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Interpreting graphs in 0-simple semigroups with involution with applications to computational complexity and the finite basis problem

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Finite Basis Problem: an Overview

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First, a few words about “standard” techniques for plain semigroups.

The techniques are:

- Syntactic analysis;
- Critical semigroup method;
- Use of inherently nonfinitely based semigroups.

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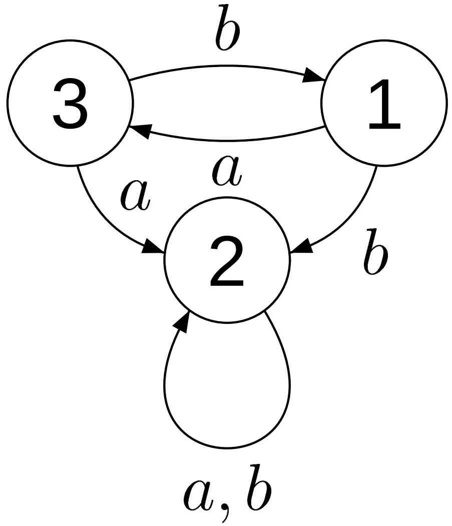

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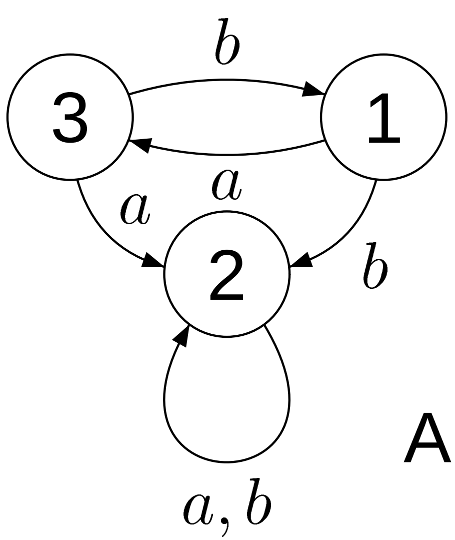

$$\langle a, b \mid aba = a,\ bab = a,\ a^2 = b^2 = 0\rangle$$

A difficulty: ID-CHECK(S) may be hard.

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Critical semigroup method: given S, try to construct for each sufficiently large n a semigroup S_n such that $S_n \notin \operatorname{var} S$ but every n-generated subsemigroup of S_n belongs to $\operatorname{var} S$.

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A difficulty: the membership checking for $\operatorname{var} S$ may be hard.

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Inherently nonfinitely based semigroups: given S, try to find an inherently nonfinitely based semigroup within $\mathrm{var}\, S$. (A finite semigroup T is inherently nonfinitely based if no finitely based locally finite variety can contain T.)

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A difficulty: not so many inherently nonfinitely based semigroups exist (there is a complete classification of them).

Unary Semigroup Case

- Syntactic analysis: basically fails as combinatorics of unary words becomes rather difficult even in the presence of the law $(xy)' = y'x'$ and almost impossible in the absence of this law – look at $((xy')'zx)'(zy')'$.

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- Critical semigroup method: works pretty well (Auinger & $\sim$, still in progress). Concrete applications – all binary relations on a finite non-singleton set with inversion; all 2×2-matrices over a finite field with at least 3 elements with transposition, etc.

Interpreting Graphs

Recall that to each graph $G = \langle V, \sim \rangle$ we have assigned its unary adjacency semigroup $A(G)$ – the Rees matrix semigroup over the trivial group with the adjacency matrix of the graph a sandwich matrix equppied with an additional unary operation (reversion):

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Theorem 1. The assignment $G \mapsto A(G)$ induces an injective join-preserving map from the lattice of all universal Horn classes of graphs to the lattice of subvarieties of the variety generated by all (unary) adjacency semigroups.

Interpreting Graphs

A uH-class $\mathbf{H}$ cannot be finitely axiomatized iff there exists an infinite descending chain of uH-classes

$$\mathbf{H}_1 \supset \mathbf{H}_2 \supset \cdots \subset \mathbf{H}_n \supset \ldots$$

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Since the map of Theorem 1 is order-preserving and injective, it sends each such chain to an infinite descending chain of varieties. Hence this map sends uH-classes that cannot be finitely axiomatized to nonfinitely based varieties of unary semigroups!

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One can think that the converse my be also true: finitely axiomatized uH-classes of graphs are sent to finitely based varieties of unary semigroups. This is not the case!

