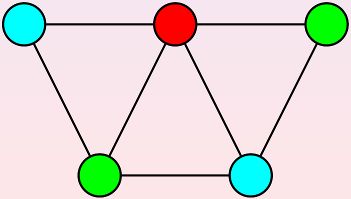

Jackson–McKenzie's Result

Nešetřil and Pultr (1978) have shown that a graph G is 3-colorable iff G can be approximated by Co_3 in the following sense:

- there is at least one homomorphism from G to Co_3;
- for each pair of distinct vertices a, b of G, there is a homomorphism $\varphi : G \to Co_3$ with $\varphi(a) \neq \varphi(b)$;
- for each pair of non-adjacent vertices a, b of G, there is a homomorphism $\varphi : G \to Co_3$ such that $\varphi(a)$ and $\varphi(b)$ are not adjacent in Co_3.

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If G is a graph with n vertices, then $S(G)$ has n^2+5n+5 elements. In particular, the semigroup $S(Co_3)$ has 55 elements. The structure of the semigroups $S(G)$ is rather transparent: they are *local semilattices*, that is, $eS(G)e$ satisfies $xy \simeq yx$ and $x \simeq x^2$ for every idempotent $e \in S(G)$.

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Jackson and I (The algebra of adjacency patterns: Rees matrix semigroups with reversion, about to be submitted) used a similar idea for unary semigroups.

Here the construction is pretty easy: Given a graph $G = \langle V; E \rangle$, its unary adjacency semigroup $A(G)$ is defined on the set $(V \times V) \cup \{\mathbf{0}\}$; the multiplication rule is

$$(x, y)(z, t) = \begin{cases} (x, t) & \text{if } (y, z) \in E, \\ \mathbf{0} & \text{if } (y, z) \notin E; \end{cases}$$

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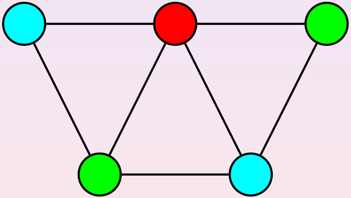

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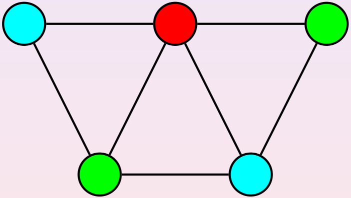

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- T_n, the monoid of all total transformations;
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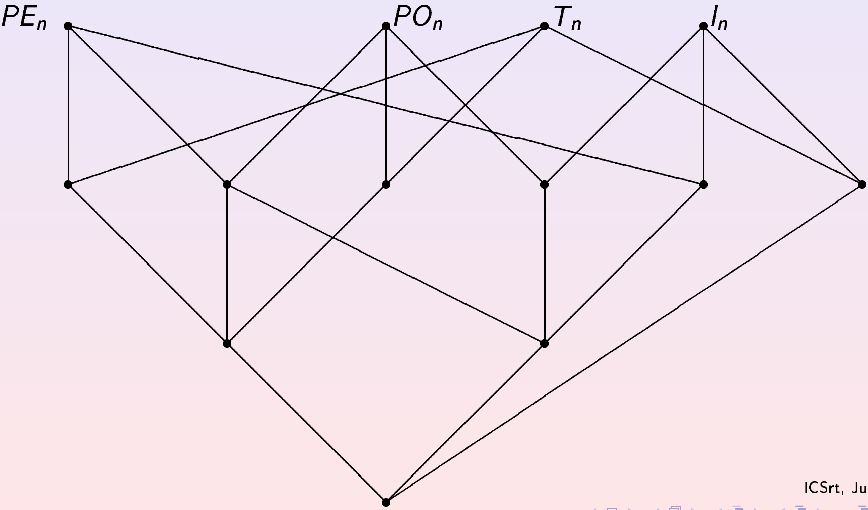

