Semigroup identities of groups: Shirshov's problems and group radicals

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RTC, July 16, 2011

Maltsev's Identities

A semigroup identity is a pair of semigroup words (u, v) usually written as a formal equality $u \doteq v$. It is called non-trivial if $u \neq v$.

Maltsev (1953) observed that every nilpotent group satisfies a non-trivial semigroup identity while the free metabelian group with two generators does not.

Moreover, he proved that the variety of all nilpotent groups of class $\leq c$ can be defined by a single semigroup identity.

Let $X_0 = x$, $Y_0 = y$, and for $k > 0$ let $X_k = X_{k-1} z_k Y_{k-1}$, $Y_k = Y_{k-1} z_k X_{k-1}$. Then a group G is nilpotent of class c iff G satisfies the identity $X_c \doteq Y_c$.

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Shirshov's Identities

Shirshov (1963) considered another interesting sequence of semigroup identities. He started with the identity $xy \doteq yx$ and applied the substitution $x \mapsto xy$, $y \mapsto yx$.

The n-th identity is $T_n \doteq \overleftarrow{T_n}$ where T_n is the n-th Thue-Morse word and $\overleftarrow{T_n}$ is its mirror image.

Shirshov denoted by $\mathbf{N}^{(k)}$ the group variety defined by the identity $T_k \doteq \overleftarrow{T_k}$ and referred to groups from $\mathbf{N}^{(k)}$ as ν_k-groups.
He gave a complete and transparent description of finite ν-groups: a finite group G is a ν-group iff G is an extension of a nilpotent group of odd order by a 2-group.

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Engel Groups

Shirshov's goal however was to understand the relations between ν-groups and Engel groups.

Recall the standard notation for iterated commutators: $[x, {}_1y] = [x, y] = x^{-1}y^{-1}xy$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$.

The variety $\mathbf{E}^{(k)}$ of all k-Engel groups is defined by the (group) identity $[x, {}_ny] \doteq 1$.

Obviously, $\mathbf{E}^{(1)} = \mathbf{N}^{(1)}$ is the variety of all Abelian groups. It is easy to see that $\mathbf{E}^{(2)} = \mathbf{N}^{(2)}$. Shirshov proved that $\mathbf{E}^{(3)} \subset \mathbf{N}^{(3)}$. Moreover, $\mathbf{E}^{(3)}$ can be defined by two semigroup identities, namely,

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Shirshov adds that if the two last questions both have negative answers then every Engel group would be locally nilpotent.

Shirshov's Problems: A Discussion

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Do there exist Engel groups which are not ν-groups?

Shirshov means here bounded Engel groups (groups from $\bigcup_k \mathbf{E}^{(k)}$).

In fact, I do not even know if $\mathbf{E}^{(4)}$ is contained in any $\mathbf{N}^{(k)}$. Havas and Vaughan-Lee have recently proved that $\mathbf{E}^{(4)}$ is locally nilpotent (G. Havas, M. R. Vaughan-Lee. 4-Engel groups are locally nilpotent. IJAC, Vol.15 (2005) 649–682).

Ol'shanskij and Storozhev have constructed a 2-generated group which satisfies a non-trivial semigroup identity but is not a periodic extension of a locally soluble group (A. Yu. Ol'shanskij, A. Storozhev. A group variety defined by a semigroup law. J. Aust. Math. Soc., Ser. A, Vol.60 (1996) 255–259).

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We have seen some sequences of words and identities.

Can we speak of their limits in some reasonable sense?

Yes, we can!

Let A be a finite alphabet, A^+ the set of all (semigroup) words over A—the free semigroup over A. Define the function $d : A^+ \times A^+ \to \mathbb{R}_+ \cup \{0\}$ as follows:

$$d(u, v) = 2^{-r(u,v)}$$

where $r(u, v)$ is the minimum size of a semigroup violating $u \simeq v$. It is easy to see that d is a distance on A^+.

Example: $d(x, x^2) = \frac{1}{4}$ since the identity $x \simeq x^2$ fails in the 2-element group.

