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Interpreting graphs in 0-simple semigroups with involution with applications to computational complexity and the finite basis problem

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Overview

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- Computational complexity of the (Pseudo)variety Membership Problem (PMP)

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In Lecture 1 I'll explain our motivation while in Lectures 2 and 3 I'll present some new techniques that has proved to be very efficient for semigroups equipped with an additional unary operation

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- is easy to explain;
- is hard to prove;
- has many surprising connections,
- is related to my recent research.

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Hydra automata

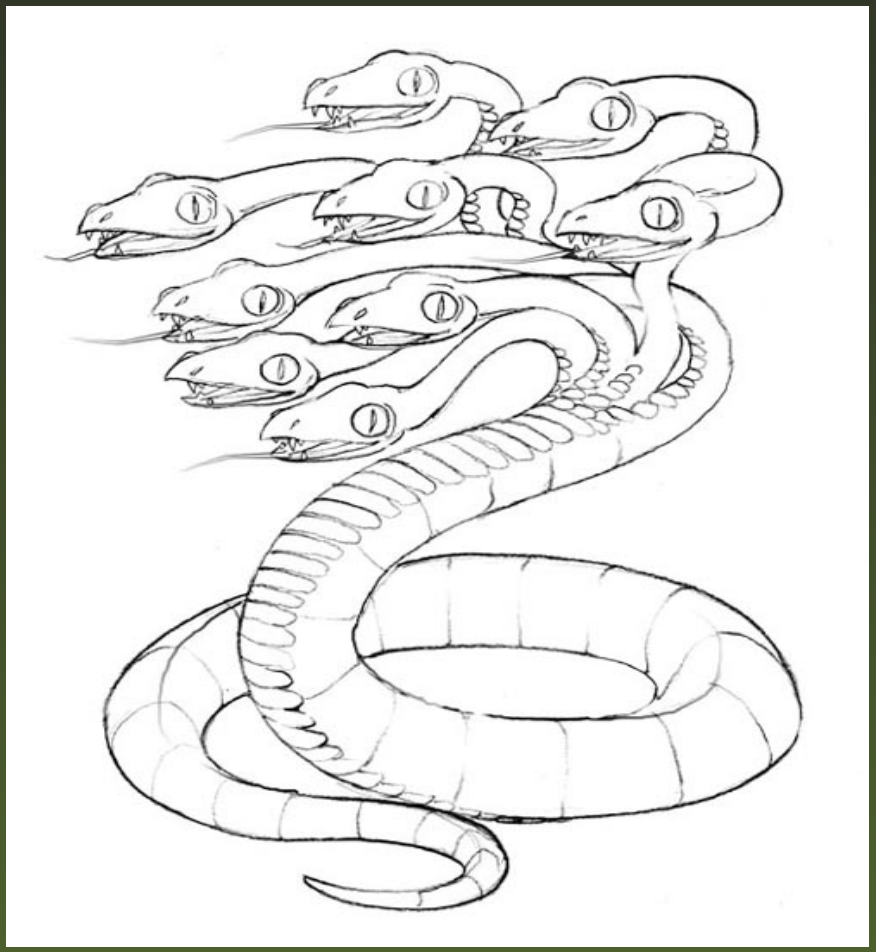

Figure 1: A 9-head hydra

Hydra automata

Figure 3: A 7-head hydra automaton

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Figure 2: A 7-head hydra automaton