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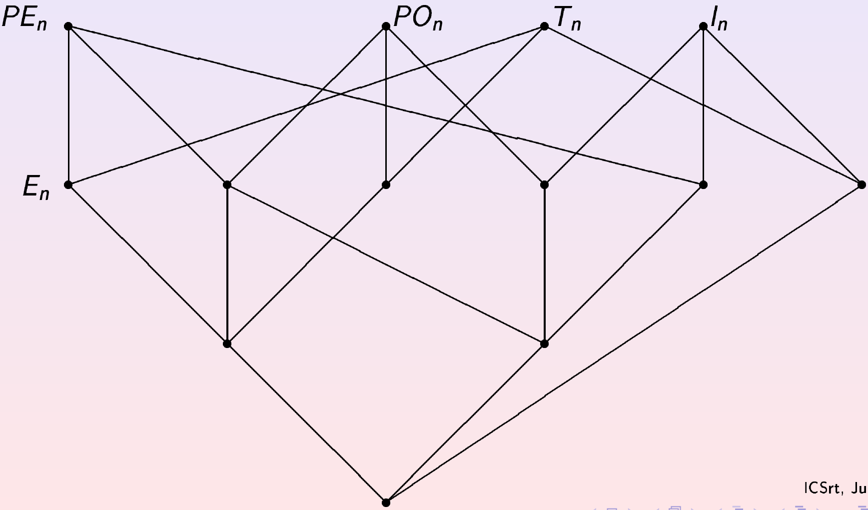

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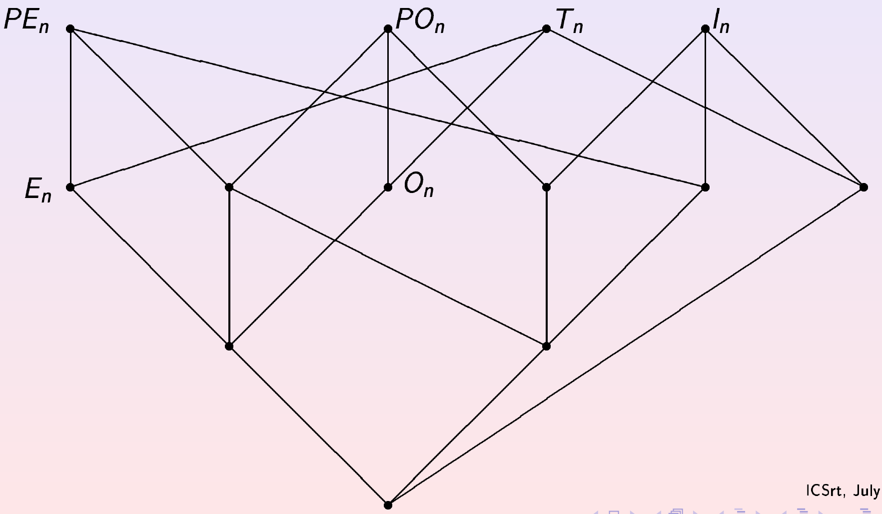

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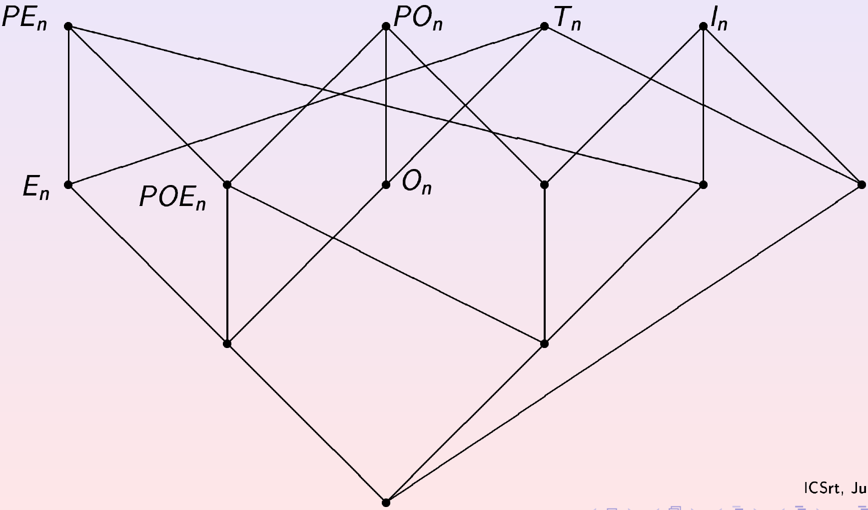

