Careful Synchronization of Partial Automata

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SATA, Sept 1st, 2008

Complete automata

First consider a complete DFA: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q is the state set
- Σ is the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function

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A DFA $\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

In symbols, $|\delta(Q, w)| = 1$. Here $|\delta(Q, w)| = \{\delta(q, w) \mid q \in Q\}$.
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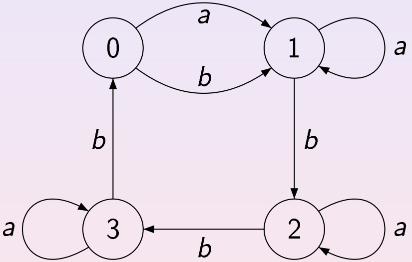

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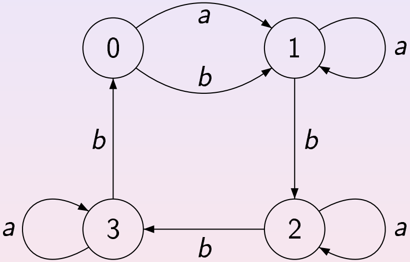

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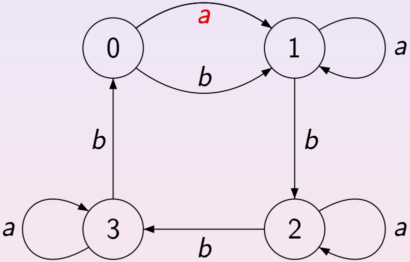

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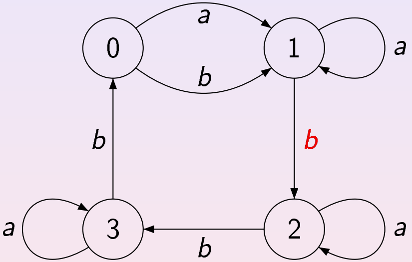

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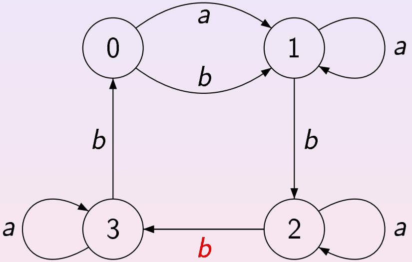

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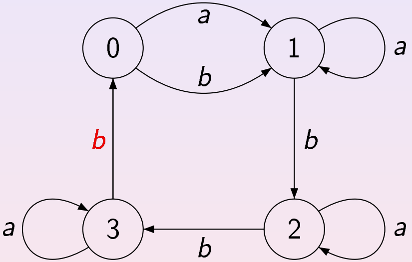

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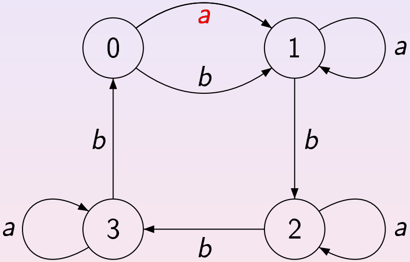

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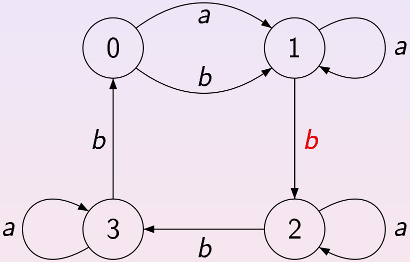

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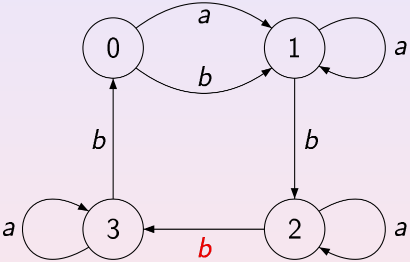

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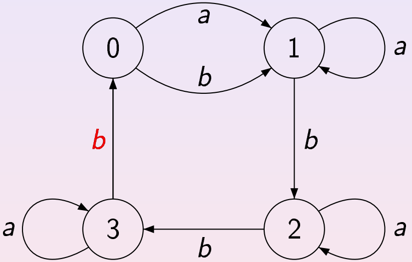