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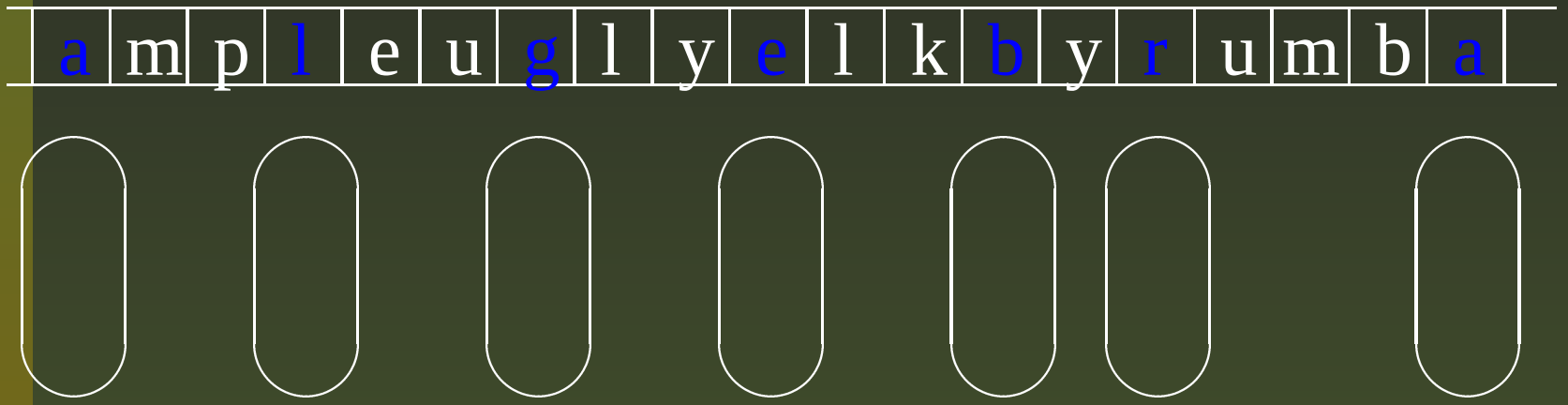

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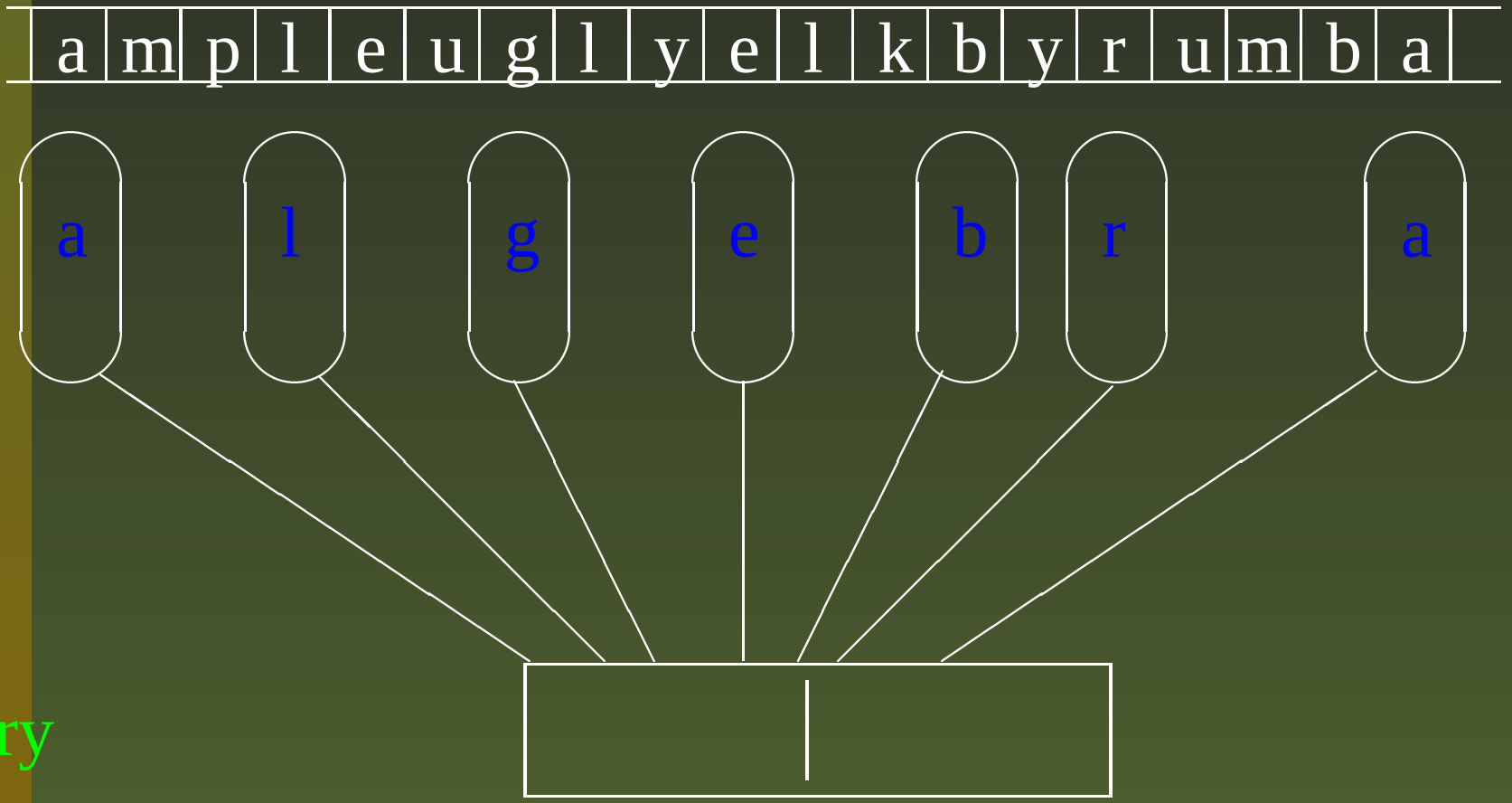