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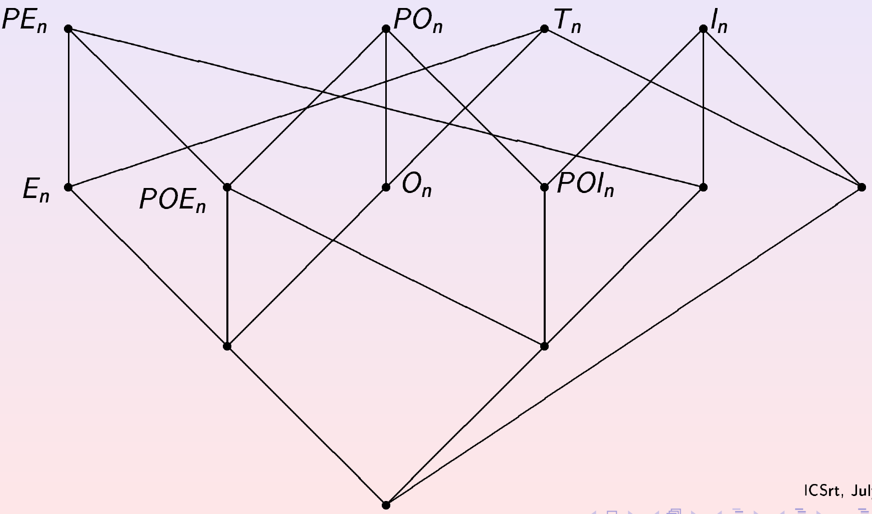

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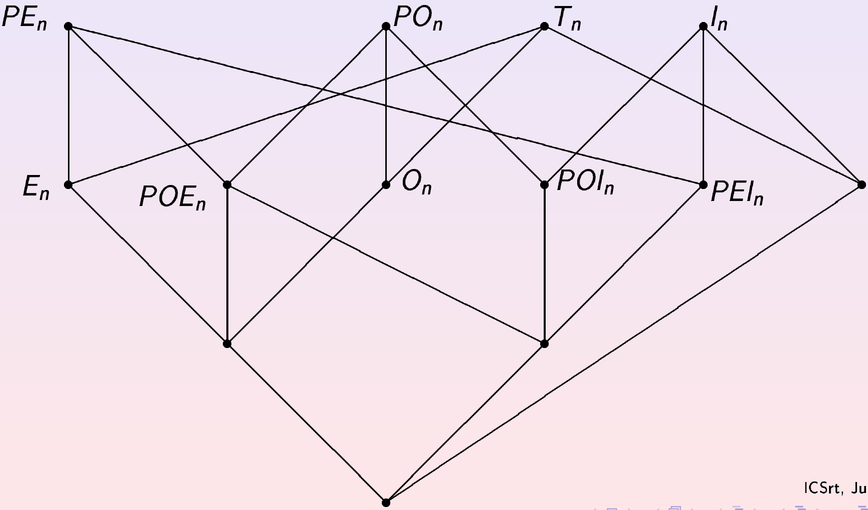

