The Finite Basis Problem for Unary Semigroups
(some new directions)

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Basic definitions

Unary semigroup — an algebraic structure with one binary associative operation and a few unary operations. Examples are abound amongst both 'concrete' and 'abstract' semigroups.

If $\mathbf{S} = \langle S; \cdot, f_1, f_2, \dots \rangle$ is a unary semigroup, we can interpret in $\mathbf{S}$ terms build from variables and the operation symbols $\cdot$, f_1, f_2, $\dots$. An interpretation of a term in $\mathbf{S}$ is defined as soon as each variable of the term gets evaluated at some element of the set S. Then the term also gets a value in S.

A unary semigroup identity is just a couple of appropriate terms. We use the notation $u \mathrel{\widehat{=}} v$ and say 'identity' for brevity. $\mathbf{S}$ satisfies $u \mathrel{\widehat{=}} v$ (or: $u \mathrel{\widehat{=}} v$ holds in $\mathbf{S}$) if the terms u and v get the same value under **every** interpretation in $\mathbf{S}$.

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Finitely Based Semigroups

Example: The identities $(xy)^T \doteq y^T x^T$ and $x \doteq (x^T)^T$ hold in the unary semigroup $\langle M_n(K); \cdot, ^T \rangle$ of all $n \times n$-matrices over a field K (the unary operation ^{T} means the usual transpose) while the identity $x \doteq xx^T x$ does not hold in this unary semigroup.

An identity $u \doteq v$ follows from a set Σ of identities if every unary semigroup $\mathbf{S}$ that satisfies each identity in Σ satisfies $u \doteq v$ as well. A unary semigroup $\mathbf{S}$ is finitely based if all identities holding in S follow from some finite set of such identities (called an identity basis for $\mathbf{S}$).

Example contd: The identities $(xy)^T \doteq y^T x^T$ and $x \doteq (x^T)^T$ form an identity basis for the unary semigroup $\langle M_n(K); \cdot, ^T \rangle$ if the field K is infinite. (Auinger-Dolinka-V., JEMS, 2012)

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Nonfinitely Based Semigroups

If a unary semigroup is not finitely based, it is said to be nonfinitely based.

Example contd: The unary semigroup $\langle M_n(K); \cdot, ^T\rangle$ is nonfinitely based if the field K is finite. (Auinger-Dolinka-V., JEMS, 2012)

Thus, finiteness of $\langle M_n(K); \cdot, ^T\rangle$ implies non-finiteness of its identity basis and vice versa. It is a good example of somewhat surprising interplays between finiteness and non-finiteness that drive the whole area.

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A *variety* given by an identity system Σ is a class of all unary semigroups satisfying all identities in Σ. A variety $\mathcal{V}$ is locally finite if every finitely generated unary semigroup in $\mathcal{V}$ is finite. The least variety containing a given unary semigroup $\mathbf{S}$ is denoted by $\operatorname{Var}\mathbf{S}$ and is said to be generated by $\mathbf{S}$.

Fact: The variety generated by a finite unary semigroup is locally finite (Birkhoff).

A finite unary semigroup is said to be inherently nonfinitely based if it is not contained in any locally finite finitely based variety. By the above fact a finite unary semigroup $\mathbf{S}$ is nonfinitely based (and even inherently nonfinitely based) if $\operatorname{Var}\mathbf{S}$ contains an inherently nonfinitely based unary semigroup.

Inherently Nonfinitely Based Semigroups

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Inherently Nonfinitely Based Semigroups – Examples

The property of being inherently nonfinitely based is much stronger than the property of being nonfinitely based and also behaves more regularly. For instance, the former property is inherited within the class of finite unary semigroups under taking oversemigroups, homomorphic preimages, and direct products while none of the operations preserve the latter property.

Example contd: Suppose K is a finite field. The unary semigroup $\langle M_n(K); \cdot, ^T \rangle$ is inherently nonfinitely based if and only if $n > 2$ or $n = 2$ and $|K| \not\equiv 3 \pmod 4$. (Auinger-Dolinka-V., JEMS, 2012)

Thus, the unary semigroup of 2×2-matrices with transposition is inherently nonfinitely based when entries of the matrices come from the field with 2,4,5,8,9, ... elements and is not inherently nonfinitely based (though still nonfinitely based) when the matrices are taken over the field with 3,7,11, ... elements.

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Impact of Extending the Language

The Finite Basis Problem (FBP) for plain semigroups has been intensively investigated since the mid-60s (when the pioneering papers by Austin and Perkins appeared).