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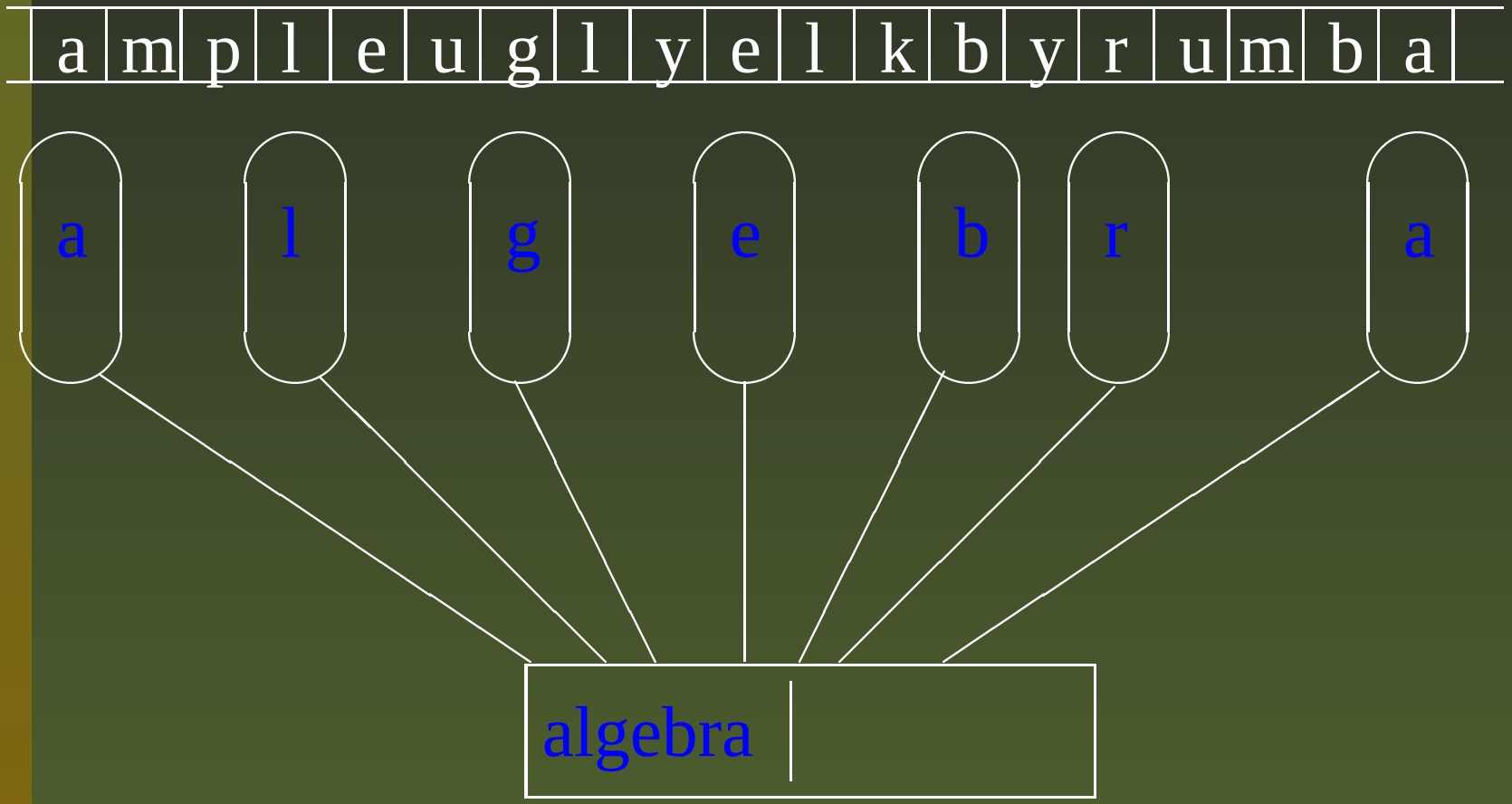

