Primitive Digraphs, Markov Chains and Synchronizing Automata

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Definitions and Terminology

We consider complete deterministic finite automata (DFA) $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ where Q stands for the state set, Σ is the input alphabet, and $\delta : Q \times \Sigma \to Q$ is a (total) transition function.

To simplify notation we often write $q \,.\, w$ for $\delta(q, w)$ and $P \,.\, w$ for $\{\delta(q, w) \mid q \in P\}$.

$\mathscr{A}$ is called synchronizing if there is a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves $\mathscr{A}$ in one particular state no matter at which state in Q it started: $q \,.\, w = q' \,.\, w$ for all $q, q' \in Q$. In short, $|Q \,.\, w| = 1$.

Any w with this property is a reset word for $\mathscr{A}$.

Other names:

- for automata: directable, cofinal, collapsible, etc;
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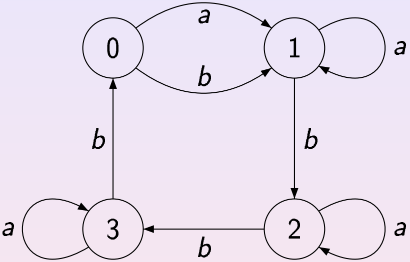

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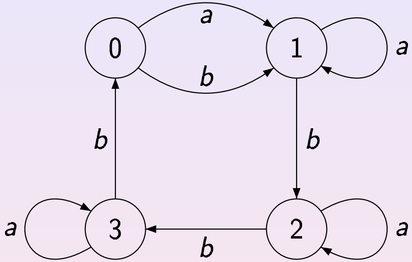

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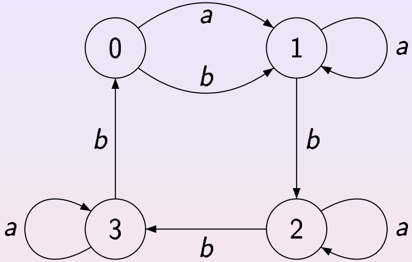

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The idea of synchronization is pretty natural and of obvious importance: we aim to restore control over a device whose current state is not known.

Think of a satellite which loops around the Moon and cannot be controlled from the Earth while "behind" the Moon (Černý's original motivation).

Independently, the same notion was discovered in coding theory by Shimon Even (Test for synchronizability of finite automata and variable length codes, IEEE Trans. Inform. Theory 10 (1964) 185–189). The name synchronizing seems to have originated from Even's paper.

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Černý Conjecture

The Černý conjecture is the claim that every synchronizing automaton with n states possesses a reset word of length $(n-1)^2$. The validity of the conjecture is main open problem of the area and arguably one of the most long-standing open problems in combinatorial theory of finite automata.

Define the Černý function $C(n)$ as the maximum reset threshold for synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

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Why is the problem so surprisingly difficult?

One of the reasons: "slowly" synchronizing automata turn out to be extremely rare. Only one infinite series of n-state synchronizing automata with reset threshold $(n-1)^2$ is known (due to Černý, 1964), with a few (actually, 8) sporadic examples for $n \leq 6$.

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Advantage of Being Old

Thus, the pattern is:

$(n-1)^2$ the first gap the "island" the second gap

The second gap first appears at 9 states and grows rather regularly with the number of states. The size of the island depends only on the parity of the number of states.

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Exponents of Non-negative Matrices

A non-negative matrix A is said to be primitive if some power A^k is positive. The minimum k with this property is called the exponent of A, denoted $\exp A$.

Helmut Wielandt proved in 1950 that for any primitive $n \times n$-matrix A, one has $\exp A \leq n^2 - 2n + 2 = (n-1)^2 + 1$, and this bound is tight. Possible exponents of $n \times n$-matrices were intensively studied in the 1960s, and it was discovered that two extreme values are each attained by a unique matrix, then there is a gap followed by an island followed by another gap. The sizes of the gaps and of the island perfectly match the sizes of the gaps and of the islands in possible reset thresholds of synchronizing automata with n states — basically one has the same picture shifted by 1. Clearly, this cannot be a mere coincidence.

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Digraphs and Matrices

A directed graph (digraph) is a pair $D = \langle V, E \rangle$.

- V set of vertices
- $E \subseteq V \times V$ set of edges

This definition allows loops but excludes multiple edges.