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Example. Let F denote the anti-reflexive, anti-symmetric graph on four vertices consisting of two disjoint edges: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

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Observe that the variety $\mathbf{A}$ generated by all adjacency semigroups is locally finite as it is generated by the adjacency semigroup of the following generator for the class of all graphs:

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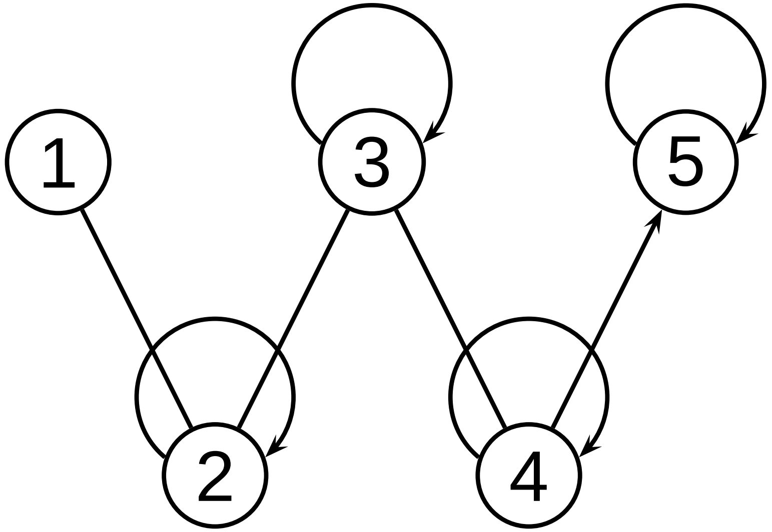

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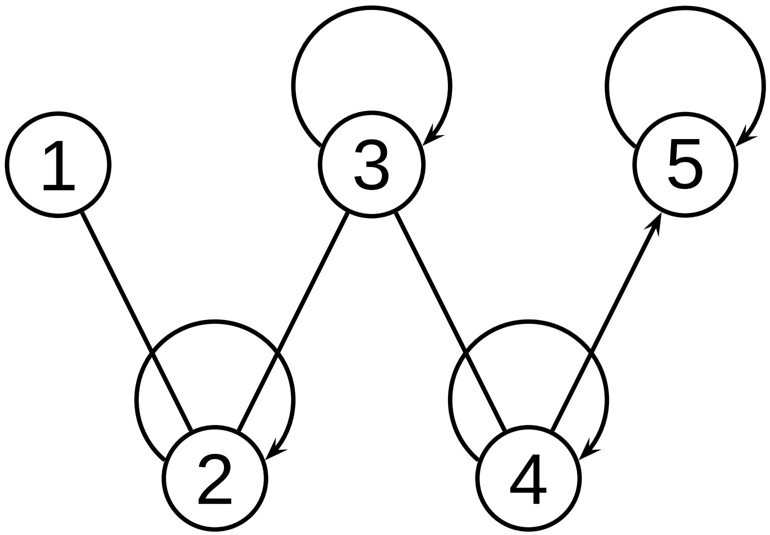

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since $\mathbf{A}$ contains an inherently nonfinitely based semigroup, it is not finitely based while the class of all graphs is finitely axiomatized.

Interpreting Graphs

The adjacency semigroup of the 3-cycle $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(which as a semigroup is simply the Brandt semigroup B_3) is also inherently nonfinitely based.

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The direct product of $A(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ with the adjacency semigroup $A(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ of the two element chain is also inherently nonfinitely based. Note that as semigroups, these are simply B_2 and A_2.

Interpreting Graphs

The situation changes when we pass to reflexive graphs. Recall that when restricted to reflexive graphs, the map induced by $G \mapsto A(G)$ is “nearly” surjective: the lattice of subvarieties of the variety $\mathbf{A_{ref}}$ generated by adjacency semigroups satisfying $xx'x = x$ is obtained from the lattice of all uH-classes of reflexive graphs by inserting just one new element.

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Besides that, a nice aspect of adjacency semigroups of reflexive graphs, is that the unary operation preserves $\mathcal{J}$-classes, a fact that can be equationally captured. This allows us to show that $\mathbf{A_{ref}}$ is finitely based.

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Hence finitely axiomatized uH-classes of reflexive graphs correspond to finitely based varieties of unary semigroups and vice versa.

Interpreting Graphs

As an application we can show , for example, that the adjacency semigroup of the graph $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ generates a limit (=minimal nonfinitely based variety of unary semigroups. The variety has only 5 subvarieties.

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Finally, restricting to reflexive and symmetric graphs, we recover Auinger's classification of varieties of combinatorial strict *-regular semigroups.

And very finally ...