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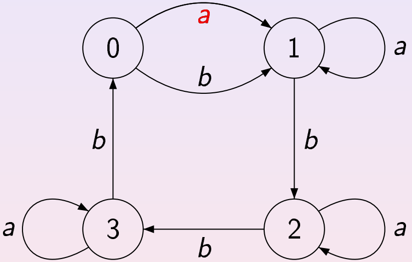

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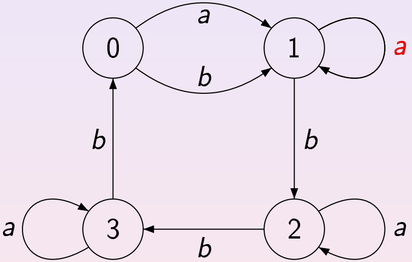

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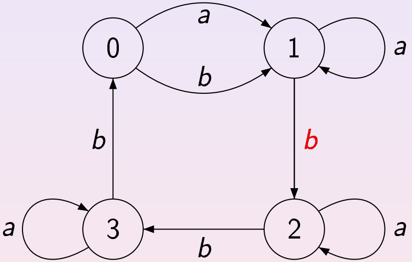

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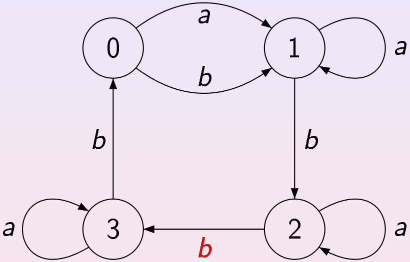

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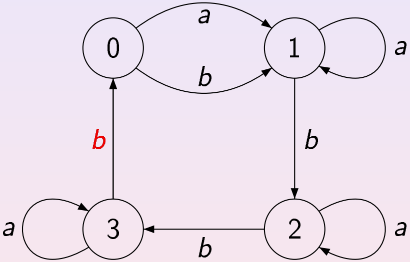

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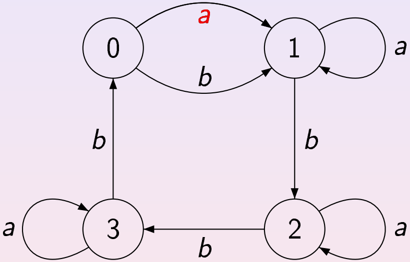

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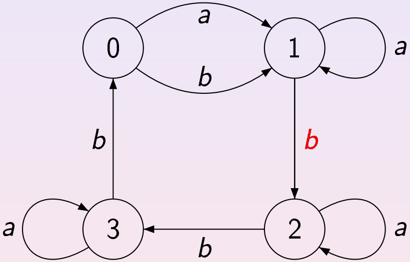

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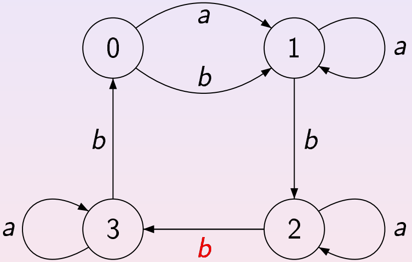

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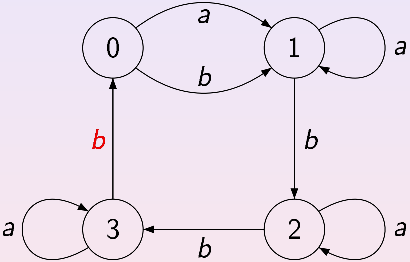

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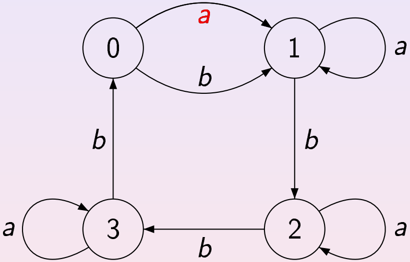

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The idea of synchronization is pretty natural and of obvious importance: we aim to restore control over a device whose current state is not known.

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Now consider a partial DFA: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

The only (but essential!) difference is that now the transition function δ is allowed to be partial, that is $\delta(q, a)$ can be undefined for some $q \in Q$ and $a \in \Sigma$.