The matrix of a digraph $D = \langle V, E \rangle$ is just the incidence matrix of the edge relation, that is, a $V \times V$-matrix whose entry in the row v and the column v' is 1 if $(v, v') \in E$ and 0 otherwise.

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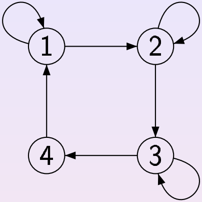

(with respect to the chosen numbering of its vertices) is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Conversely, given an $n \times n$-matrix $P = (p_{ij})$ with non-negative real entries, we assign to it a digraph $D(P)$ on the set $\{1, 2, \dots, n\}$ as follows: (i, j) is an edge of $D(P)$ if and only if $p_{ij} > 0$.

This 'two-way' correspondence allows us to reformulate in terms of digraphs several important notions and results which originated in the classical Perron–Frobenius theory of non-negative matrices.

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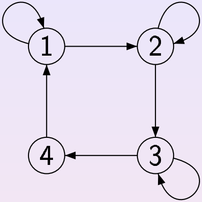

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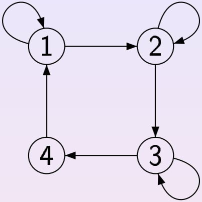

(with respect to the chosen numbering of its vertices) is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Conversely, given an $n \times n$-matrix $P = (p_{ij})$ with non-negative real entries, we assign to it a digraph $D(P)$ on the set $\{1, 2, \dots, n\}$ as follows: (i, j) is an edge of $D(P)$ if and only if $p_{ij} > 0$.

This 'two-way' correspondence allows us to reformulate in terms of digraphs several important notions and results which originated in the classical Perron–Frobenius theory of non-negative matrices.

Digraphs and Colorings

By a coloring of a digraph we mean assigning labels from an alphabet Σ to edges such that the digraph labeled this way becomes a DFA.

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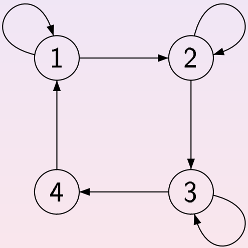

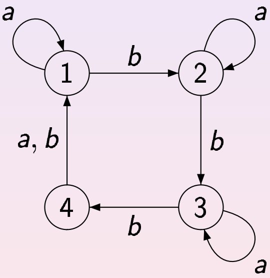

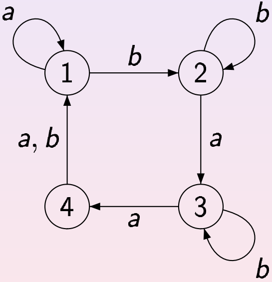

Primitive Digraphs

A digraph D is primitive if D is strongly connected and the greatest common divisor of the lengths of all cycles in D is equal to 1.

Adler, Goodwyn, Weiss (Equivalence of topological Markov shifts, Israel J. Math. 27 (1977) 49–63):
Underlying digraphs of strongly connected synchronizing automata are primitive.

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Exponents

A digraph D is primitive iff there exists t such that for each pair of vertices there exists a path between them of length exactly t. (This is equivalent to saying that the t-th power of the matrix of D is positive.) The least t with this property is called the exponent of the digraph D and is denoted by $\gamma(D)$.

1950, Wielandt: The exponent of every primitive digraph on n vertices is not greater than $(n-1)^2+1$ and this bound is tight.

1964, Dulmage–Mendelsohn: There is exactly one primitive digraph on n vertices with exponent $(n-1)^2+1$ and exactly one primitive digraph on n vertices with exponent $(n-1)^2$.

If $n > 4$ is even, then there is no primitive digraph D on n vertices such that $n^2-4n+6 < \gamma(D) < (n-1)^2$.

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Exponents vs Reset Lengths

Exponents of primitive digraphs with 9 vertices vs reset thresholds of 2-letter strongly connected synchronizing automata with 9 states

N	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51
# of primitive digraphs with exponent N	1	1	0	0	0	0	0	1	1	2	0	0	0	0	3
# of 2-letter synchronizing automata with reset threshold N	0	1	0	0	0	0	0	1	2	3	0	0	0	4	4

The Wielandt Automaton

The Wielandt automaton $\mathscr{W}_n$ is a (unique) coloring of the Wielandt digraph W_n with $\gamma(W_n) = (n-1)^2 + 1$. The Wielandt digraph has n vertices $1, 2, \dots, n$, say, and the following $n+1$ edges: $(i, i+1)$ for $i = 1, \dots, n-1$, $(n, 1)$, and $(n, 2)$.