What happens with the finite basis property when we enhance the vocabulary by additional unary operations?

More precisely: let $\mathbf{S}_{\mathrm{plain}} = \langle S; \cdot \rangle$ be a (plain) semigroup, $\mathbf{S}_{\mathrm{unary}} = \langle S; \cdot, f_1, f_2, \dots \rangle$ a unary semigroup on the same base set. How are the answers to the FBP for $\mathbf{S}_{\mathrm{plain}}$ and $\mathbf{S}_{\mathrm{unary}}$ related?

On the one hand, the expressive power of the equational language increases. Hence $\mathbf{S}_{\mathrm{unary}}$ has more identities than $\mathbf{S}_{\mathrm{plain}}$ so that the former may have more chance to become nonfinitely based.

On the other hand, the inference power of the languages increases too. Hence one can imagine the situation when some identity of $\mathbf{S}_{\mathrm{plain}}$ does not follow from a system Σ as a 'plain' identity but follows from Σ as a 'unary' identity. This indicates that $\mathbf{S}_{\mathrm{unary}}$ may be finitely based even if $\mathbf{S}_{\mathrm{plain}}$ is not.

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Impact of Extending the Language – Examples

The cumulative effect of the trade-off between increased expressivity and increased inference power is hard to predict in general.

Example: The one-relator monoid presentation $\langle a, b \mid ab^2a = 1\rangle$ defines in fact an infinite group **G**. If one considers **G** as a unary semigroup (with the group inversion as the unary operation), then it is finitely based; moreover, the single law $x^2y^2 \simeq y^2x^2$ can be used as its identity basis. On the other hand, the semigroup reduct of **G** is nonfinitely based (Isbell).

Example: The wreath product of the countably generated free group of exponent 4 with the countably generated free abelian group is nonfinitely based as a group (Yu. Kleiman) but satisfies no non-trivial plain semigroup identity whence the semigroup reduct of this product is finitely based.

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Example: The one-relator monoid presentation $\langle a, b \mid ab^2a = 1\rangle$ defines in fact an infinite group **G**. If one considers **G** as a unary semigroup (with the group inversion as the unary operation), then it is finitely based; moreover, the single law $x^2y^2 \doteq y^2x^2$ can be used as its identity basis. On the other hand, the semigroup reduct of **G** is nonfinitely based (Isbell).

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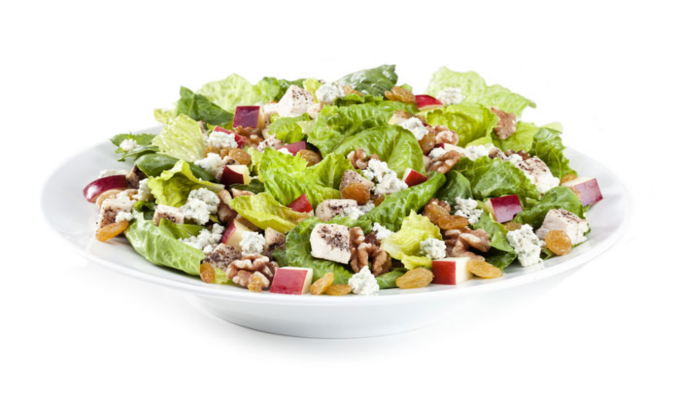

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Let A be a finite set with at least 3 elements. Is it possible to define an infinite sequence $f_1, f_2, \ldots$ of operations on A such that the algebraic structure $\langle A; f_1, f_2, \ldots, f_n \rangle$ is nonfinitely based when n is odd and finitely based when n is even?

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Here the operations $f_1, f_2, \dots$ are **not** supposed to be unary. In fact, it is clear that $f_1, f_2, \dots$ cannot be of a bounded arity.

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Theorem (Willard, Acta Sci. Math. Szeged, 1999)

The Chautauqua Problem has a positive solution for $|A| = 9$. Moreover, the sequence $f_1, f_2, \ldots$ can be chosen such that $\langle A; f_1, f_2, \ldots, f_n \rangle$ is inherently nonfinitely based when n is odd and finitely based when n is even.

