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Interpreting graphs in 0-simple semigroups with involution with applications to computational complexity and the finite basis problem

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- the Finite Basis Problem (FBP)

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- A positive answer to some instance of FBP gives an efficient algorithm for the corresponding instance of PMP $\Rightarrow$ systematically investigate FBP!
- For many important instances, the answer to FBP is negative $\Rightarrow$ develop some “equational” approach to PMP for nonfinitely based pseudovarieties!

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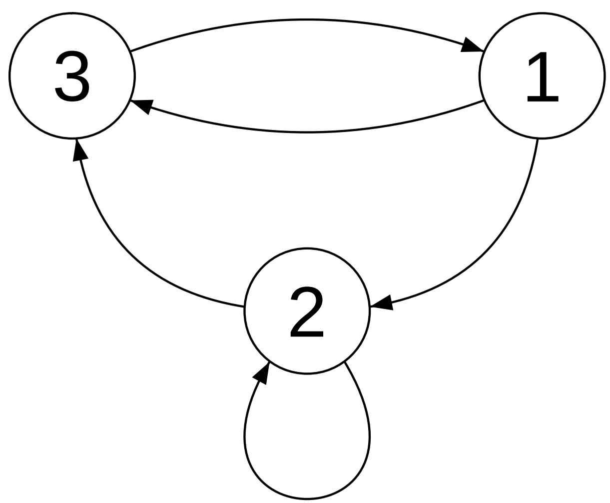

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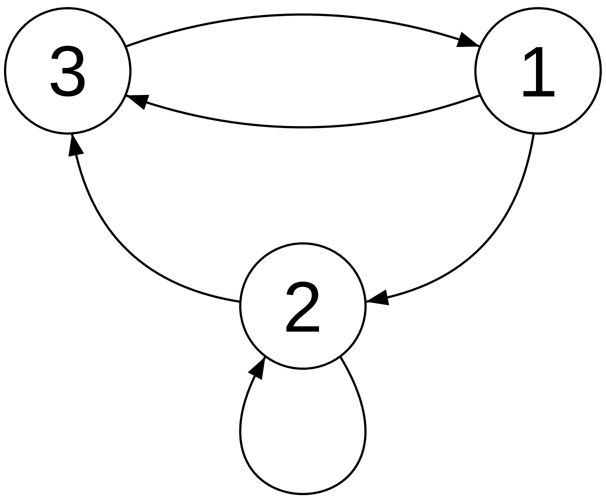

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In other words, $A(G)$ is defined on the set $(V \times V) \cup \{\mathbf{0}\}$ and the multiplication rule is

$$(i,j)(k,\ell) = \begin{cases} (i,\ell) & \text{if } j \sim k, \\ \mathbf{0} & \text{if } j \nsim k; \end{cases}$$

$$a\mathbf{0} = \mathbf{0}a = \mathbf{0} \text{ for all } a \in A(G).$$

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New idea: to equip $A(G)$ with an additional unary operation (reversion):

$$(i,j)' = (j,i), \quad 0' = 0$$

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- $x \sim y \to y \sim x$ (symmetry of G) is equivalent to $A(G) \models (xy)' = y'x'$;
- G is an anti-chain (satisfies $x \sim y \to x = y$ and $x \sim x$) if and only if $A(G)$ is a combinatorial Brandt semigroup.

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A universal Horn sentence is a sentence of one of the following two forms:

$$(\forall x_1 \forall x_2 \ldots) \left(\left(\mathop{\&}_{1 \le i \le n} \Phi_i \right) \to \Phi_0 \right),$$

$$(\forall x_1 \forall x_2 \ldots) \left(\bigvee_{0 \le i \le n} \neg \Phi_i \right)$$

where the Φ_i are of the form $x_j \sim x_k$ or $x_j = x_k$.

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  - complete looped graphs (add $x \sim y$)
- Simple graphs (anti-reflexifity + symmetry)
- n-colorable simple graphs (no finite axiomatization)

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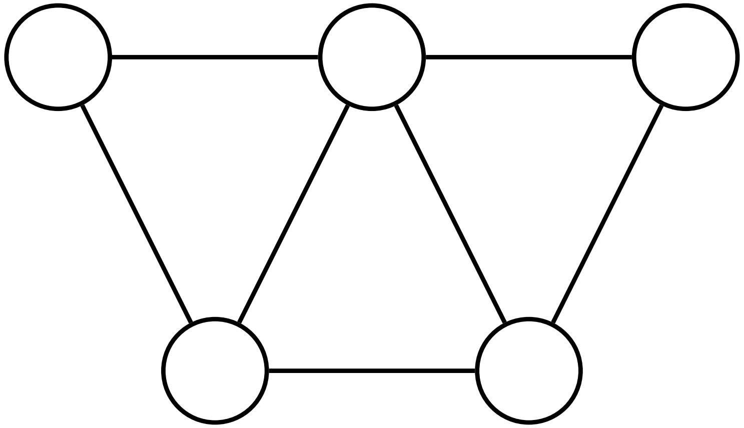