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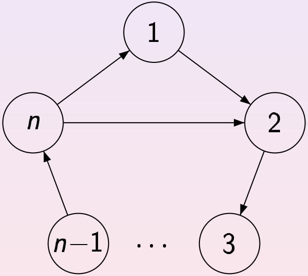

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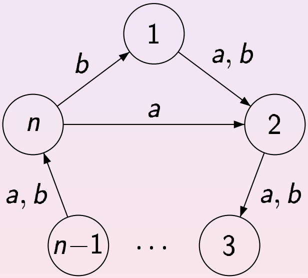

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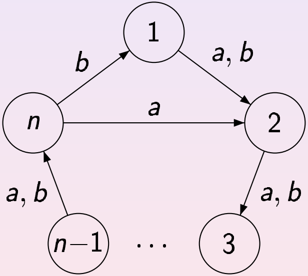

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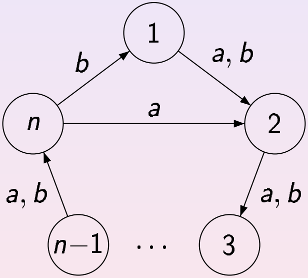

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In a similar way, each digraph with large exponent generates slowly synchronizing automata.

Observation

Let a strongly connected synchronizing automaton with n states and reset threshold t be a coloring of a digraph D. Then

$$\gamma(D) \leq t + n - 1.$$

Colorings of Digraphs with Large Exponents

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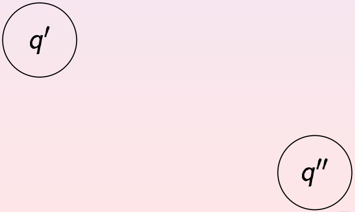

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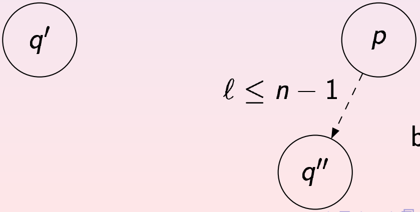

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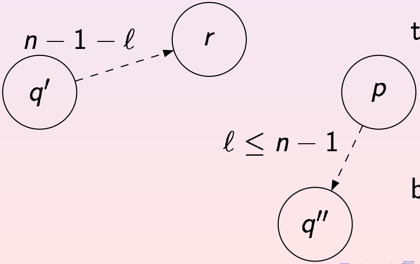

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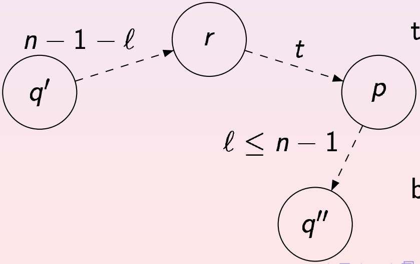

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$$\gamma(D) \leq t + n - 1.$$

For instance, the reset threshold t of the Wielandt automaton $\mathscr{W}_n$ must satisfy

$$t \geq \gamma(W_n) - n + 1 = (n-1)^2 + 1 - n + 1 = n^2 - 3n + 3,$$

and it is easy to find a reset word of length $n^2 - 3n + 3$.

The Černý Automaton

There are slowly synchronizing automata that cannot be obtained as colorings of a digraph with large exponent. For instance, the Černý automaton $\mathscr{C}_n$ has reset threshold $(n-1)^2$ while its underlying digraph has exponent $n-1$.