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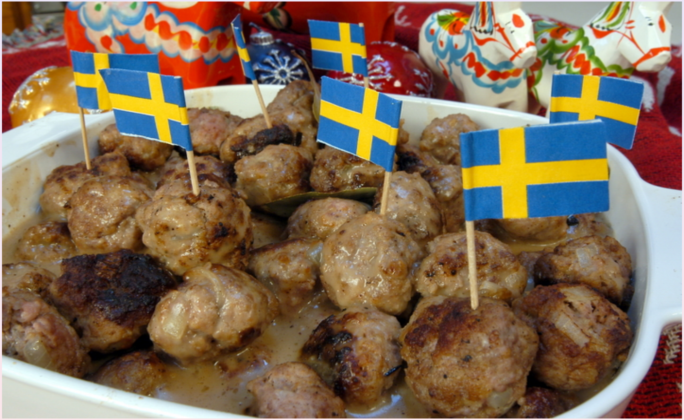

Uppsala, September 1st, 2012

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Let $\mathbf{S} = \langle S; \cdot \rangle$ be a finite semigroup. For which maximum number $k = k(\mathbf{S})$ is it possible to define a sequence $f_1, f_2, \ldots, f_k$ of **unary** operations on S such that the unary semigroup $\langle S; \cdot, f_1, f_2, \ldots, f_n \rangle$ is (inherently) nonfinitely based when n is odd and finitely based when n is even?

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The essence of the problem: to what extent does a semigroup influence the equational properties of its unary enrichments?

A Recent Example — Restriction Semigroups

The set $I(X)$ of all partial injective transformations of a set X has two natural unary operations. If a is a partial injective transformation then a^* and a^+ denote the restrictions of the identity mapping to the range and respectively the domain of a. The unary semigroup $\langle I(X); \cdot, {}^*, {}^+ \rangle$ satisfies the identities

$$x^+x \doteq x,\ (x^+y)^+ \doteq x^+y^+,\ x^+y^+ \doteq y^+x^+,\ xy^+ \doteq (xy)^+x,\ (x^+)^* \doteq x^+;$$
$$xx^* \doteq x,\ (yx^*)^* \doteq y^*x^*,\ x^*y^* \doteq y^*x^*,\ y^*x \doteq x(yx)^*,\ (x^*)^+ \doteq x^*.$$

Unary semigroups $\langle S; \cdot, {}^*, {}^+ \rangle$ satisfying the 10 identities above are nowadays termed restriction semigroups. Another mass example of a restriction semigroup: take an inverse semigroup $\langle S; \cdot, {}^{-1} \rangle$ and define $x^* := x^{-1}x$ and $x^+ := xx^{-1}$.

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Restriction Semigroup of a Digraph

With every digraph $G = (V, E)$, where E (the edge set) is just a relation on V (the vertex set), one can associate a natural restriction semigroup $\mathbf{S}(G)$ of partial maps on the set V. It is generated (as a restriction semigroup) by the edge maps φ_e where e runs over E. If $e = (u, v)$, then $u\varphi_e = v$ and $w\varphi_e$ is undefined for all vertices $w \neq u$.

 — for this digraph, $\mathbf{S}(G)$ consists of 4 elements: $\alpha, \alpha^+, \alpha^*, 0$. We denote this restriction semigroup by $\mathbf{B}_0$.

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- the vertex maps χ_v for each vertex v such that $u\chi_v = v$ if $u = v$ and is undefined otherwise,
- the path mappings (a path mapping is the compositions of the edge mappings for the consecutive edges forming the path),
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Brandt Semigroup as a Restriction Semigroup

Similarly, for the graph  (where the undirected edge represent two oppositely directed edges), $\mathbf{S}(G)$ consists of 5 elements: α, β, $\alpha^{+} = \beta^{*}$, $\alpha^{*} = \beta^{+}$, 0. Its semigroup reduct is nothing but the 5-element Brandt semigroup. We denote this restriction semigroup by $\mathbf{B}_2$.

Recall that the semigroup reduct of $\mathbf{B}_2$ is finitely based (Trahtman–Reilly). Adding two natural unary operations produces an inherently nonfinitely based unary semigroup. But the Brandt semigroup is in fact an inverse semigroup and as such it is finitely based (E. Kleiman). Thus, adding the operation $x \mapsto x^{-1}$ restores the finite basis property.

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When Does the INFB Property Persist?

Recall: Sapir has proved that no finite inverse semigroup is inherently nonfinitely based (as a unary semigroup). A classic example of an inherently nonfinitely based plain semigroup (also due to Sapir) is the 6-element Brandt monoid which is an inverse semigroup. Hence adding the inverse operation (which is the same as transposition in the standard matrix representation

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Question

Under which conditions adding a ‘good’ unary operation preserves the property of being INFB?