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Profinite Completions

We have seen some sequences of words and identities.
Can we speak of their limits in some reasonable sense?
Yes, we can!

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So $\langle A^+, d\rangle$ becomes a metric space.

Its completion $\overline{A^+}$ is called the free profinite semigroup over A and its elements (limits of Cauchy sequences of words) are profinite words.

Similarly, one defines the free profinite group over A (as the completion of the free group over A with respect to an analogous metric).

Warning: While the free semigroup A^+ embeds into the free group over A, the free profinite semigroup $\overline{A^+}$ is "much bigger" than the free profinite group over A and contains uncountably many disjoint copies of the latter.

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$\mathbf{G}_{2'}$, the pseudovariety of all groups of odd order, is defined by $x^{2^\omega - 1} \doteq 1$ where $x^{2^\omega - 1} = \lim_{n\to\infty} x^{2^{n!}-1}$.

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Radical Pseudovarieties

Theorem (Almeida, Margolis, Steinberg, ∼, 2010)

Let $\mathbf{X}$ be a radical pseudovariety. Then there exists a profinite word w in two variables such that $\mathbf{X}$ is defined by the profinite identity $w \doteq 1$.

Remark 1. The result depends on the classification of finite simple groups.

Remark 2. This is a compactness argument; the explicit construction of w for some X may be a difficult task —in general even algorithmically undecidable.

Remark 3. For $\mathbf{G}_{\mathrm{sol}}$, an explicit construction may be derived from a recent work by T. Bandman e.a. (Two-variable identities for finite solvable groups. C. R. Acad. Sci. Paris Sér. I Math. Vol.337 (2003) 581–586) or J. N. Bray, J. S. Wilson, R. A. Wilson (A characterization of finite soluble groups by laws in two variables. Bull. London Math. Soc. Vol.37 (2005) 179–186.)

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Fitting Pseudovarieties

We have similar (but more complicated) results for Fitting pseudovarieties, i.e. pseudovarieties satisfying the second property in the definition of a radical class but not the third.

If $\mathbf{X}$ is a Fitting pseudovariety, then for every finite group G the $\mathbf{X}$-radical $G_{\mathbf{X}}$ of G exists but the subgroup $(G/G_{\mathbf{X}})_{\mathbf{X}}$ may be non-trivial.

Example: $\mathbf{G}_{\mathrm{nil}}$, the class of all finite nilpotent groups.

$\mathbf{G}_{\mathrm{nil}}$ is defined by the profinite Engel identity $[x,_\omega y] \doteq 1$, where $[x,_\omega y] = \lim_{n\to\infty}[x,_{n!} y]$.

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Characterization of the Radical

Let **X** be a Fitting pseudovariety. We say that the **X**-radical is characterized by a set W of profinite words if, for every finite group G,

$$G_{\mathbf{X}} = \{a \in G : \forall b_1, \ldots, b_r \in G \; \forall w \in W, \; w(a, b_1, \ldots, b_r) = 1\}.$$

The number $r + 1$ is the arity of the characterization.

Examples:
The $\mathbf{G}_{\mathrm{nil}}$-radical is characterized by the profinite word $[x_2, {}_\omega x_1]$.
The $\mathbf{G}_2$-radical is characterized by the profinite word $[x_2, {}_\omega x_1]\, x_1^{2^\omega}$.

In both cases we have a singleton binary characterization.

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Is there a singleton binary characterization of the solvable radical?

It follows for a result by R. Guralnick e.a. (Thompson-like characterization of the solvable radical. J. Algebra, Vol.300 (2006) 363–375) that the solvable radical admits a binary characterization but, perhaps, involving infinitely many profinite words.

T. Bandman e.a. (Engel-like characterization of radicals in finite dimensional Lie algebras and finite groups. Manuscripta Math., Vol.119 (2006) 365–381) have formulated a general conjecture whose validity would imply a positive answer to the above problem.

They established the analog of the conjecture for finite-dimensional Lie algebras while J. S. Wilson (Characterization of the solvable radical by a sequence of words. J. Algebra, Vol.326 (2011) 286–289) has recently proved their conjecture for the class of finite linear groups.

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