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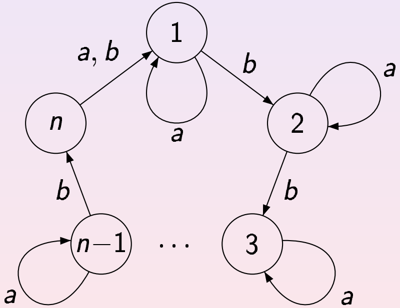

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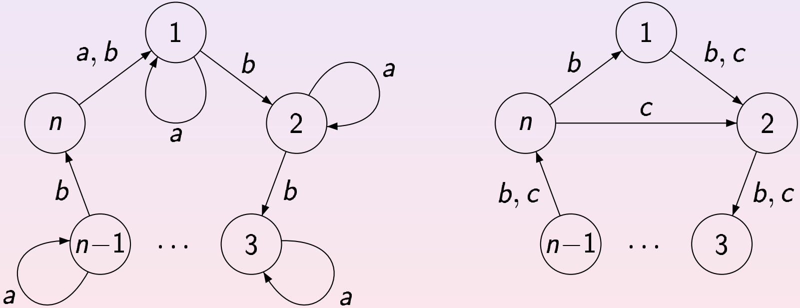

However, $\mathscr{C}_n$ becomes $\mathscr{W}_n$ under the action of b and $c = ab$.

The Černý Automaton

Let w be a shortest reset word for $\mathscr{C}_n$. It must end with a and every other occurrence of a in w is followed by an occurrence of b. Thus, $w = w'a$ where w' can be rewritten into a word v over the alphabet $\{b, c\}$. Since w' and v act in the same way, the word vc is a reset word for $\mathscr{W}_n$. Hence $|v| \geq n^2 - 3n + 2$.

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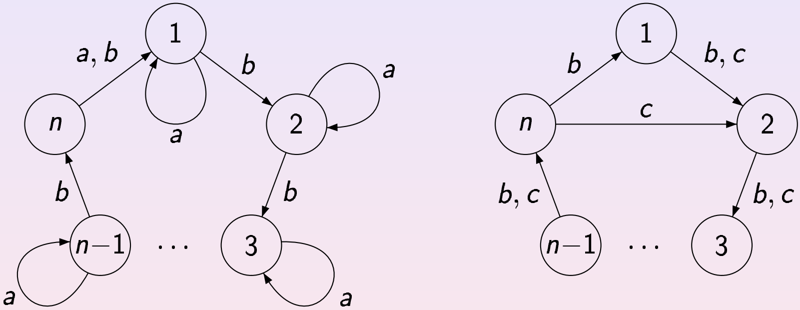

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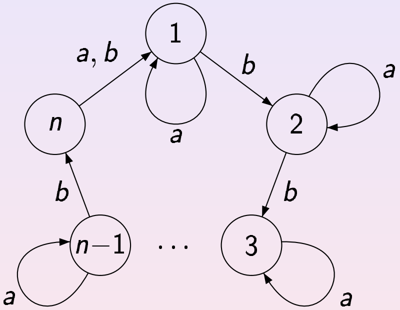

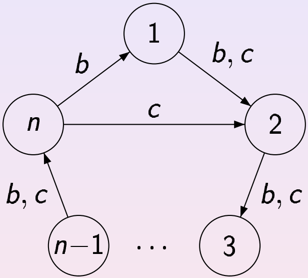

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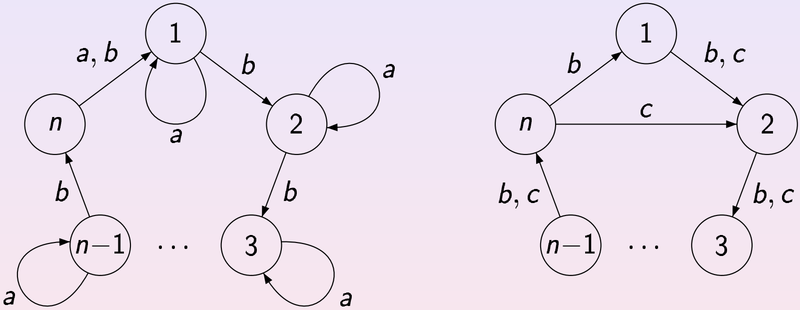

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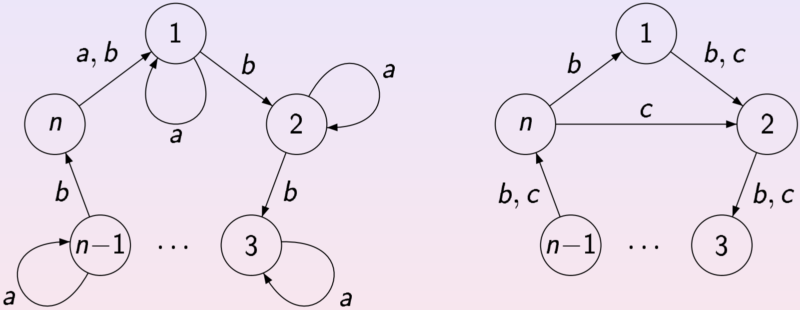

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Thus, it is the Wielandt digraph that stays behind the Černý automaton!