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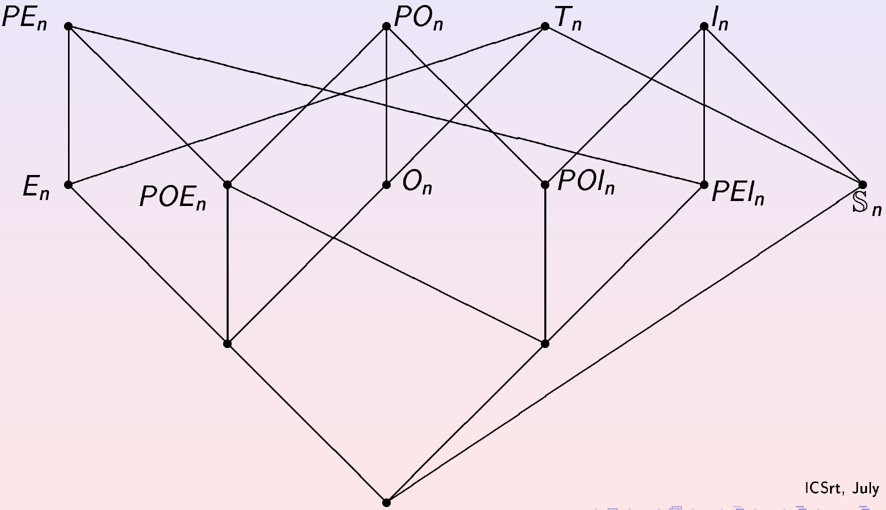

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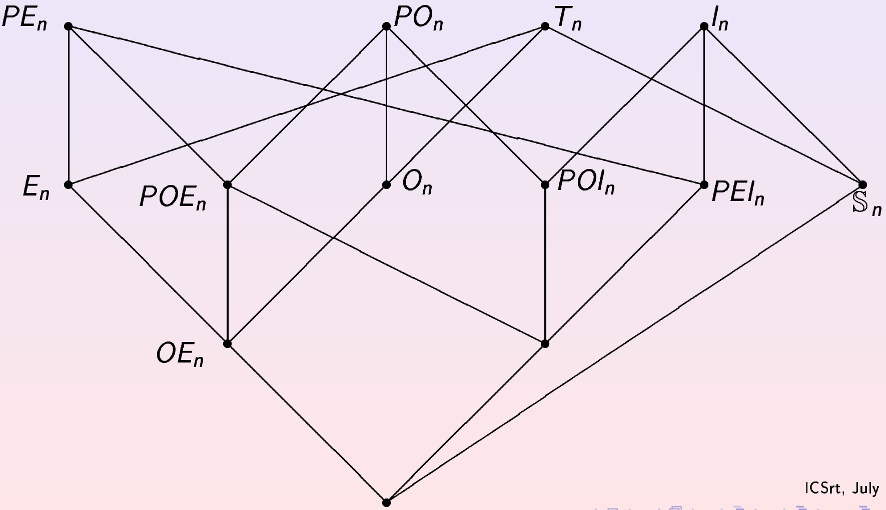

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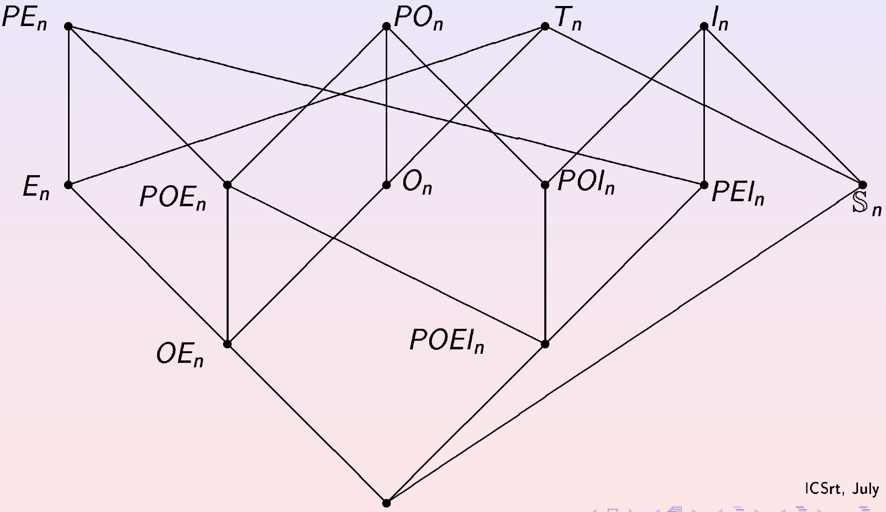

