Synchronizing Automata and the Road Coloring Theorem

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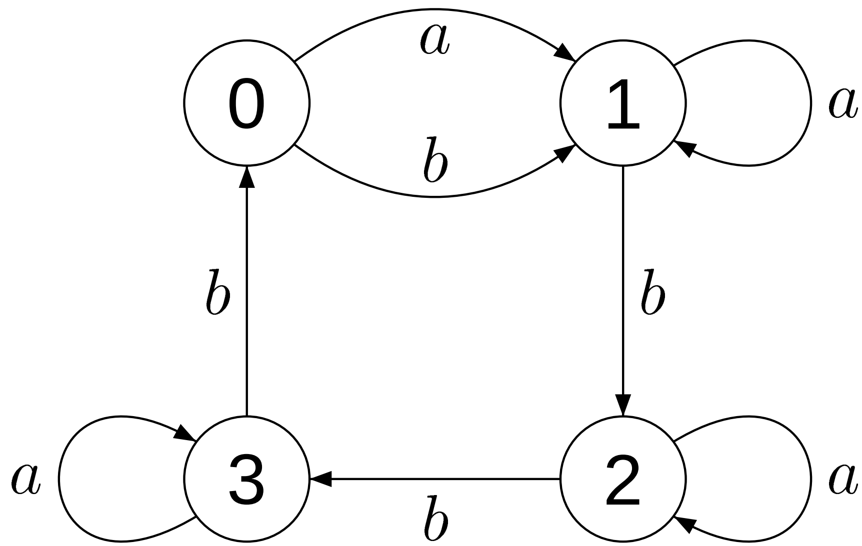
Other names:

- for automata: directable, cofinal, collapsible, etc;
- for words: directing, recurrent, synchronizing, etc.

Synchronizing automata



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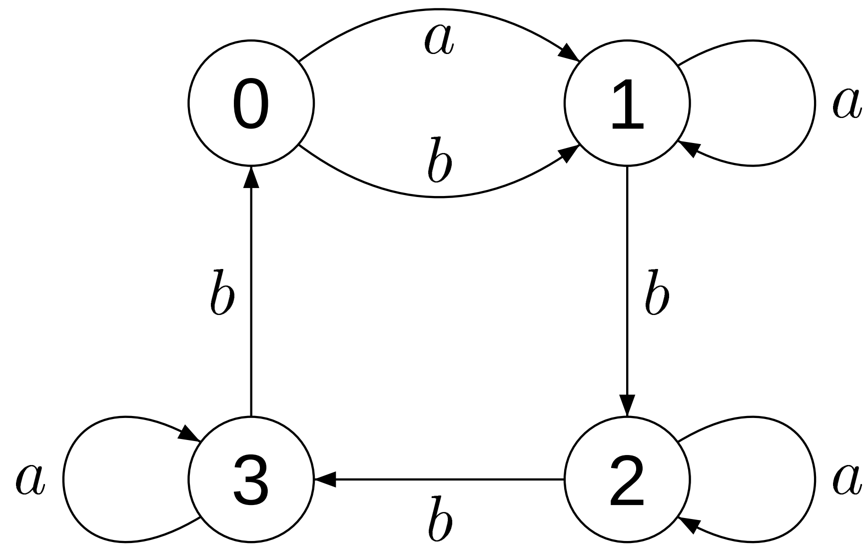

A reset word is $abbbabbba$. Applying it at any state brings the automaton to the state 1.

Synchronizing Automata

The notion was formalized in 1964 in a paper by Jan Černý (Poznámka k homogénnym eksperimentom s konečnými automatami, Matematicko-fyzikalny Časopis Slovensk. Akad. Vied, 14, no.3, 208–216 [in Slovak]) though implicitly it had been around since at least 1956 (see proceedings).

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Think of a satellite which loops around the Moon and cannot be controlled from the Earth while “behind” the Moon (Černý’s original motivation).

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Example: A. E. Laemmel, B. Rudner, Study of the application of coding theory, Report PIBEP-69-034, Polytechnic Inst. Brooklyn, Dept. Electrophysics, Farmingdale, N.Y., 94 pp.; more examples in proceedings.

Re-inventing by Engineers

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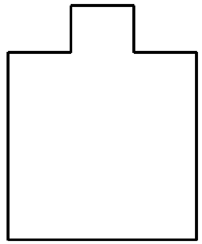

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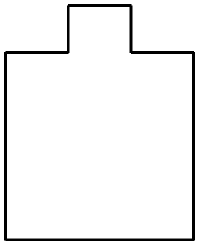

Such parts arrive at manufacturing sites in boxes and they need to be sorted and oriented before assembly.

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Assume that only four initial orientations of the part shown above are possible, namely, the following ones:

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Suppose that prior the assembly the part should take the “bump-left” orientation (the second one in the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

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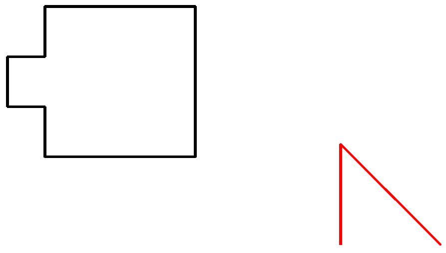
A high obstacle is high enough so that any part on the belt encounters this obstacle by its rightmost low angle.



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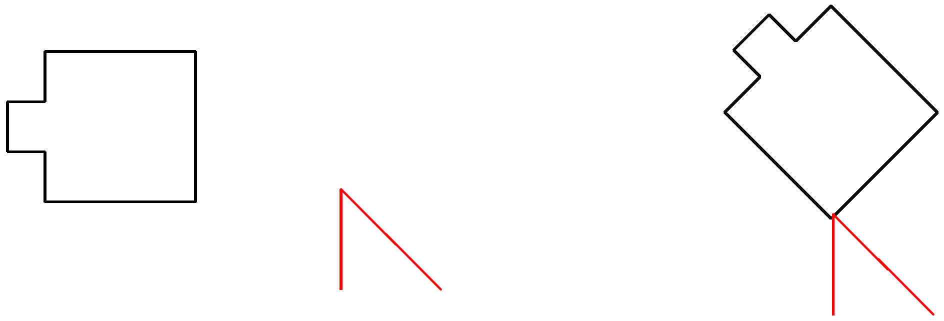

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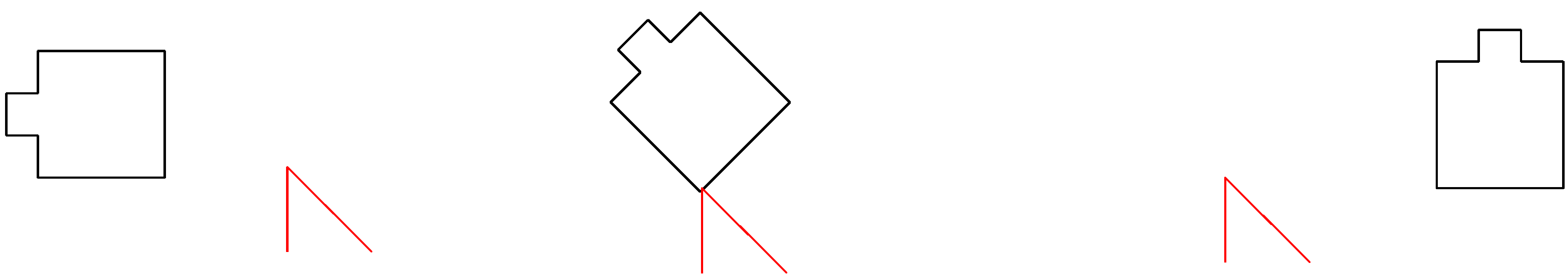

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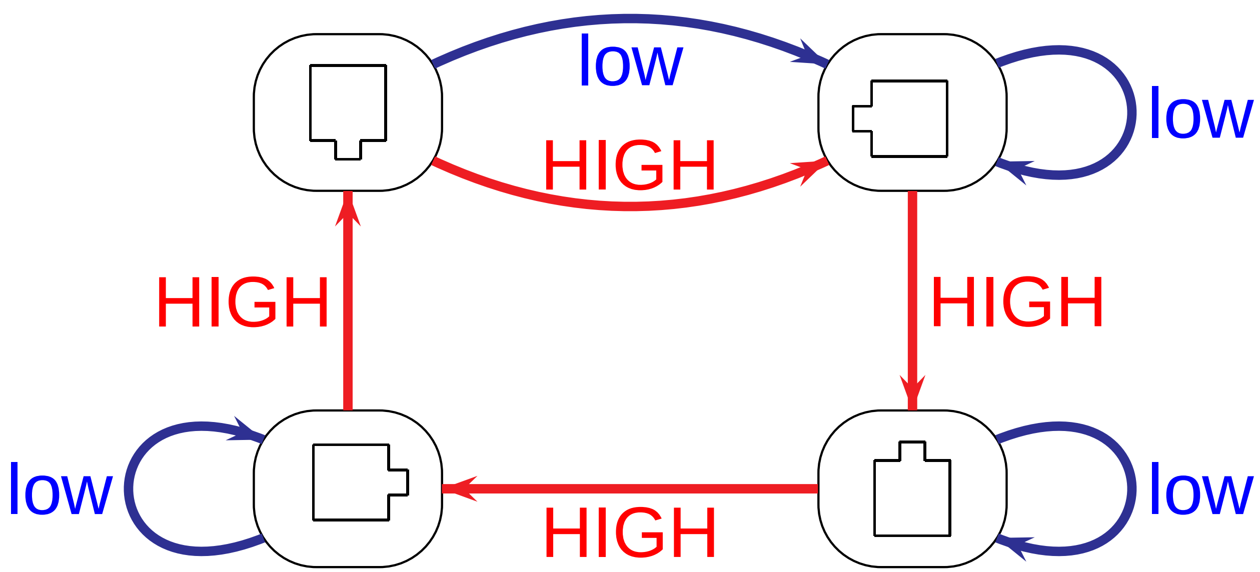
The following schema summarizes how the obstacles effect the orientation of the part in question:



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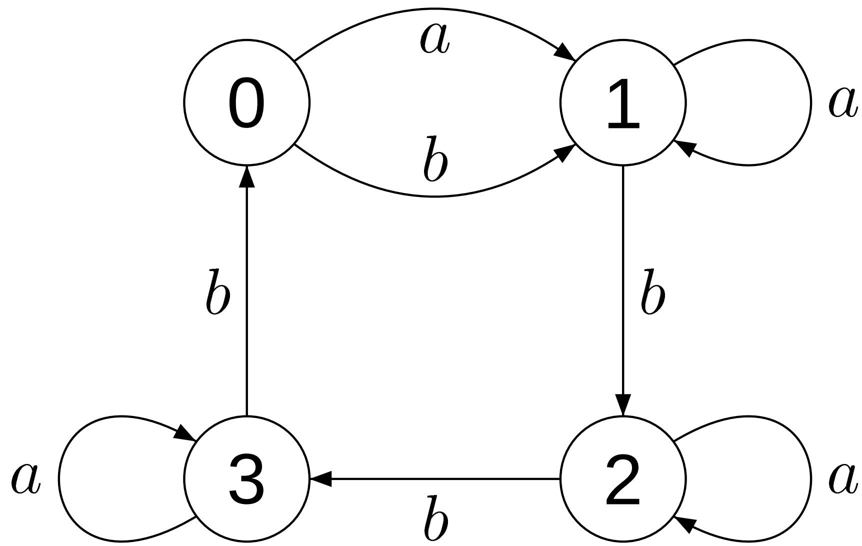

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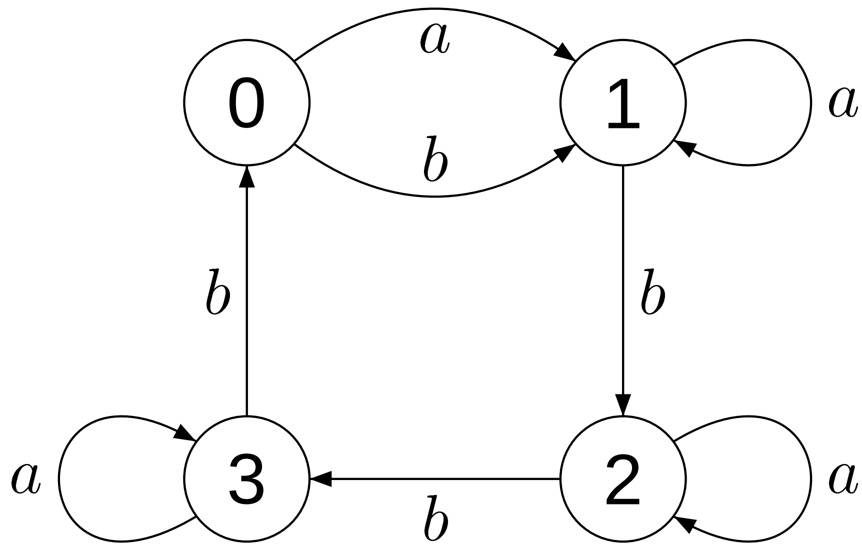

– this was our example of a synchronizing automaton, and we saw that $abbbabbba$ is a reset sequence of actions. Hence the series of obstacles

low-HIGH-HIGH-HIGH-low-HIGH-HIGH-HIGH-low

yields the desired sensorless orienter.

Possible Use in Biocomputing

In DNA-computing, there is a fast progressing work by Ehud Shapiro’s group on “*soup of automata*” (Programmable and autonomous computing machine made of biomolecules, Nature 414, no.1 (November 22, 2001) 430–434; DNA molecule provides a computing machine with both data and fuel, Proc. National Acad. Sci. USA 100 (2003) 2191–2196, etc).

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- Synchronizing automata constitute an interesting combinatorial object. Their studies are mainly motivated by the Černý conjecture. We discuss the Černý conjecture in Part II.
- Yet another mathematical motivation for studying synchronizing automata comes from symbolic dynamics. In Part III we present a recent breakthrough in the area—a (positive) solution to the Road Coloring Problem found by Trahtman.

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The *power automaton* $\mathcal{P}(\mathscr{A})$ of a given DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$:
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A $w \in \Sigma^*$ is a reset word for the DFA $\mathscr{A}$ iff w labels a path in $\mathcal{P}(\mathscr{A})$ starting at Q and ending at a singleton.

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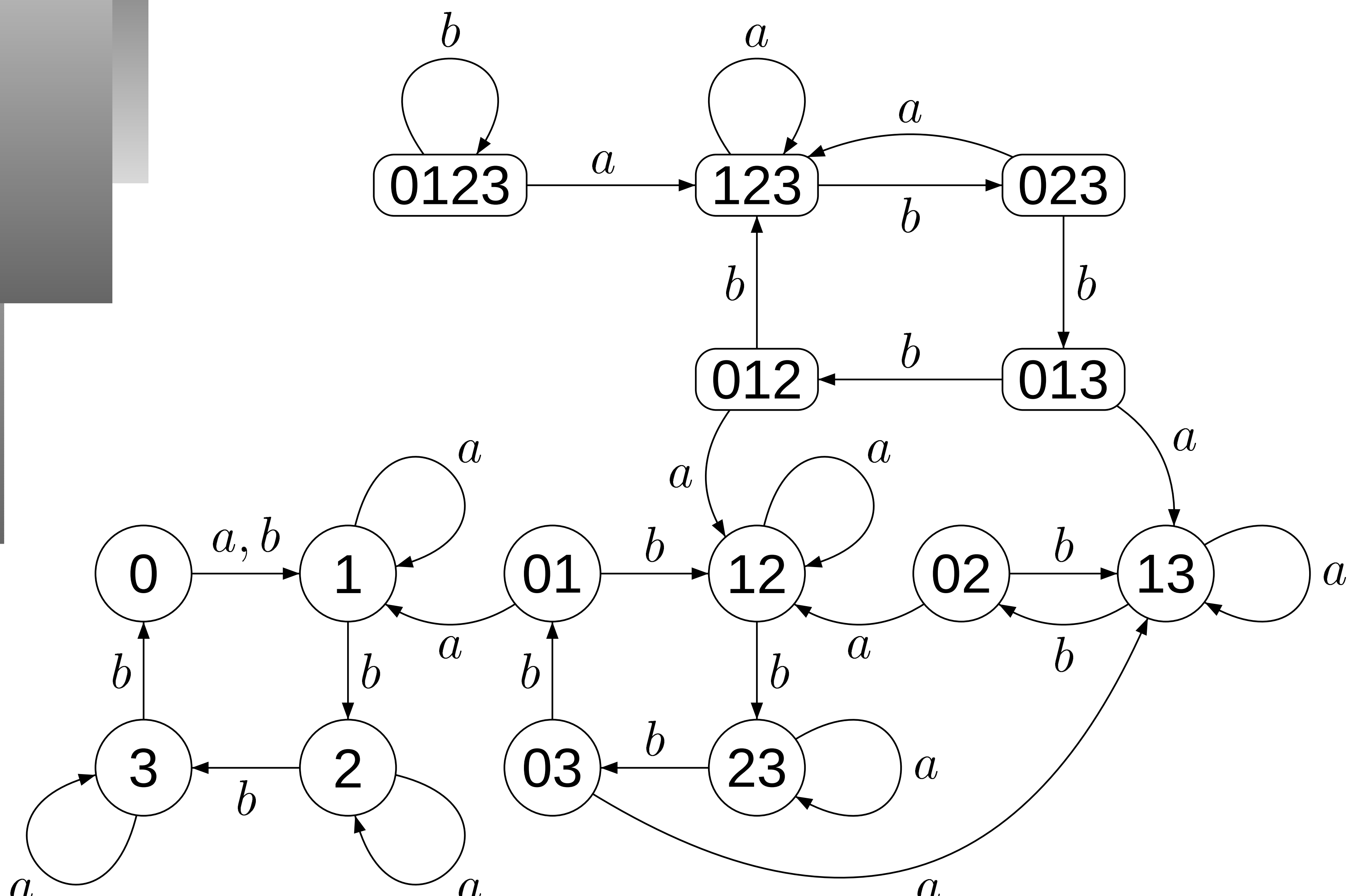