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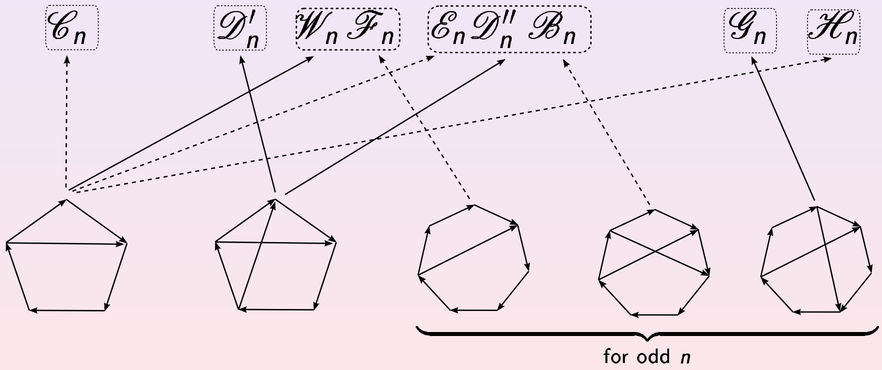

Extensibility

How to get upper bounds for reset threshold?

For $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, a subset $P \subset Q$ is extensible if $P \supseteq R \, . \, w$ for some $w \in \Sigma^*$ of length at most $n = |Q|$ and some $R \subseteq Q$ with $|R| > |P|$. It was conjectured for some time that in synchronizing automata every proper non-singleton subset is extensible. Observe that this would imply the Černý conjecture.

Indeed, some letter a should sent two states q, q' to the same state p. Let $P_0 = \{q, q'\}$ and, for $i > 0$, let P_i be such that $|P_i| > |P_{i-1}|$ and $P_{i-1} \supseteq P_i \, . \, w_i$ for some word w_i of length $\leq n$. Then in at most $n - 2$ steps the sequence $P_0, P_1, P_2, \ldots$ reaches Q and

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For $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, a subset $P \subset Q$ is extensible if $P \supseteq R\,.\,w$ for some $w \in \Sigma^*$ of length at most $n = |Q|$ and some $R \subseteq Q$ with $|R| > |P|$. It was conjectured for some time that in synchronizing automata every proper non-singleton subset is extensible. Observe that this would imply the Černý conjecture.

Indeed, some letter a should sent two states q, q' to the same state p. Let $P_0 = \{q, q'\}$ and, for $i > 0$, let P_i be such that $|P_i| > |P_{i-1}|$ and $P_{i-1} \supseteq P_i\,.\,w_i$ for some word w_i of length $\leq n$. Then in at most $n - 2$ steps the sequence $P_0, P_1, P_2, \ldots$ reaches Q and

$$Q\,.\,w_{n-2}w_{n-3}\cdots w_1 a = \{p\},$$

that is, $w_{n-2}w_{n-3}\cdots w_1 a$ is a reset word. The length of this reset word is at most $n(n-2) + 1 = (n-1)^2$.

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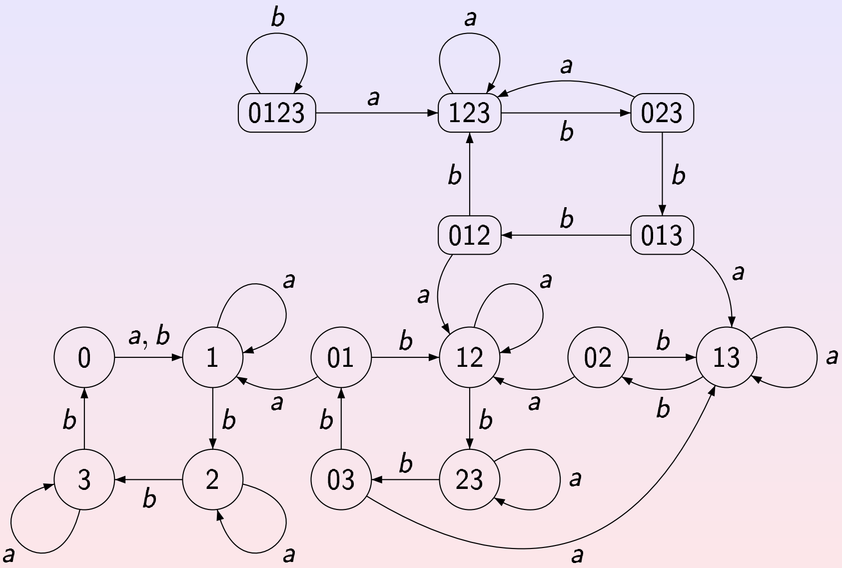
Example