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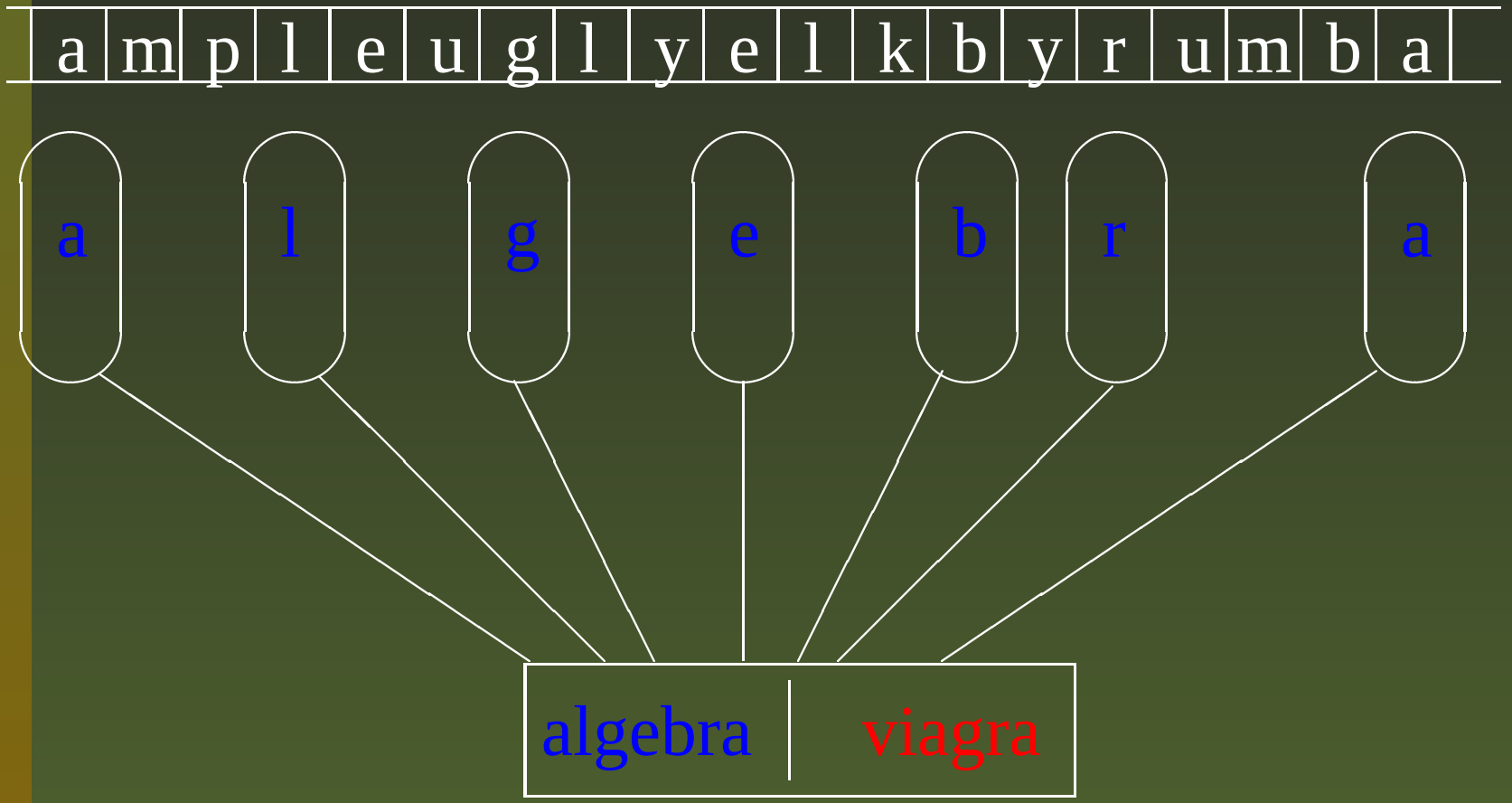