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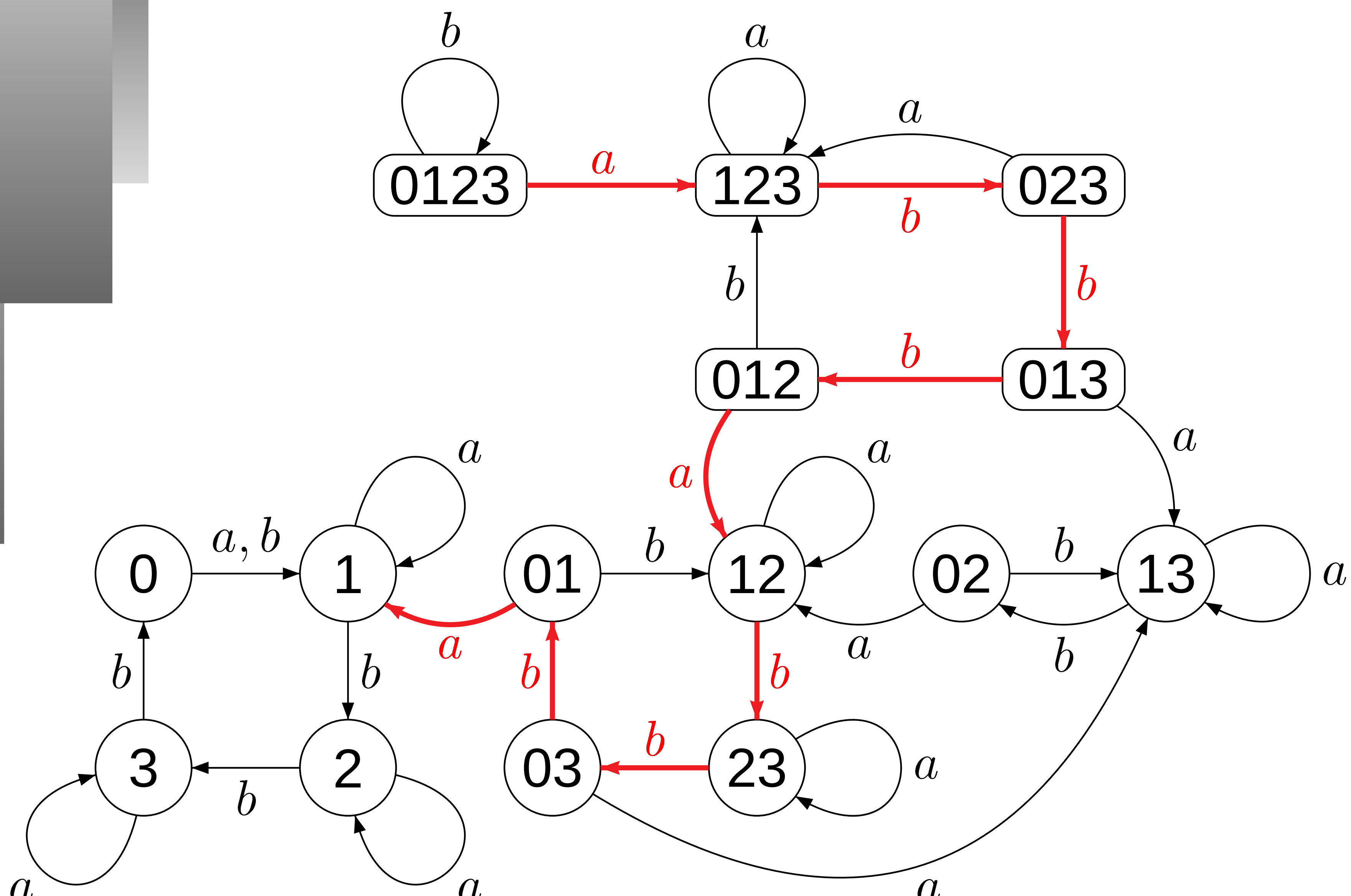

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The following result by Černý gives a polynomial algorithm:

Proposition. *A DFA* $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ *is synchronizing iff for every* $q, q' \in Q$ *there exists a word* $w \in \Sigma^*$ *such that* $\delta(q, w) = \delta(q', w)$.

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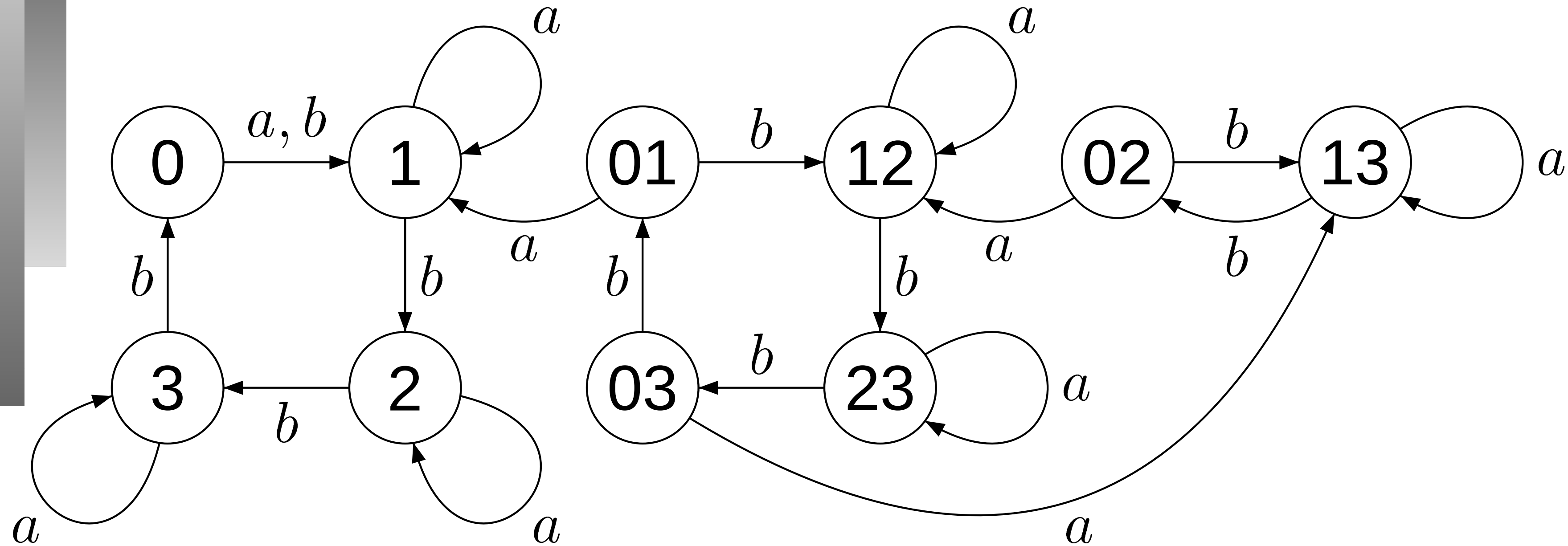

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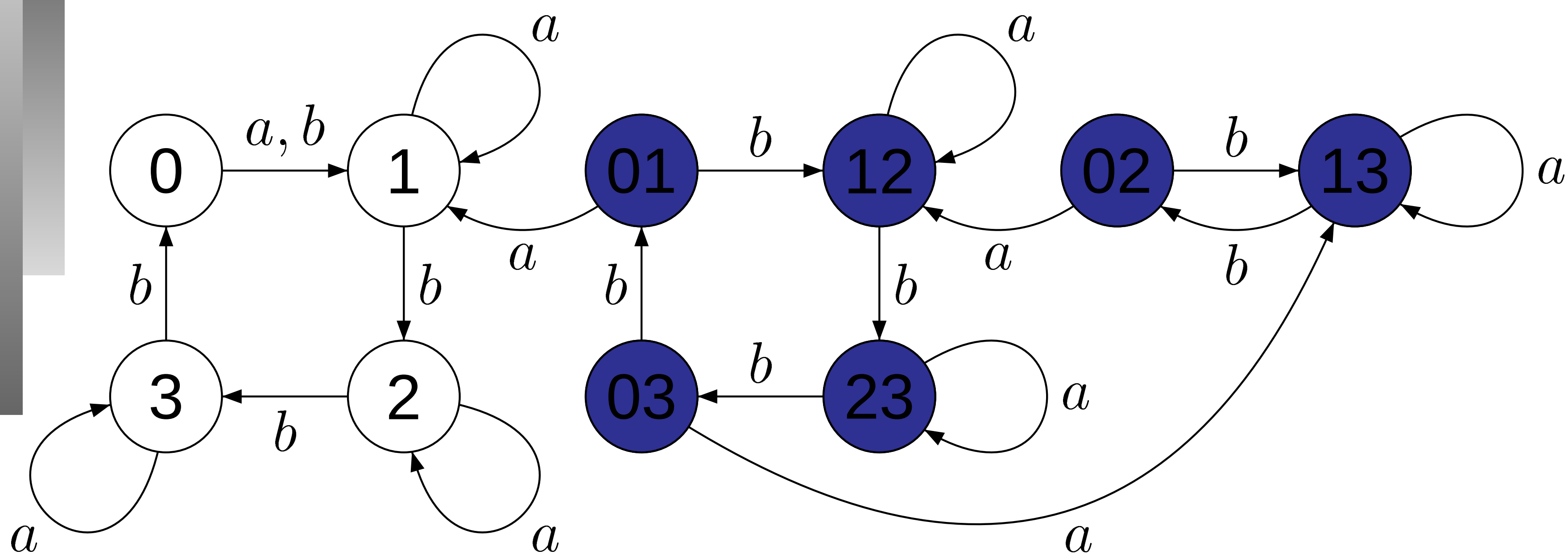