For an illustration, consider the subset automaton of the Černý automaton $\mathscr{C}_4$.

Example



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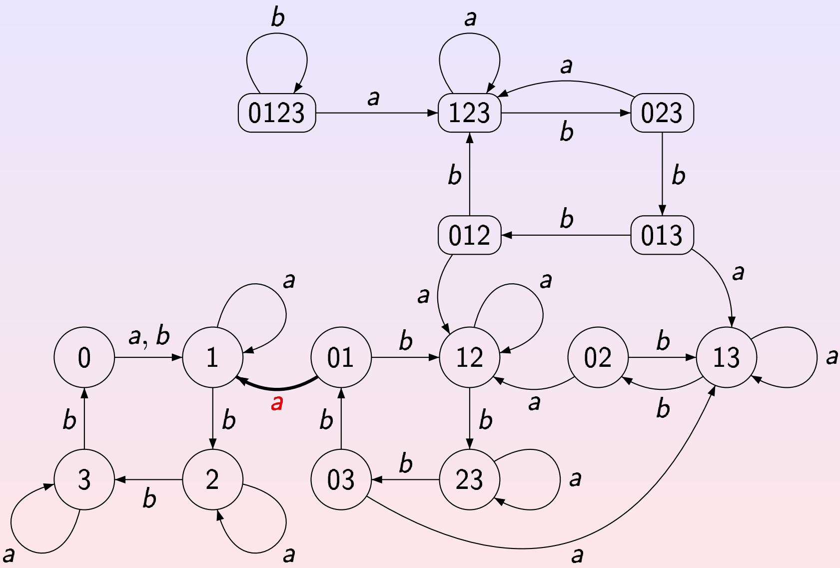

April 26, 2015

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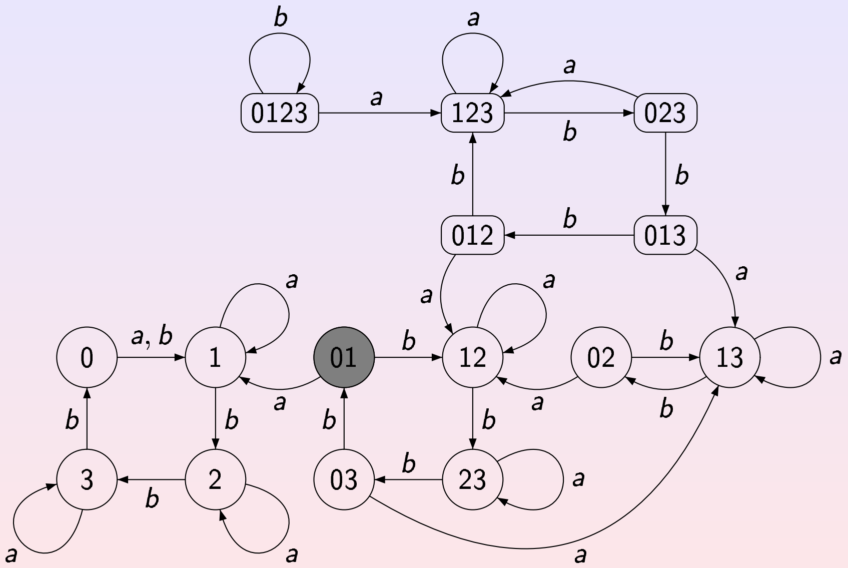