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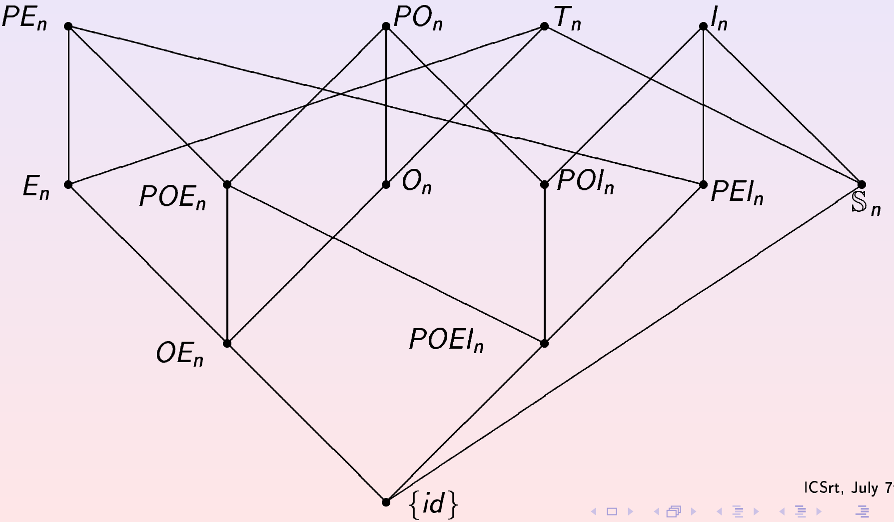

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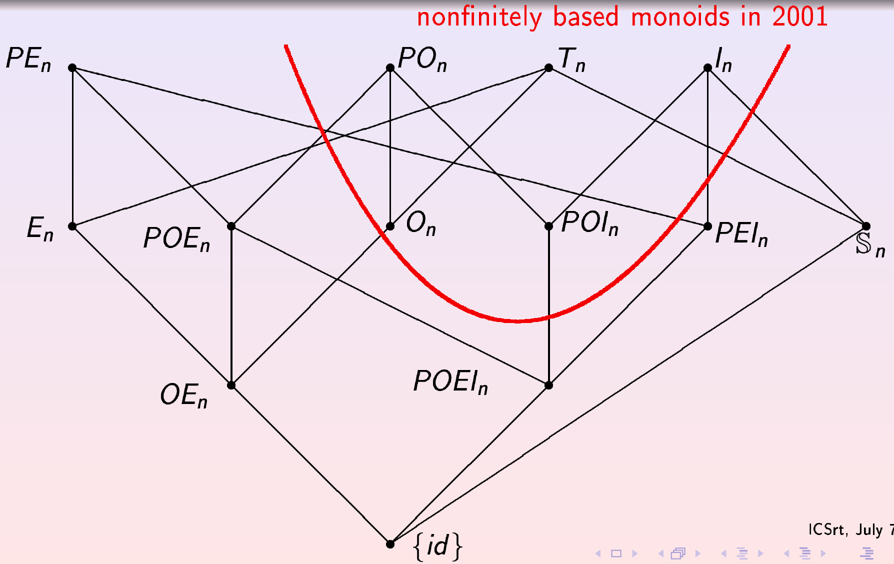