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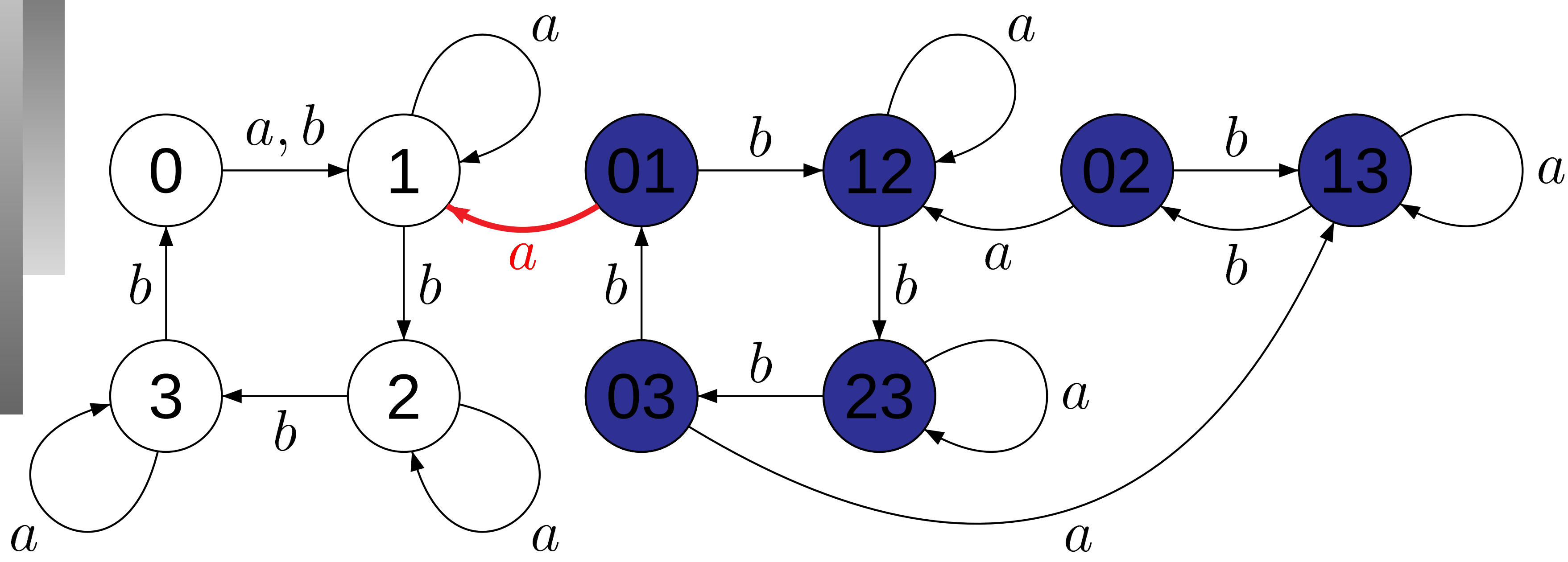

$$a,\ Q\,.\,a = \{1, 2, 3\}$$

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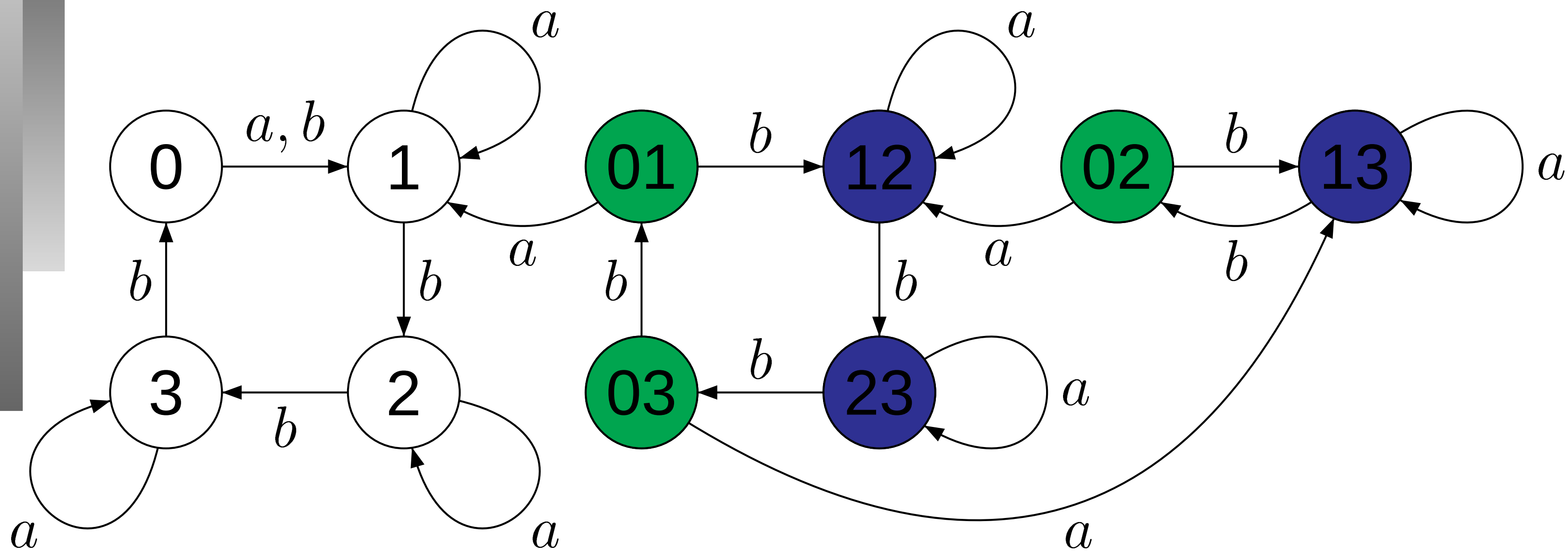

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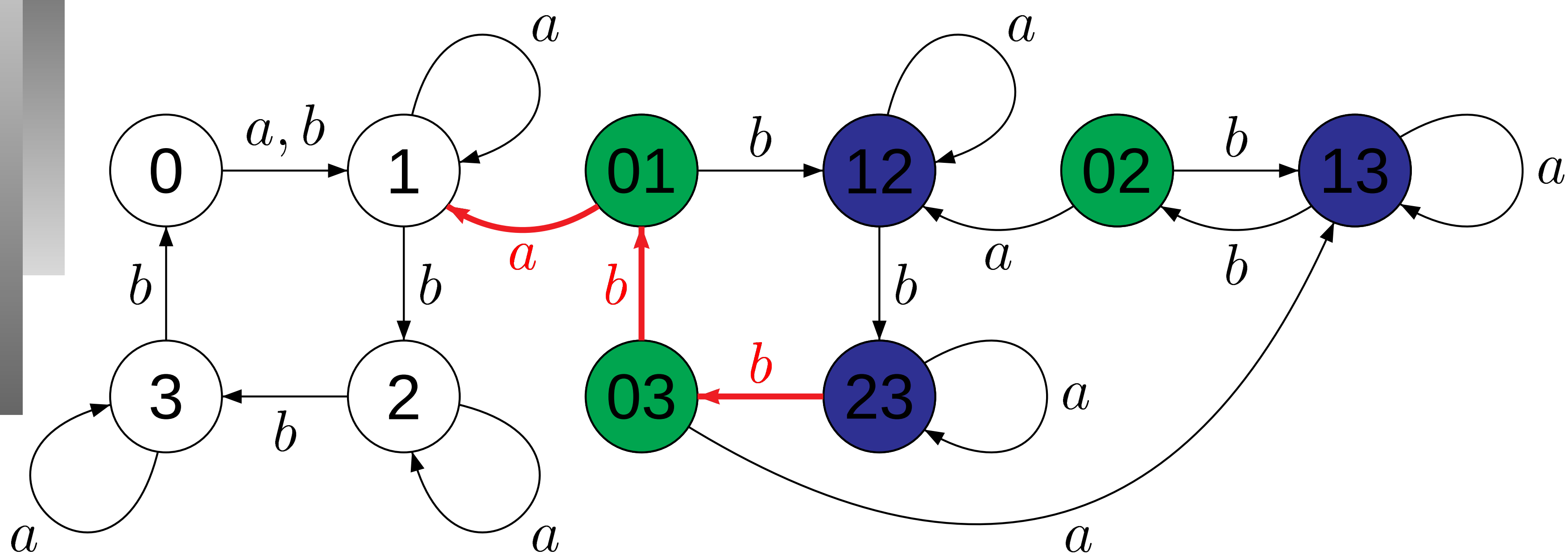