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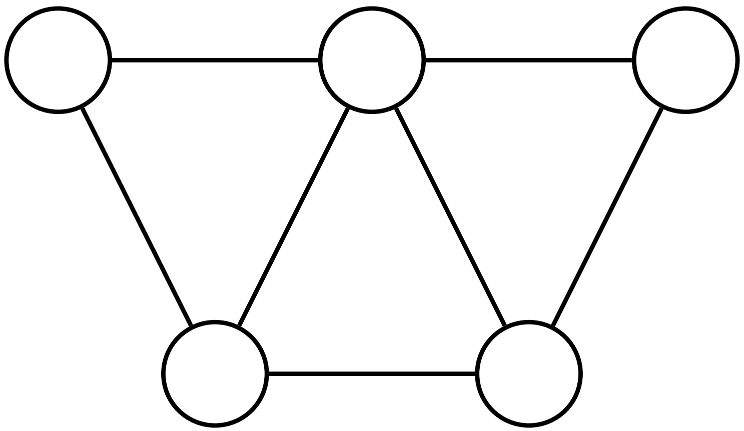

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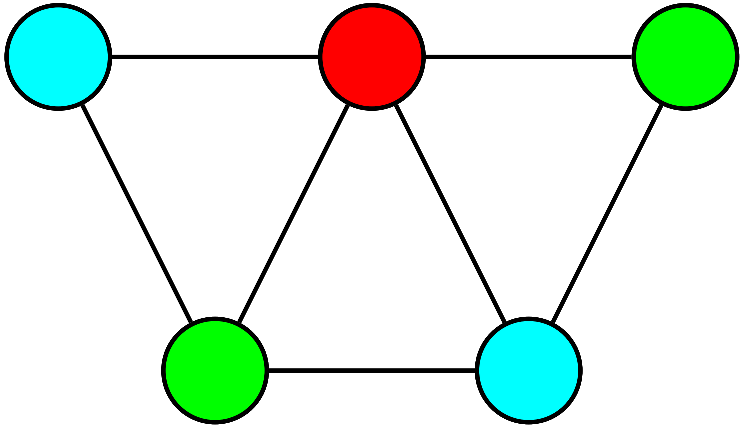

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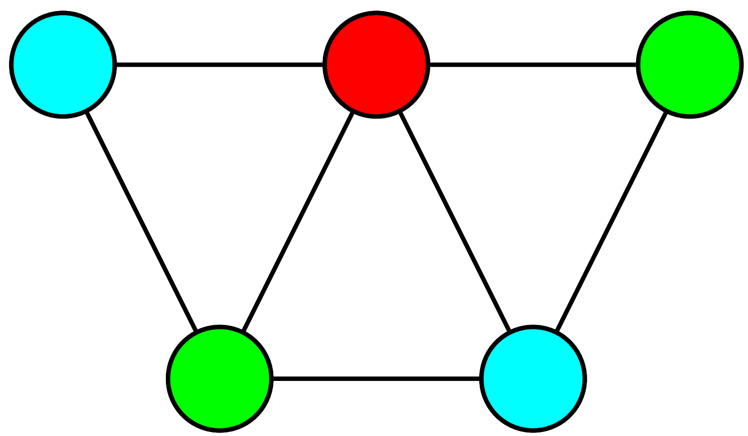

The above graph C_3 is 3-colorable but not 2-colorable. In fact, it is a generator of minimum size for the uH-class of all 3-colorable graphs (Nešetřil and Pultr, 1978).

Main results

Theorem 1. The assignment $G \mapsto A(G)$ induces an injective join-preserving map from the lattice of all universal Horn classes of graphs to the lattice of subvarieties of the variety generated by all (unary) adjacency semigroups.

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Theorem 2. When restricted to reflexive graphs, the map preserves meets as well as joins, and is “nearly” surjective in the sense that the lattice of subvarieties of the variety generated by adjacency semigroups satisfying $xx'x = x$ is isomorphic to the lattice obtained from the lattice of all universal Horn classes of reflexive graphs by inserting just one new element.