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A language $L \subseteq \Sigma^*$ is said to be *recognized* by a hydra automaton $\mathscr{H}$ if $\mathscr{H}$ accepts exactly words that are members of L. Such languages are called *piecewise testable*.

Piecewise testable languages

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Simon's hierarchy of piecewise testable languages:

$$PTL_1(\Sigma) \subset PTL_2(\Sigma) \subset PTL_3(\Sigma) \subset \ldots$$

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Syntactic monoids

For a language $L \subseteq \Sigma^*$ its *syntactic congruence* $\sim_L$ is defined by

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For a regular language L, the syntactic monoid $M(L)$ can be also defined as the *transition monoid* of the *minimal automaton* of L.

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Thus, whenever L is "given", so is $M(L)$.

Simon's theorem

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Theorem. (Imre Simon, 1972) *A language L is piecewise testable if and only if its syntactic monoid $M(L)$ is $\mathscr{J}$-trivial.*

Simon's theorem

	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	a	0	c	0
b	0	b	d	b	b	d
c	0	c	a	c	c	a
d	0	d	d	0	b	0

The monoid A_2^1

Simon's theorem

	0	1	a	b	c	d
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1	0	1	a	b	c	d
a	0	a	a	0	c	0
b	0	b	d	b	b	d
c	0	c	a	c	c	a
d	0	d	d	0	b	0

$$aA_2^1 = \{0, a, c\}$$

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b	0	b	d	b	b	d
c	0	c	a	c	c	a
d	0	d	d	0	b	0

$$A_2^1 a A_2^1 = \{0, a, b, c, d\}$$

Similarly one can verify that $A_2^1 b A_2^1 = \{0, a, b, c, d\}$ whence the monoid A_2^1 is not $\mathscr{J}$-trivial.

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If $\Sigma \supset \{a, b\}$, the language $\Sigma^* ab \Sigma^*$ is not piecewise testable.

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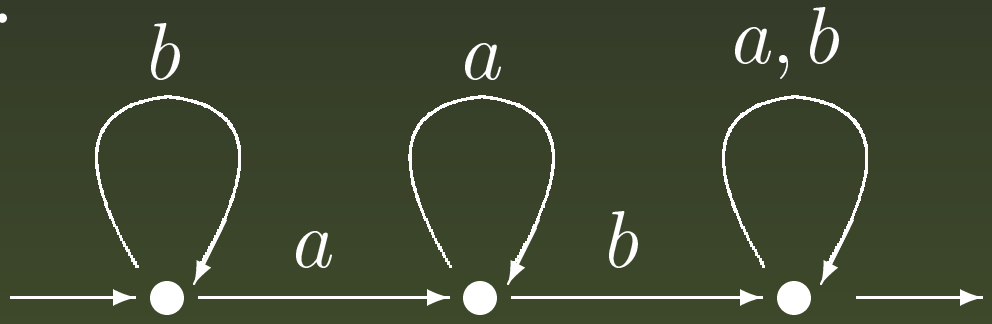
If $\Sigma = \{a, b\}$, the minimal automaton of $L = \Sigma^* ab \Sigma^*$ looks as follows:



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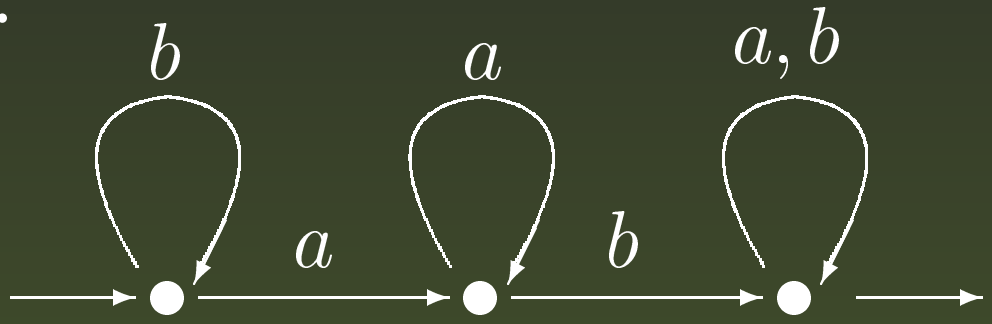

Using this, one readily calculates that

$$M(L) = \{0, 1, a, b, ba\}$$

subject to the relations $a^2 = a,\ b^2 = b, ab = 0$ and that $M(L)$ is $\mathscr{J}$-trivial. In fact, $L = \Sigma^* a \Sigma^* b \Sigma^*$.

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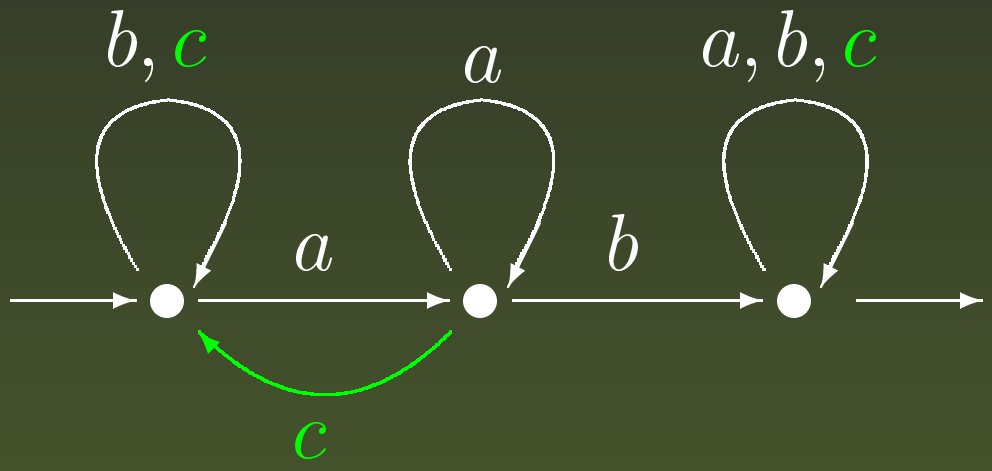
If $\Sigma = \{a, b, c\}$, the minimal automaton of $L = \Sigma^* ab \Sigma^*$ only slightly changes:



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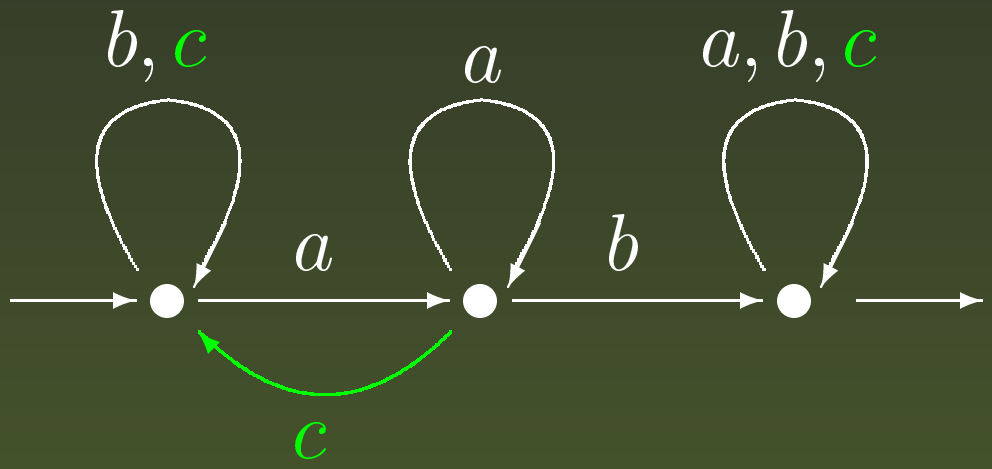