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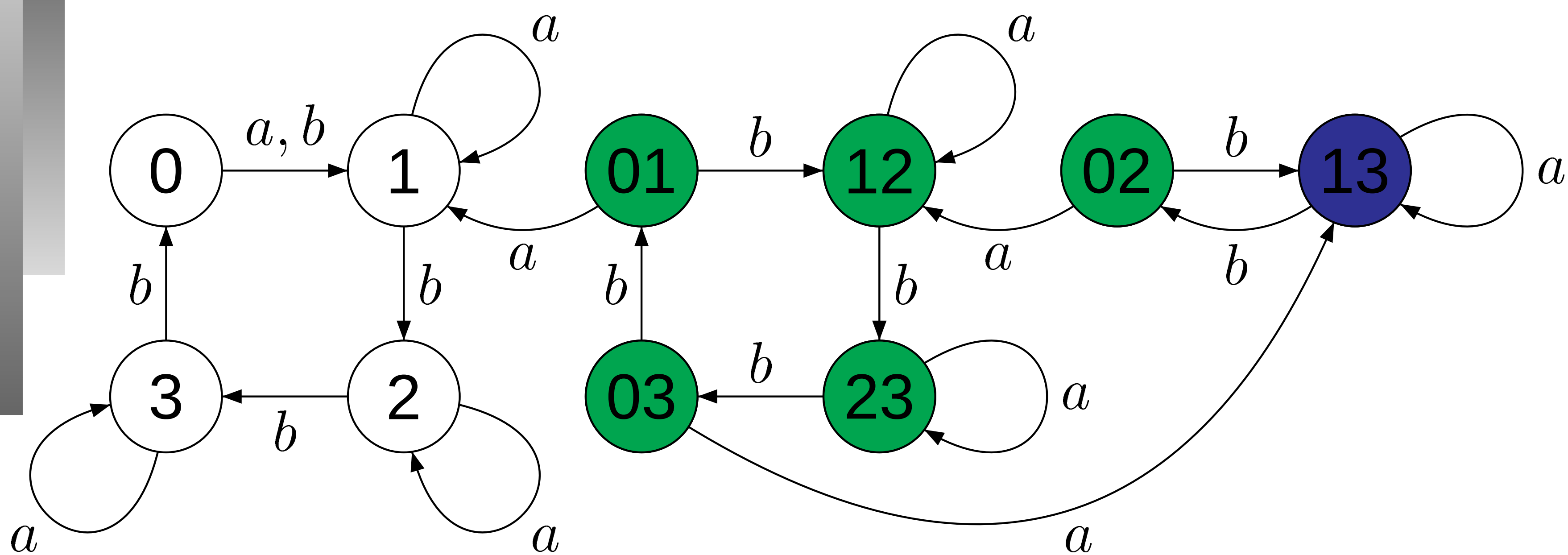

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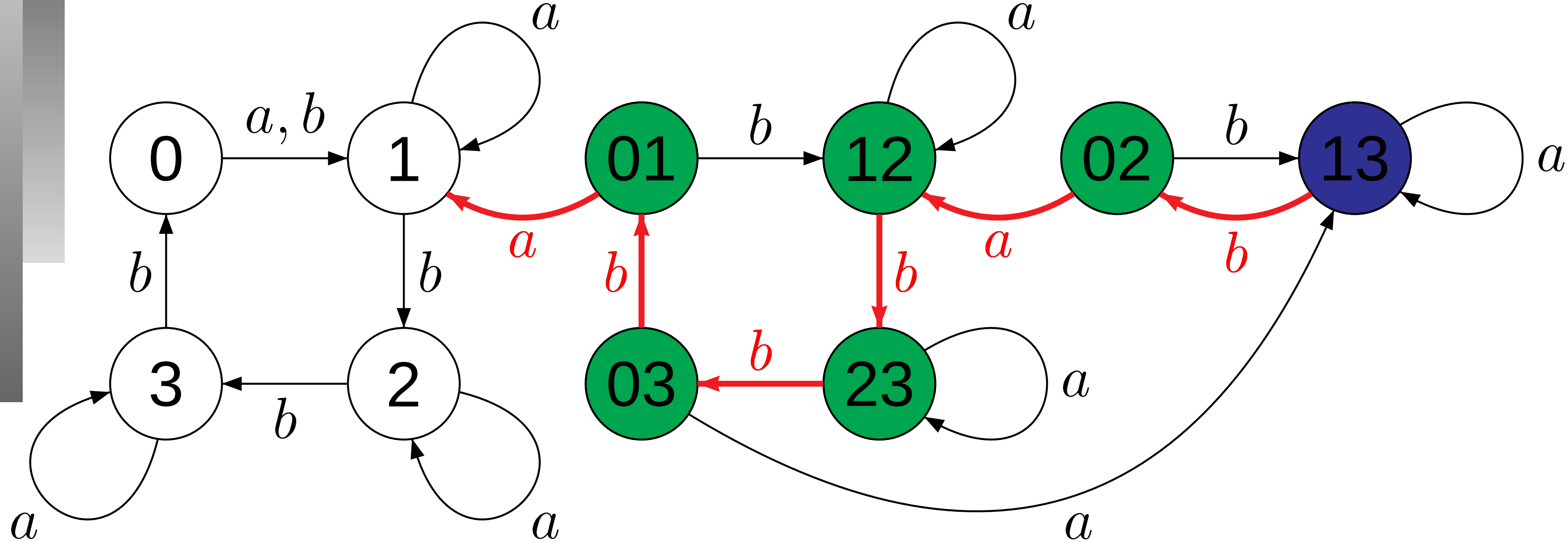

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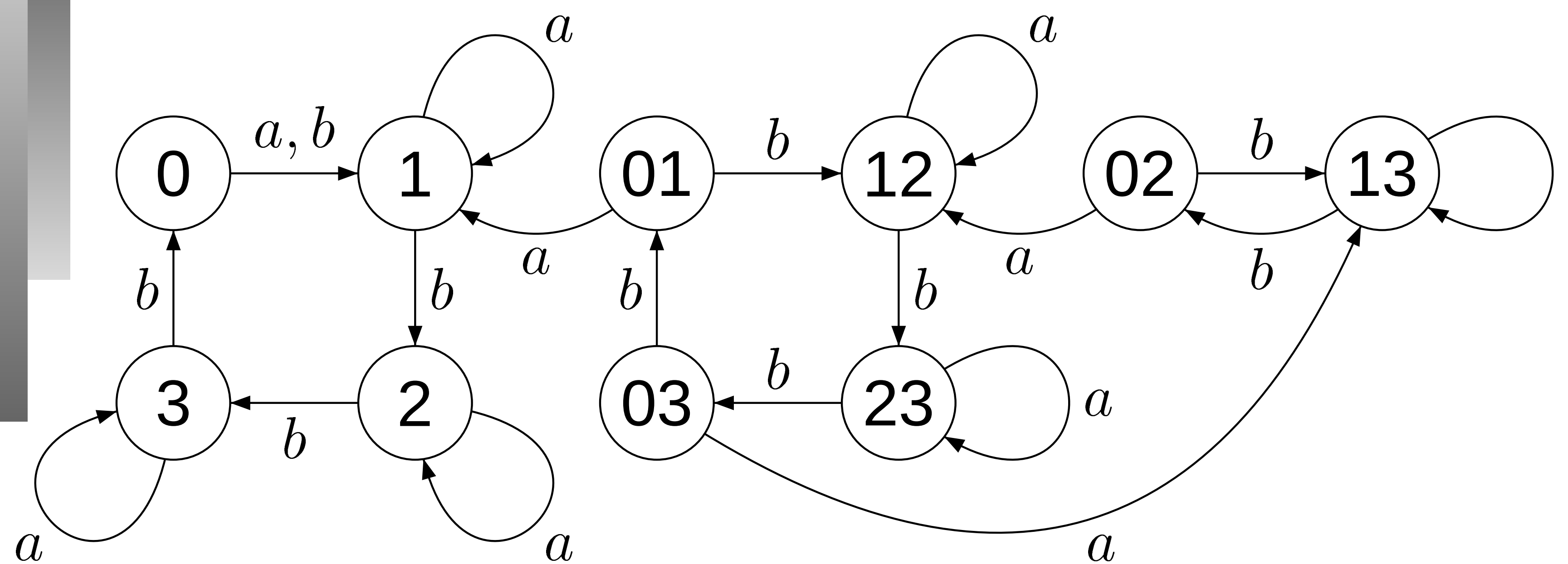

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Observe that the reset word constructed this way is of length 10 while we know a reset word of length 9 for this automaton.