An application to PMP

Theorem 1 interpreted another way: membership of a unary Rees matrix semigroup $\mathcal{M}_0'[\mathbf{1}, P]$ in the variety generated by another such unary semigroup $\mathcal{M}_0'[\mathbf{1}, Q]$ is equivalent to the membership of the graph whose adjacency matrix is P in the uH-class of the graph whose adjacency matrix is Q.

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This allows us to produce numerous examples of finite unary Rees matrix semigroups M such that recognizing membership to the (pseudo)variety generated by M is computationally difficult. In order to formulate the results precisely, we need a 5-minute tour in Complexity Theory.

The Classes P *and* NP

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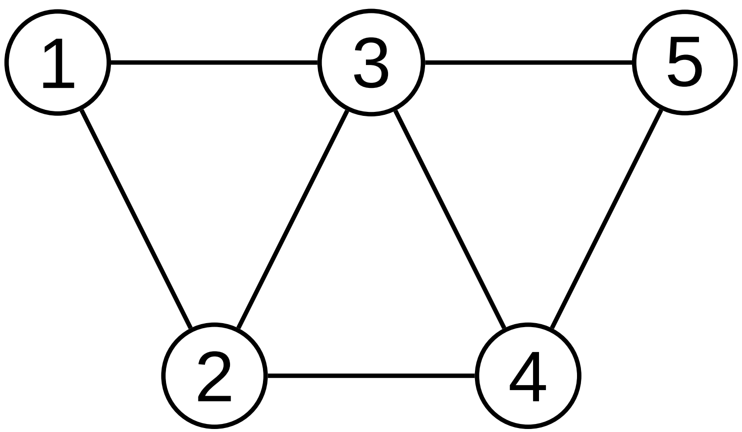
Example: 3-COLOR is $\mathbf{NP}$-complete (Levin, 1973).

An application to PMP

Recall that C_3 is a five element graph generating a uH-class equal to the class of all 3-colorable graphs:



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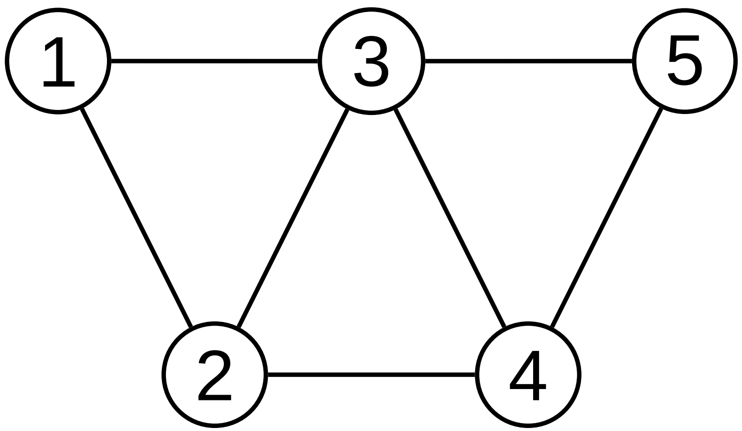

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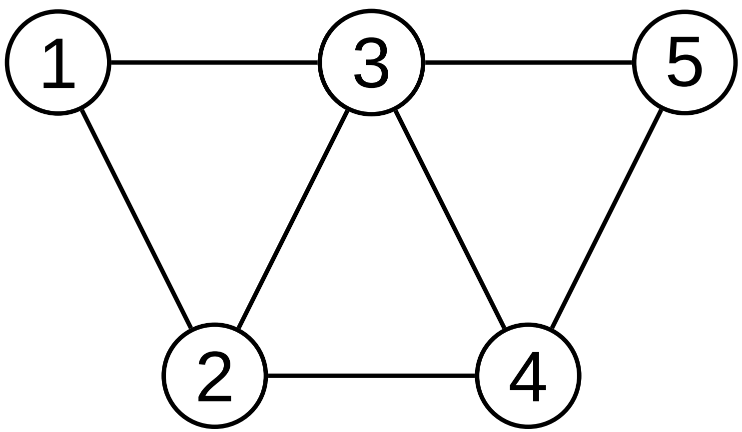

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Corollary. The membership problem for the pseudovariety generated by $A(C_3)$ is $\mathbf{NP}$-hard.

(One can't claim that the problem is $\mathbf{NP}$-complete because it is not yet clear that it belongs to the class $\mathbf{NP}$.)

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In the plain semigroup setting similar examples were produced by Jackson and McKenzie, 2006. Their smallest semigroup with analogous properties consists of 55 elements.