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Say that $w = a_1 \cdots a_\ell$ with $a_1, \ldots, a_\ell \in \Sigma$ is a careful reset word for a partial DFA $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ if:

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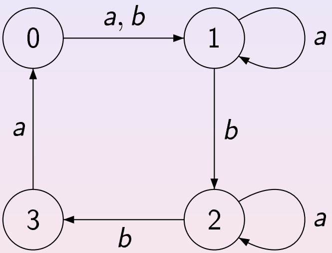

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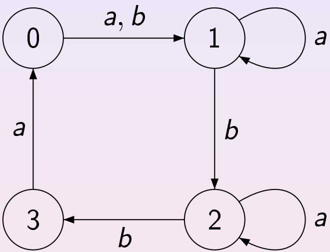

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Power automata

Question

Given an automaton, how to determine whether or not it is synchronizing?

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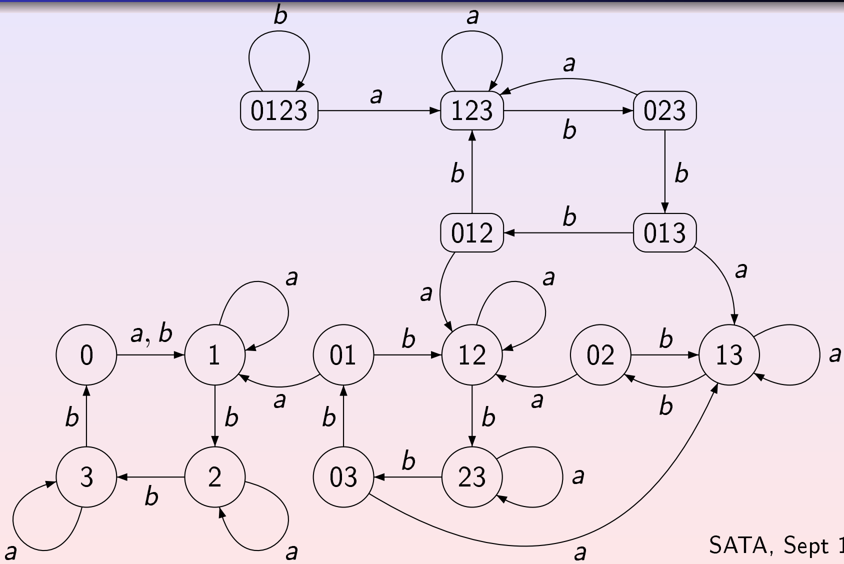
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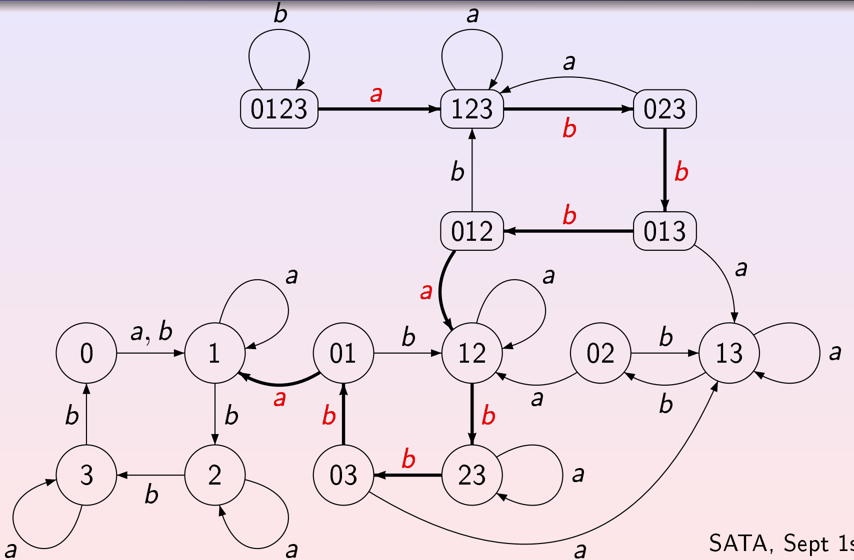