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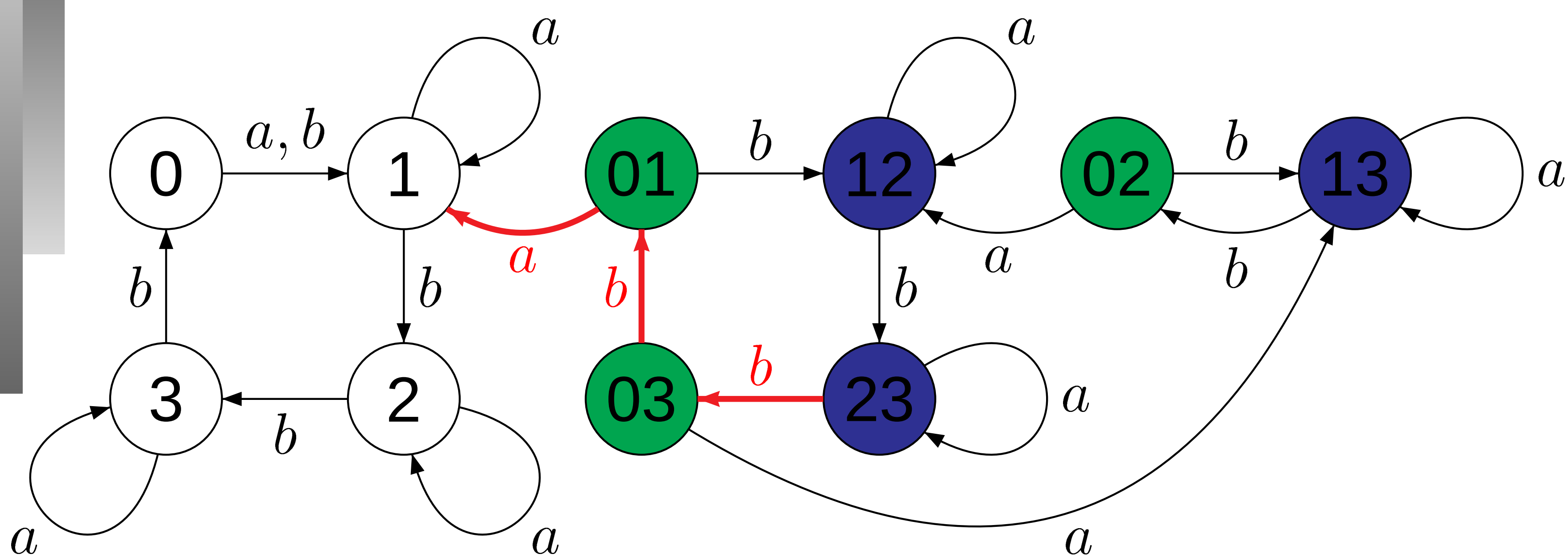
Studying Greedy Algorithm



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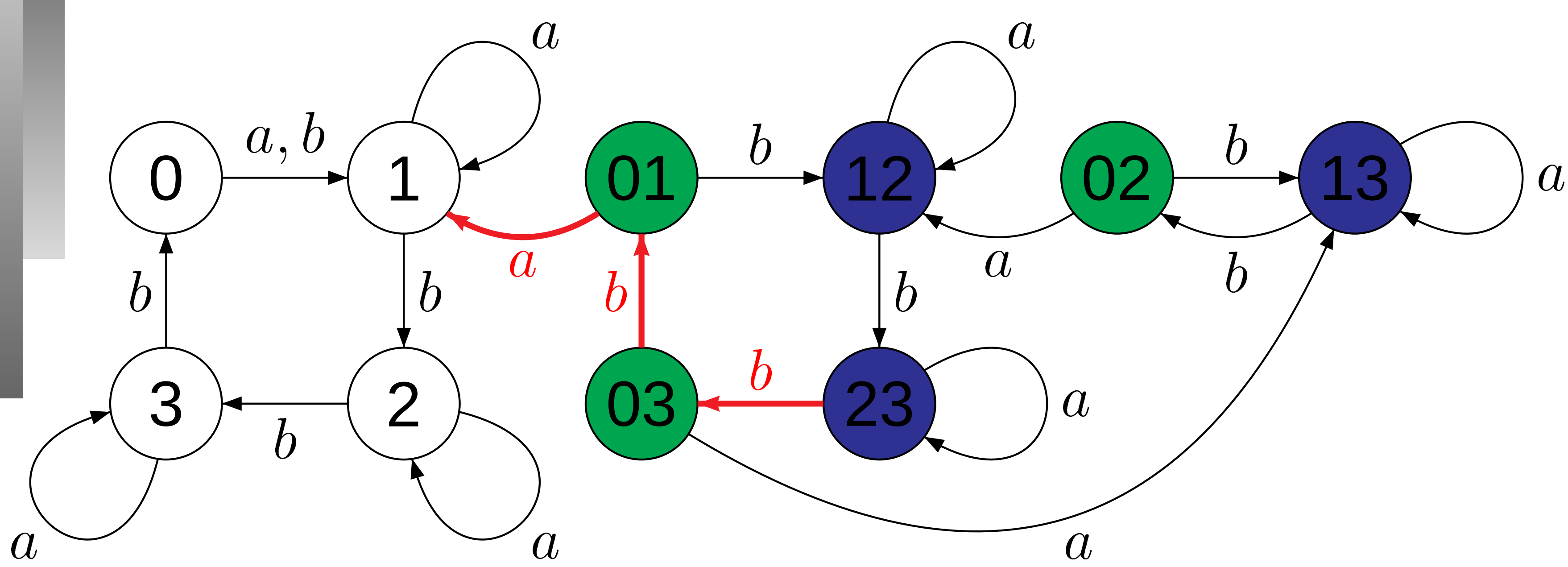

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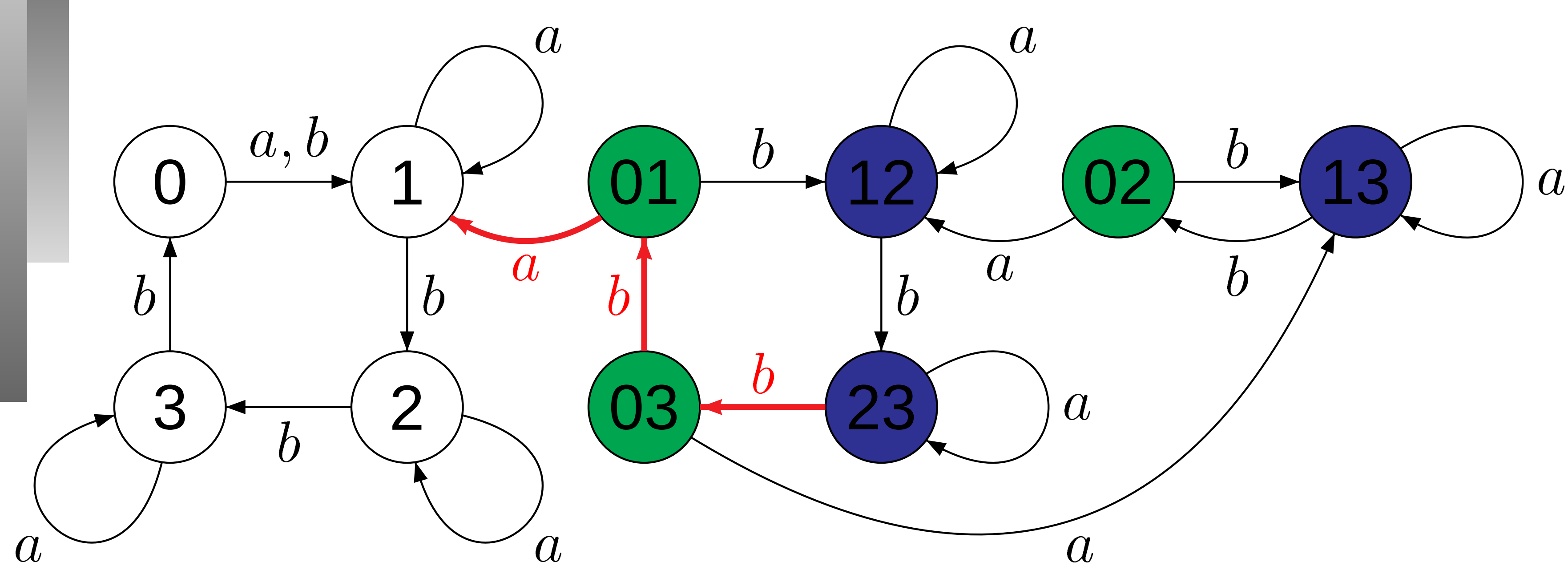

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Consider a generic step of the algorithm at which states to be compressed form a set P with $|P| = k > 1$ and let $v = a_1 \cdots a_\ell$ with $a_i \in \Sigma$, $i = 1, \ldots, \ell$, be a word of minimum length such that $|P \,.\, v| < k$.