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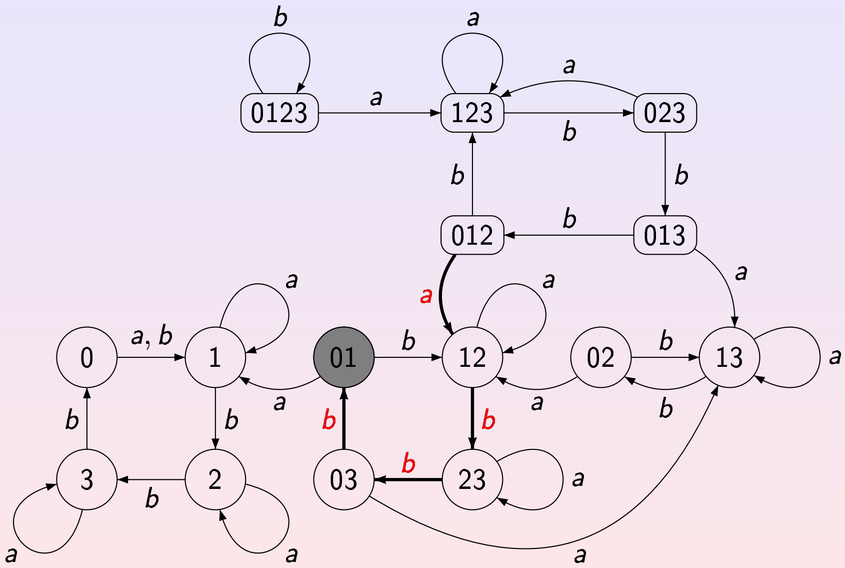

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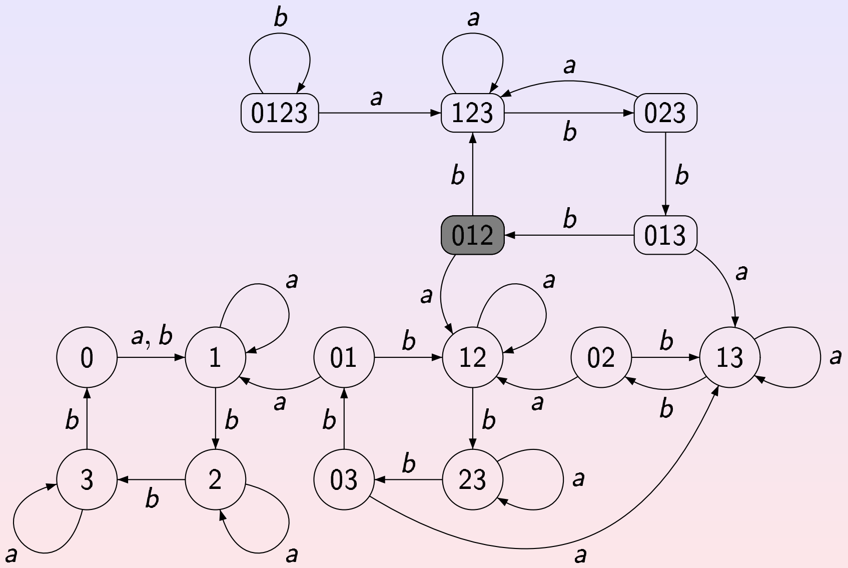

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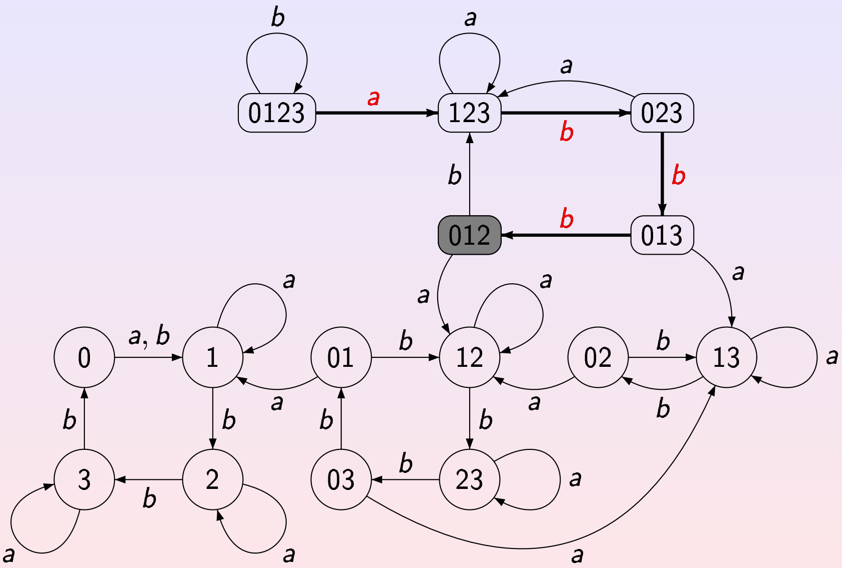