A Partial Answer

Denote by $\mathbf{SL}_3$ the 3-element involution semilattice $\langle\{a, b, 0\}; \cdot, {}^*\rangle$ in which $ab = ba = 0$, $a^* = b$, $b^* = a$, and $0^* = 0$.

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Theorem (Auinger-Dolinka-Pervukhina-V.)

Let $\mathbf{S} = \langle S; \cdot, ^*\rangle$ be an involution semigroup such that

- the semigroup reduct $\langle S, \cdot\rangle$ is inherently nonfinitely based;
- the variety $\operatorname{Var}\mathbf{S}$ contains $\mathbf{SL}_3$.

Then $\mathbf{S}$ is inherently nonfinitely based as a unary semigroup.

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This result simplifies many proofs from our earlier paper (Auinger-Dolinka-V., JEMS, 2012) and produces some new applications.

If one equips the 6-element Brandt monoid with the involution that fixes the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and swaps the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then the resulting involution monoid is inherently nonfinitely based.

Indeed, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ form a unary subsemigroup isomorphic to $\mathbf{SL}_3$.

Applications – I

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Indeed, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ form a unary subsemigroup isomorphic to $\mathbf{SL}_3$.

Applications – II

Consider the monoid $\langle T_n(K); \cdot \rangle$ of all upper-triangular $n \times n$-matrices over a finite field K. It is inherently nonfinitely based if and only if $|K| > 2$ and $n > 3$ (Goldberg-V.)

If one equips $\langle T_n(K); \cdot \rangle$ with the skew involution (the reflection with respect to the secondary diagonal), then the resulting involution monoid is inherently nonfinitely based if and only if $|K| > 2$ and $n > 3$.

Indeed, the 'if' part follows since the matrices e_{11}, e_{nn}, and 0 form a unary subsemigroup isomorphic to $\mathbf{SL}_3$. The 'only if' part holds because a finitely generated involution semigroup is finitely generated as a plain semigroup as well.

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Regular Case

We have seen that the presence of $\mathbf{SL}_3$ in the variety generated by a finite involution semigroup is sufficient for preserving the INFB property. In the regular case, we are able to show that it is also necessary.

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Theorem (Auinger-Dolinka-Pervukhina-V.)

Let $\mathbf{S} = \langle S; \cdot, ^* \rangle$ be a finite involution semigroup such that

- the semigroup reduct $\langle S, \cdot \rangle$ is regular;
- the variety $\operatorname{Var} \mathbf{S}$ does not contain $\mathbf{SL}_3$.

Then $\mathbf{S}$ is **not** inherently nonfinitely based as a unary semigroup.

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Again, this simplifies some proofs from (Auinger-Dolinka-V., JEMS, 2012). In particular, regular *-semigroups (that is, involution semigroups satisfying $xx^*x \doteq x$) can't be inherently nonfinitely based since $xx^*x \doteq x$ fails in $\mathbf{SL}_3$.

Conjecture

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Find an efficient characterization of inherently nonfinitely based involution semigroups.

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Conjecture (Auinger)

A finite involution semigroup $\mathbf{S} = \langle S; \cdot, ^* \rangle$ is inherently nonfinitely based as a unary semigroup if and only if:

- the semigroup reduct $\langle S, \cdot \rangle$ is inherently nonfinitely based as a plain semigroup;
- the variety $\operatorname{Var} \mathbf{S}$ contains $\mathbf{SL}_3$.

Conclusion

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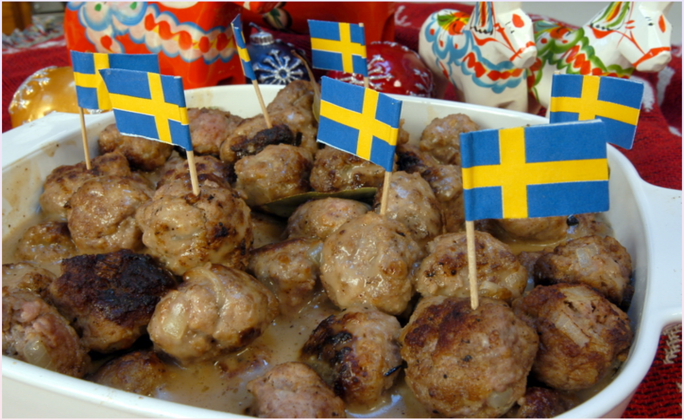

. . . our challenge no.2 for today!

Uppsala, September 1st, 2012