Studying Greedy Algorithm

The sets $P_1 = P,\ P_2 = P_1 \,.\, a_1,\ \ldots,\ P_\ell = P_{\ell-1} \,.\, a_{\ell-1}$ are k-subsets of Q.

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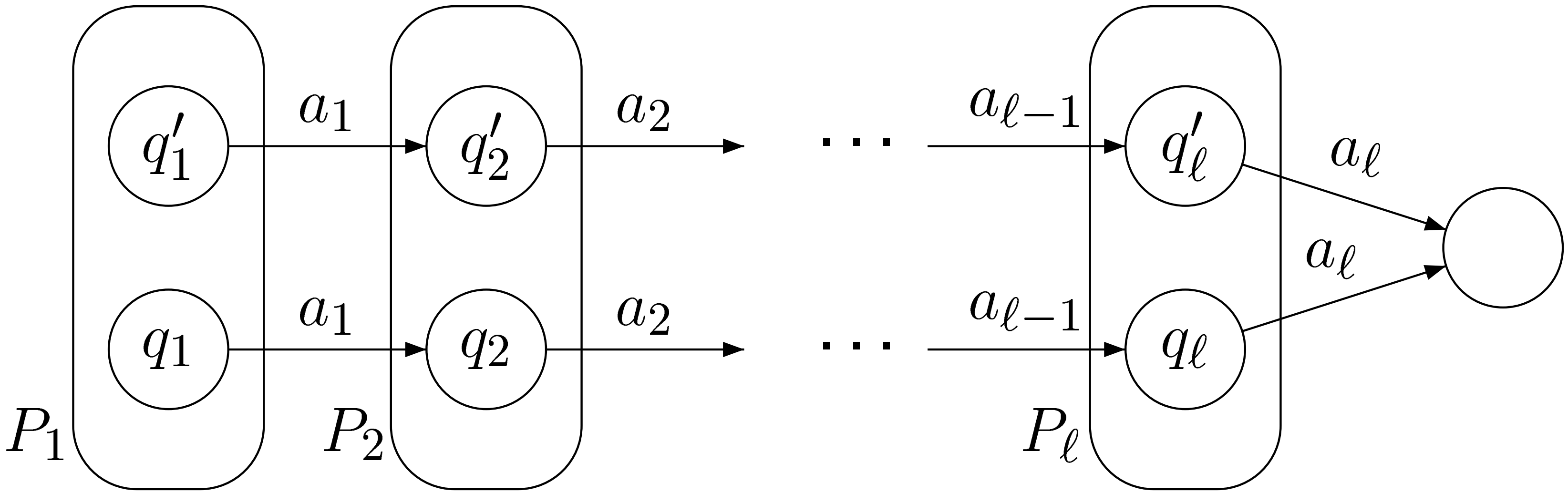

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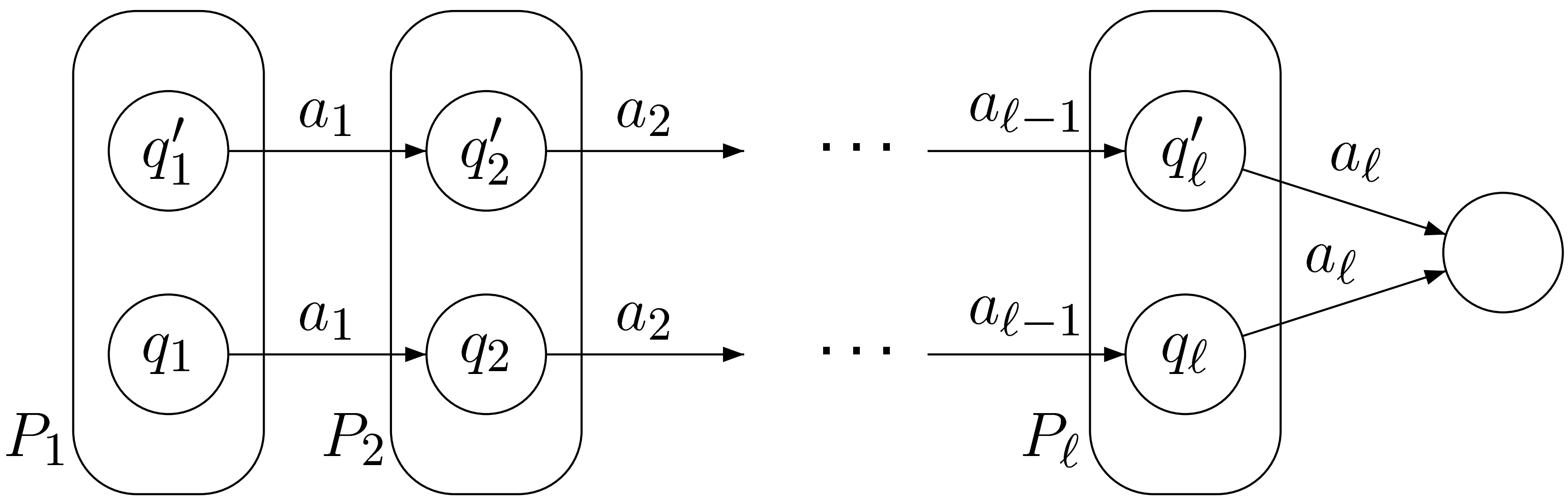

The condition that v is a word of minimum length with $|P \,.\, v| < |P|$ implies $R_i \not\subseteq P_j$ for $1 \leq j < i \leq \ell$.

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We identify Q with $\{1, 2, \ldots, n\}$ and assign to each k-subset $I = \{i_1, \ldots, i_k\}$ the following polynomial $D(I)$ in variables $x_{i_1}, \ldots, x_{i_k}$ over the field of rationals.

Combinatorial Configuration

$$I = \{i_1, \ldots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

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• the polynomials $D(T_1), \ldots, D(T_s)$ (derived from the "standard" sequence) generate the linear space spanned by all polynomials of the form $D(I)$.

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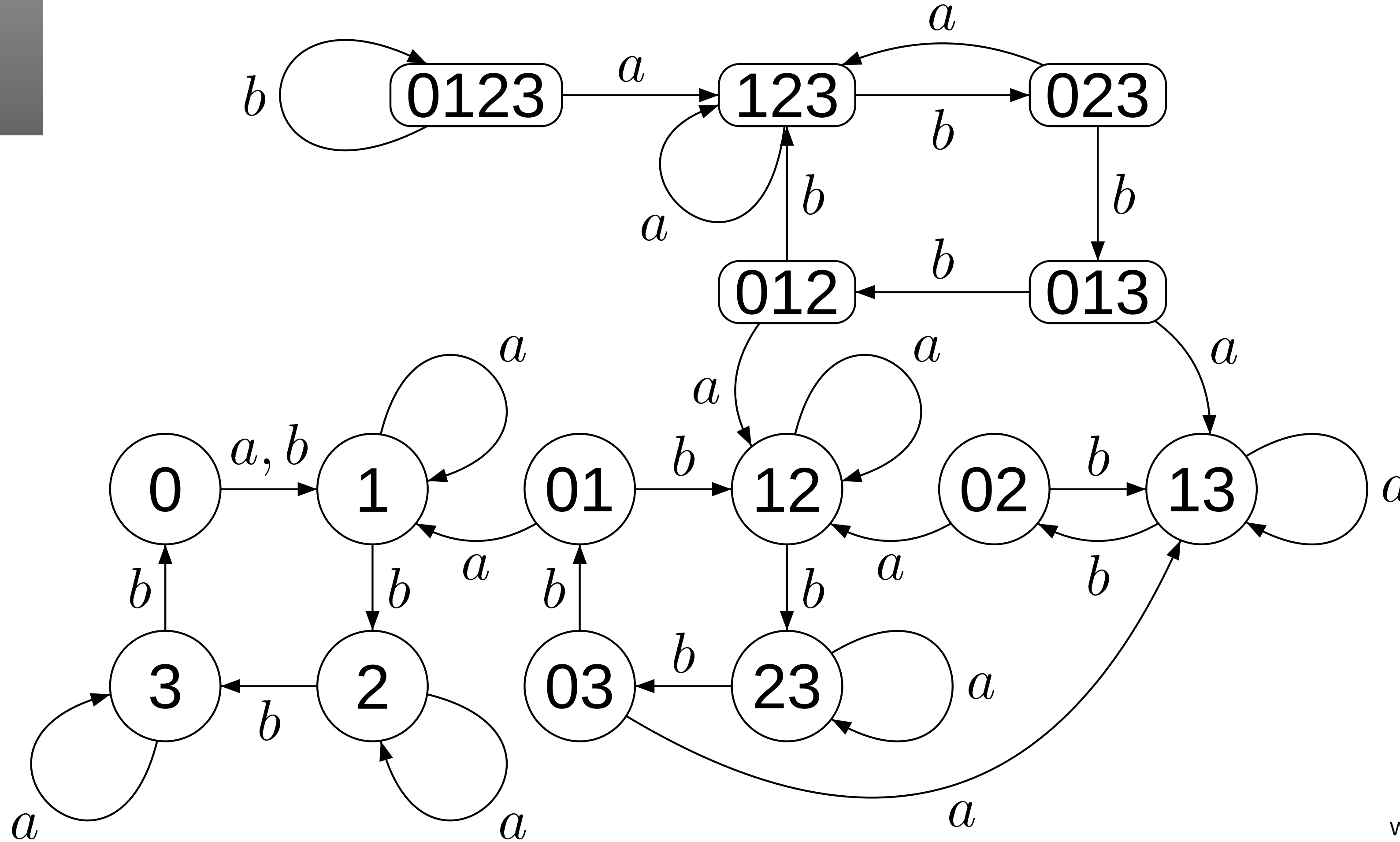