SATA, Sept 1st, 2008

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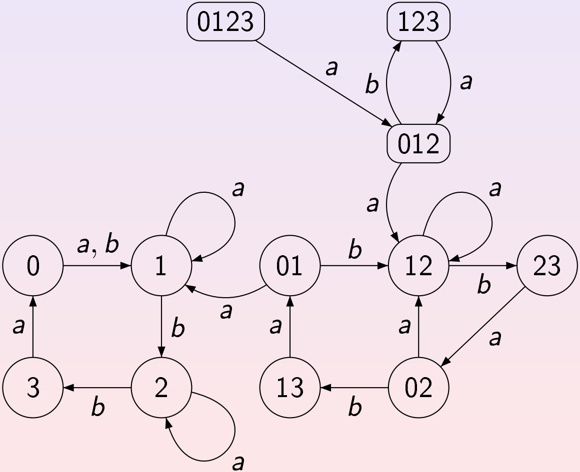
- states are the non-empty subsets of Q,
- $\delta(P, a) = \{\delta(p, a) \mid p \in P\}$ provided $\delta(p, a)$ is defined for each $p \in P$; otherwise $\delta(P, a)$ is undefined.

$w \in \Sigma^*$ is a careful reset word for $\mathscr{A}$ iff w labels a path in $\mathcal{P}'(\mathscr{A})$ starting at Q and ending at a singleton.

An example



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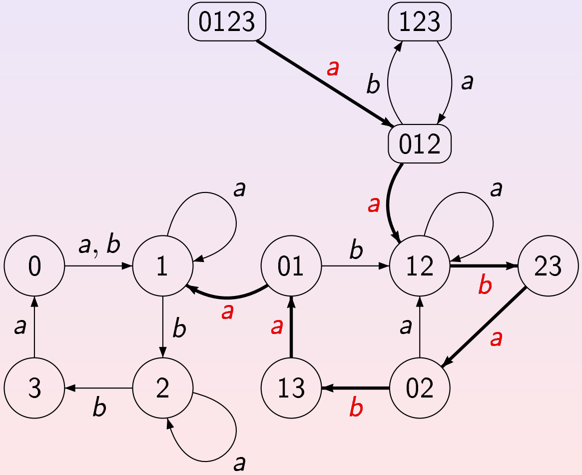

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Efficiency: complete case

Algorithms arising from the power set construction are exponential w.r.t. the size of $\mathscr{A}$.

For the complete case, the following result by Černý gives a polynomial algorithm:

Proposition (Černý, 1964)

A complete DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is synchronizing iff for every $q, q' \in Q$ there exists a word $w \in \Sigma^*$ such that $\delta(q, w) = \delta(q', w)$.

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Upper bounds

The power automaton construction gives an upper bound for the length of the shortest reset word of a given synchronizing automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ (complete or partial): if the automaton has n states, it has a reset word of length at most $2^n - n - 1$. (The shortest path in $\mathcal{P}'(\mathscr{A})$ starting at Q and ending at a singleton does not visit any state twice and ends at the first singleton it reaches.)

For the complete case, we know a much better bound due to Pin:

Theorem (Pin, 1983)

A complete synchronizing automaton with n states has a reset word of length at most $\dfrac{n^3 - n}{6}$.

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We do not know any reasonable upper bound for the partial case. It definitely cannot be polynomial because there exist non-polynomial lower bounds.

The first such bound has been found by Ito and Shikishima-Tsuji:

Theorem (Ito and Shikishima-Tsuji, 2004)

There exists a series of partial carefully synchronizing automata with n states and shortest careful reset words of length $O(2^{\frac{n}{2}})$.

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Open problems

Thus, we have the following open problems:

Upper Bound Problem

Does there exists a subexponential (in # of states) upper bound for the length of careful reset words of partial carefully synchronizing automata?

Lower Bound Problem

Does there exists an exponential (in # of states) lower bound for the length of careful reset words of partial carefully synchronizing automata?

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The Černý automata

Recall that the situation is not easy even for the complete case.

We know an upper bound: there always exists a reset word of length $\dfrac{n^3 - n}{6}$. What about a lower bound?