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Applications of Extensibility

Several important results confirming the Černý conjecture for various partial cases have been proved by verifying the extensibility conjecture for the corresponding automata. This includes:

- Louis Dubuc's result for automata in which a letter acts on the state set Q as a cyclic permutation of order $|Q|$ (Sur le automates circulaires et la conjecture de Černý, RAIRO Inform. Theor. Appl., 32 (1998) 21–34 [in French]).
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Linearization

We associate a natural linear structure with each automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$. Assume that $Q = \{1, 2, \dots, n\}$ and assign to each subset $K \subseteq Q$ its characteristic vector $[K] \in \mathbb{R}^n$ (the space of n-dimensional column vectors): the i-th entry of $[K]$ is 1 if $i \in K$, otherwise the entry is 0.

For each word $w \in \Sigma^*$, its action on Q gives rise to a linear transformation of $\mathbb{R}^n$; we denote by $[w]$ the matrix of this transformation in the standard basis $[1], \dots, [n]$ of $\mathbb{R}^n$. Clearly, the matrix $[w]$ has exactly one non-zero entry in each column and this entry is equal to 1.

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For vectors $g_1, g_2 \in \mathbb{R}^n$, we denote their usual inner product by (g_1, g_2). Then for any $K, L \subset Q$, we have $([K], [L]) = |K \cap L|$. Denote by $\mathbf{1}_n$ the uniform stochastic vector in $\mathbb{R}^n$, that is, the vector with all entries equal to $\frac{1}{n}$. Then the fact that a word w extends a subset $K \subset Q$ (that is, the inequality $|K| < |K\,.\,w^{-1}|$) can be rewritten as $([K], \mathbf{1}_n) < ([w]^T[K], \mathbf{1}_n)$.

Thus, the extension method amounts to finding a state q, a letter a, and a sequence of words $w_1, w_2, \dots, w_d$ such that

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Raphaël Jungers (The synchronizing probability function of an automaton, SIAM J. Discrete Math. 26 (2011) 177–192) has suggested an interesting idea that in our notation can be described as follows: one should substitute the uniform stochastic vector $\mathbf{1}_n$ by an adaptive positive stochastic vector p which can depend on both the automaton $\mathscr{A}$ and the given proper subset $K \subset Q$ but has the property that there exists a word v of length at most $|Q|$ such that $([v]^T[K], p) > ([K], p)$. Jungers has explored this idea using techniques from linear programming and has proved that such a positive stochastic vector indeed exists for every synchronizing automaton and every proper subset.

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Markov Chains

Assume that $\Sigma = \{a_1, a_2, \dots, a_k\}$. Each positive stochastic vector $\pi \in \mathbb{R}^k_+$ defines a probability distribution on Σ. Consider a process in which an agent randomly walks on the underlying graph of $\mathscr{A}$, choosing for each move the edge labeled a_i with probability $p(a_i)$. This is a Markov chain with the transition matrix

$$S = S(\mathscr{A}, \pi) = \sum_{i=1}^{k} p(a_i)[a_i].$$

By basic properties of Markov chains, there exists the stationary distribution $\alpha \in \mathbb{R}^n_+$ of this Markov chain, that is, a unique positive stochastic vector satisfying $S\alpha = \alpha$.

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Berlinkov's Result

Theorem (Berlinkov, 2012)

Let $\mathscr{A}$ be a synchronizing automaton with n states and k letters, $\pi \in \mathbb{R}^k_+$ a positive stochastic vector, and α the stationary distribution of the Markov chain with the transition matrix $S(\mathscr{A}, \pi)$. Then there exist a state q, a letter a, and a sequence of words $w_1, w_2, \dots, w_d$ of length at most n such that

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An immediate application: a new proof of the Černý conjecture for automata with Eulerian digraphs. In this case the matrix $S(\mathscr{A}, \pi)$ is doubly stochastic whence the uniform vector $\mathbf{1}_n$ is its stationary distribution and $d \leq n - 2$.