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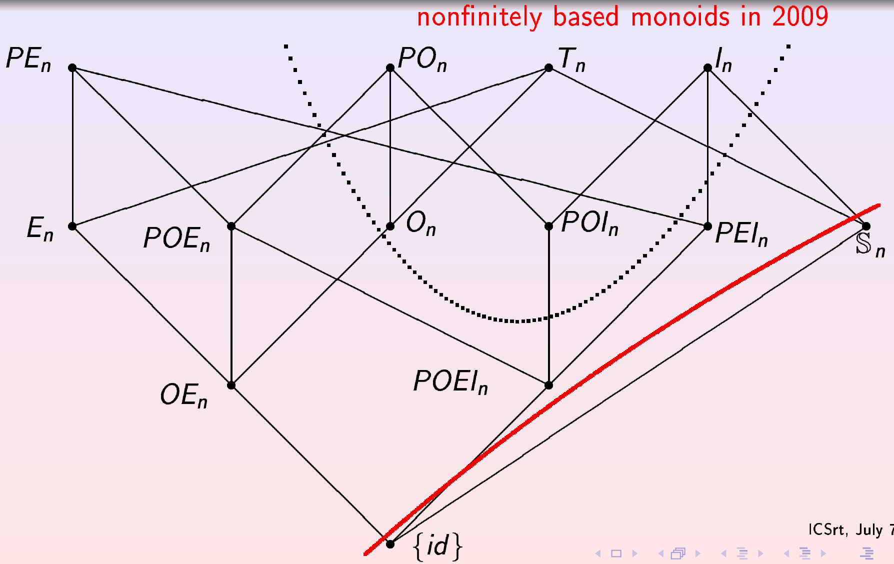

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In particular, Sapir has shown that the 6-element Brandt monoid B_2^1

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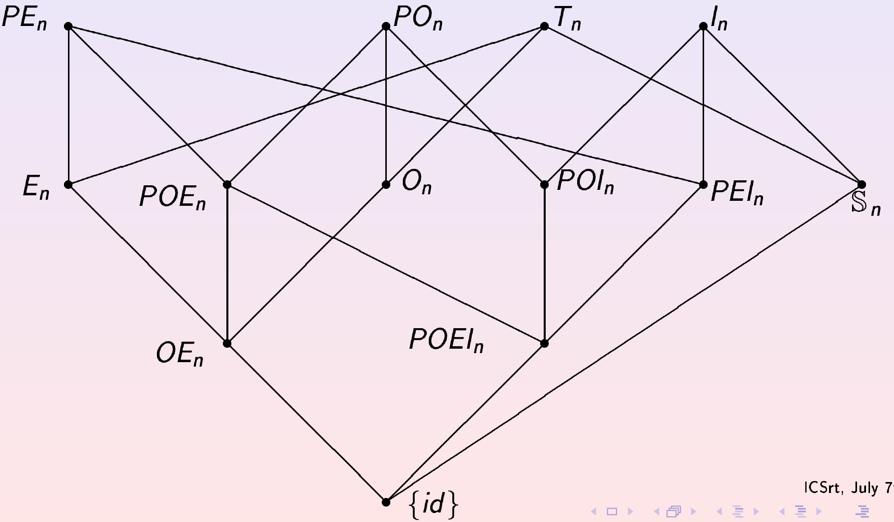

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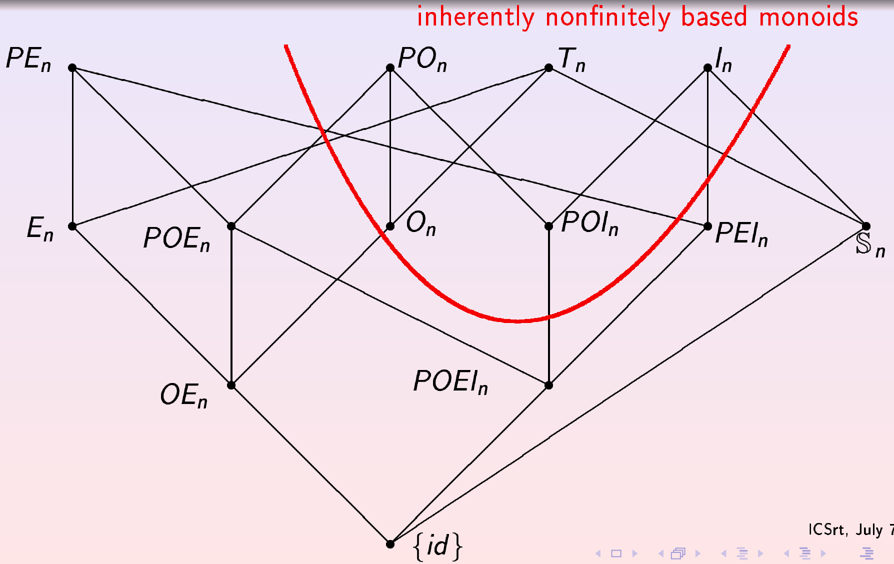