One gets

$$M(L) = \{0, 1, a, b, c, ba(= d)\} = A_2^1$$

and we already know that A_2^1 is not $\mathscr{J}$-trivial. Thus, the language L is not piecewise testable.

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- Very efficient: There are polynomial time algorithms to verify if the syntactic monoid $M(L)$ is $\mathscr{J}$-trivial when presented the minimal automaton of L. Such a description of $M(L)$ is much more compact than the Cayley table — recall that the transition monoid of an automaton with n states may consist of as many as n^n elements!

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- Transformation semigroups — Higgins, 1997.

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Simon's theorem means that the pseudovariety **J** of all finite $\mathscr{J}$-trivial monoids is generated by syntactic monoids of piecewise testable languages.

Recognizing height

Let $\mathbf{J}_h$ denote the pseudovariety of finite monoids generated by the syntactic monoids of piecewise testable languages of height $\leq h$. We have

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This is a typical instance of the **PMP** (Pseudovariety Membership Problem). The PMP has proved to systematically arise whenever one translates a "real world" (computer science) question into algebra.

Straubing's theorem

- $\mathscr{R}_n$ — the monoid of all reflexive binary relations on a set with n elements. It can be thought of as the monoid of all $n \times n$ matrices whose diagonal entries are 1 over the boolean semiring $\mathscr{B} = \langle \{0, 1\}; +, \cdot \rangle$.

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  A transformation α of a chain $\langle Q, \leq \rangle$ is *order preserving* if $q \leq q'$ implies $q.\alpha \leq q'.\alpha$ for all $q, q' \in Q$ and *extensive* if $q \leq q.\alpha$ for every $q \in Q$.

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(ii) *M divides* (is a morphic image of a submonoid of) *$\mathscr{R}_n$ for some n;*
(iii) *M divides $\mathscr{U}_n$ for some n;*
(iv) *M divides $\mathscr{C}_n$ for some n.*

This looks as a quite innocent Cayley-type theorem but in fact the proof heavily depends on Simon's theorem, and moreover, it can be shown relatively easily that the two theorems are equivalent.

Straubing's theorem

Corollary. *Each of the three sequences* $\{\mathscr{R}_n\}$, $\{\mathscr{U}_n\}$ *and* $\{\mathscr{C}_n\}$ $(n = 2, 3, \ldots)$ *generates the pseudovariety* **J** *of all finite* $\mathscr{J}$*-trivial monoids.*

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We thus have four stratifications for $\mathbf{J}$:

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Thus, for each h the pseudovariety $\mathbf{J}_h$ is generated by a single finite monoid. It easily follows from some basic universal algebra that the PMP for a (pseudo)variety generated by a single finite algebra is always decidable.

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Corollary. (Jean-Eric Pin, 1984) *For each $h = 1, 2, \ldots,$ the membership problem for the pseudovariety $\mathbf{J}_h$ is decidable, and hence, given a piecewise testable language, its height can be algorithmically determined.*

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Can we do better?

Recognizing height via identities

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d) *The monoids* $\mathscr{C}_n$ *with* $n > 4$ *are nonfinitely based.*

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Thus, there is an efficient algorithm to check if a given piecewise testable language can be recognized by a hydra automaton with 1, 2 or 3 heads, but this approach fails for larger numbers of heads.

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One may conclude that the optimal number of heads is equal to 3!

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- A positive answer to some instance of FBP gives an efficient algorithm for the corresponding instance of PMP $\Rightarrow$ systematically investigate FBP!
- For many important instances, the answer to FBP is negative $\Rightarrow$ develop some "equational" approach to PMP for nonfinitely based pseudovarieties!