In his 1964 paper Černý constructed a series $\mathscr{C}_n$, $n = 2, 3, \dots,$ of synchronizing automata over 2 letters. The states of $\mathscr{C}_n$ are the residues modulo n, and the input letters a and b act as follows:

$$\delta(0, a) = 1,\ \delta(m, a) = m \text{ for } 0 < m < n,\ \delta(m, b) = m+1 \pmod{n}.$$

The automaton in our first example is $\mathscr{C}_4$.

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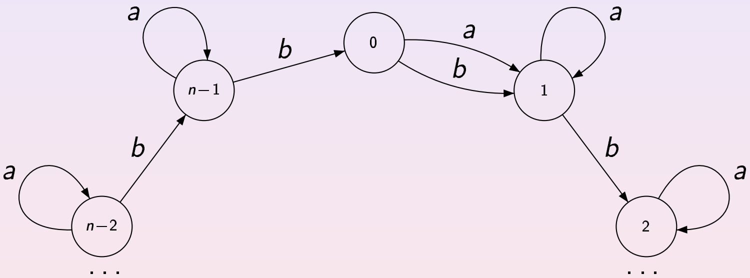

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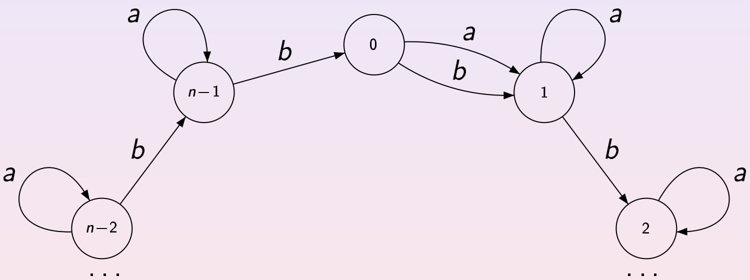

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Strongly connected synchronizing automata

Assume that a complete synchronizing automaton is strongly connected as a digraph. Such an automaton can be reset to any state. That is, to every state q of the automaton one can assign an instruction (a reset word) w_q such that following w_q one will surely arrive at q from any initial state.

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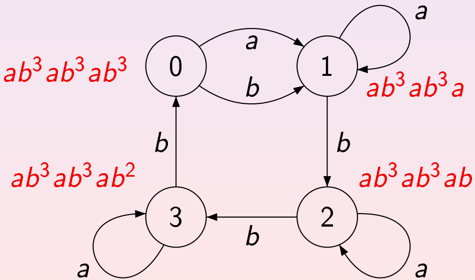

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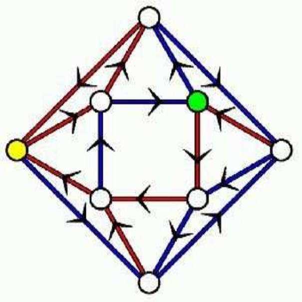

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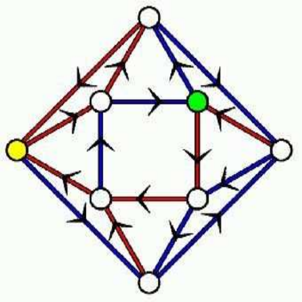

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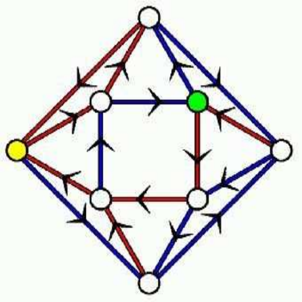

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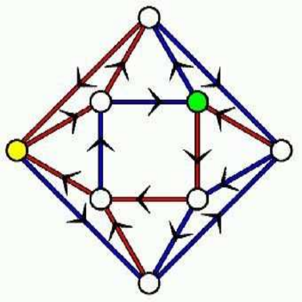

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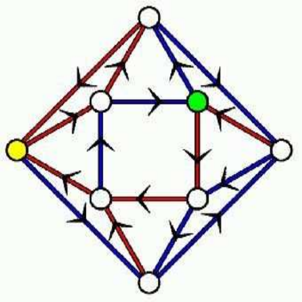

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Now suppose that we have a transport network, that is, a strongly connected digraph.

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