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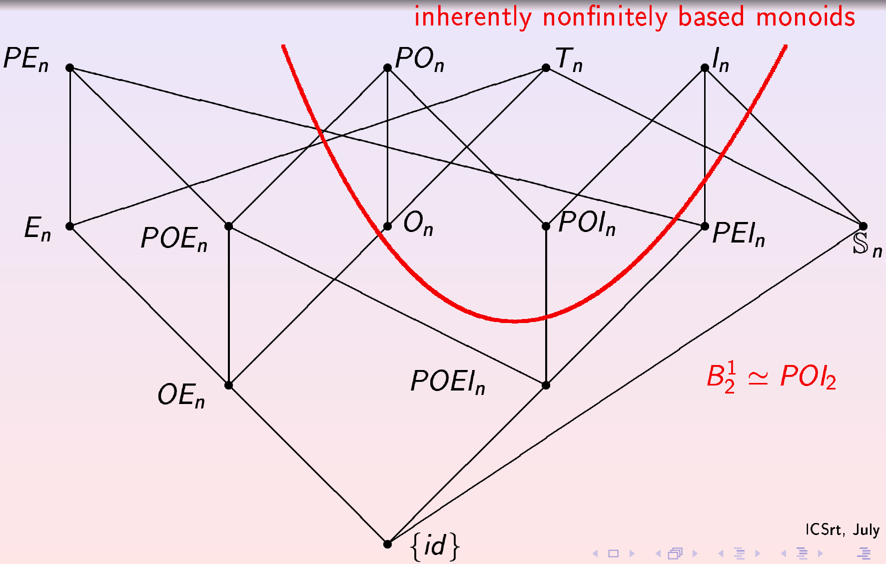

$B_2^1 \simeq POI_2$

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Surprisingly enough, a slight twist completely changes the picture: if one equips B_2^1 with the involution that fixes the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and swaps the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then the resulting involution monoid is inherently nonfinitely based (Dolinka).

Important Application

Thanks to this result, Auinger and I have finally managed to complete our study of the FBP for monoids of matrices over a finite field considered as unary semigroups under transposition.

This (and numerous further applications) can be found in K. Auinger, I. Dolinka, M. V. Volkov, Equational theories of semigroups with enriched signature, submitted, available online under http://arxiv.org/abs/0902.1155v2

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Theorem (Auinger–Volkov)

Let K be a finite field. The involution monoid $\langle \mathrm{M}_n(K), \cdot, {}^T \rangle$ is:

- nonfinitely based iff $n \geq 2$;
- inherently nonfinitely based iff either $n \geq 3$ or $n = 2$ and $|K| \not\equiv 3 \pmod{4}$.

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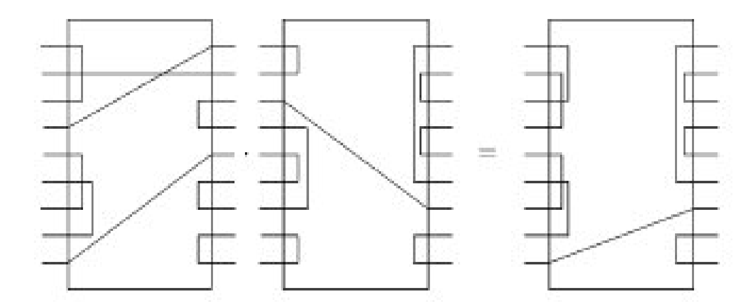

Brauer Monoids contd

Brauer monoids also have a natural involution (flipping). $\mathfrak{B}_n$ also satisfies the identity $xx^*x \triangleq x$ that excludes being inherently nonfinitely based. However, Brauer monoids turn out to be quite amenable to the method of critical semigroups.

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