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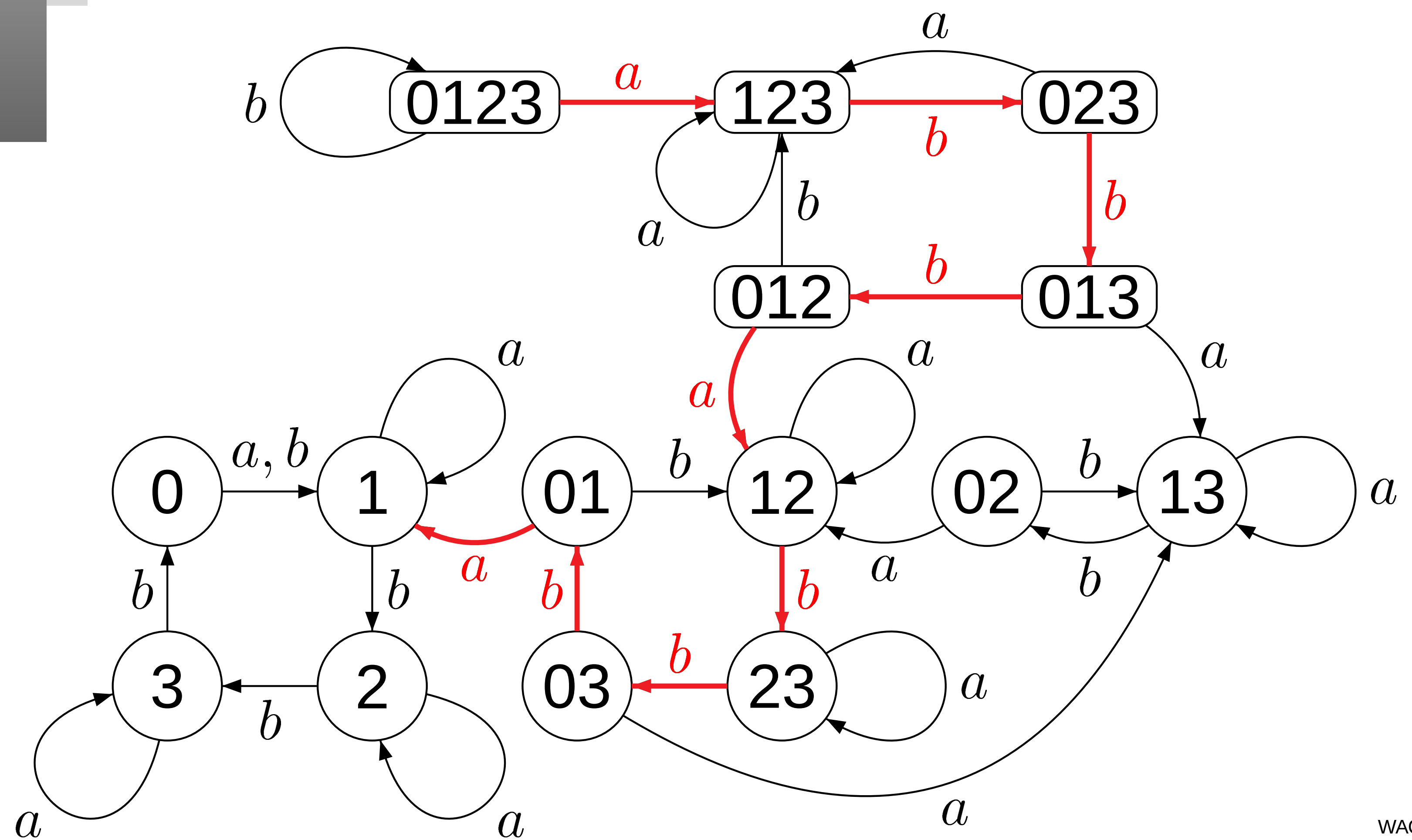

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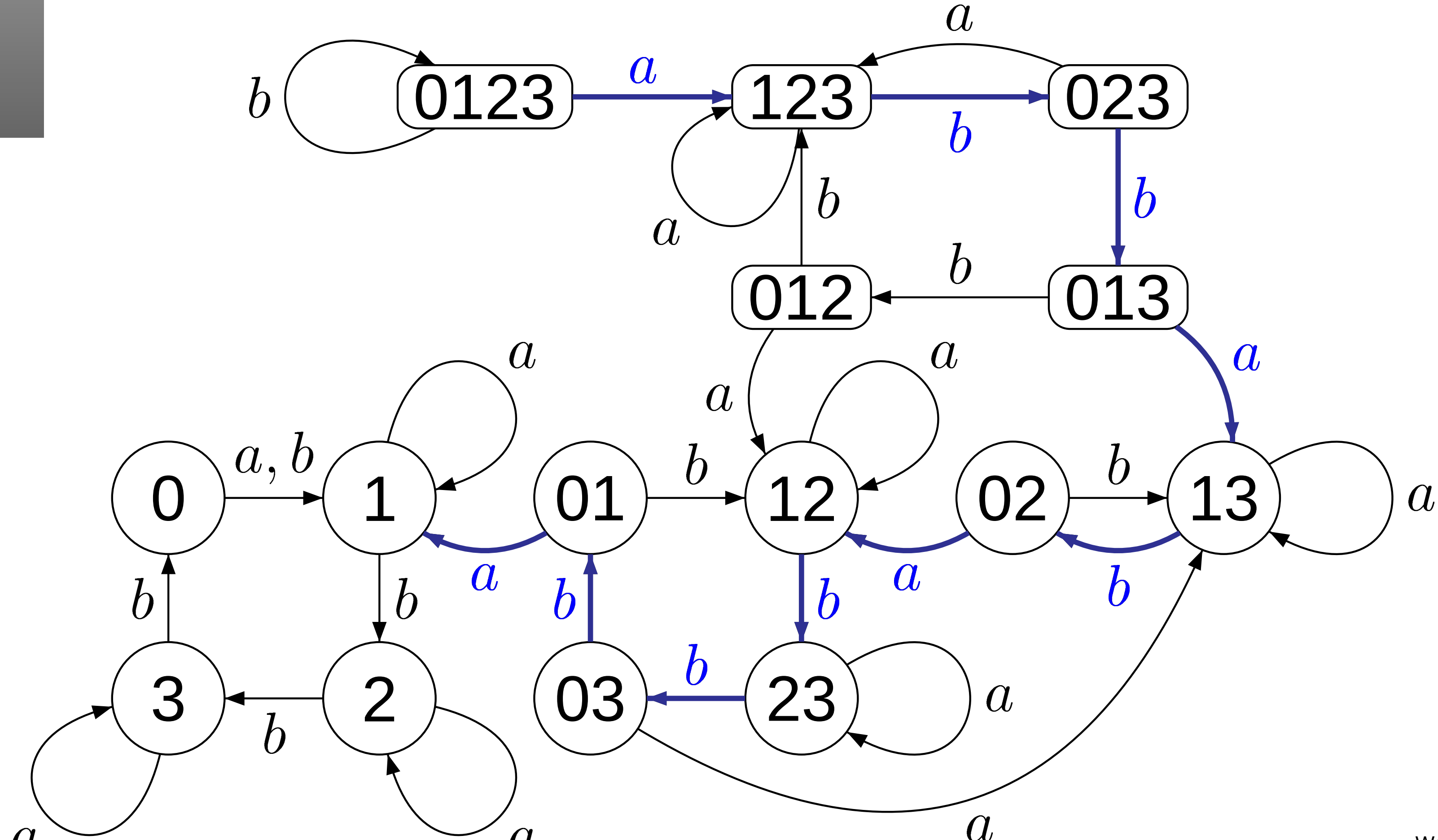

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The behaviour of the greedy algorithm on average is not yet understood; practically it behaves rather well.

Under standard assumptions (like NP $\neq$ coNP) no polynomial algorithm, even non-deterministic, can find the minimum length of reset words for synchronizing automata, see the proceedings.

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Very recently, Mikhail Berlinkov. a PhD student of mine, has shown that under NP $\neq$ P, for no k, there may exists a polynomial algorithm that, given a synchronizing automaton, produces a reset word whose length is less than $k\times$minimum possible length of a reset word.

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The automaton in the previous slide is $\mathscr{C}_4$.

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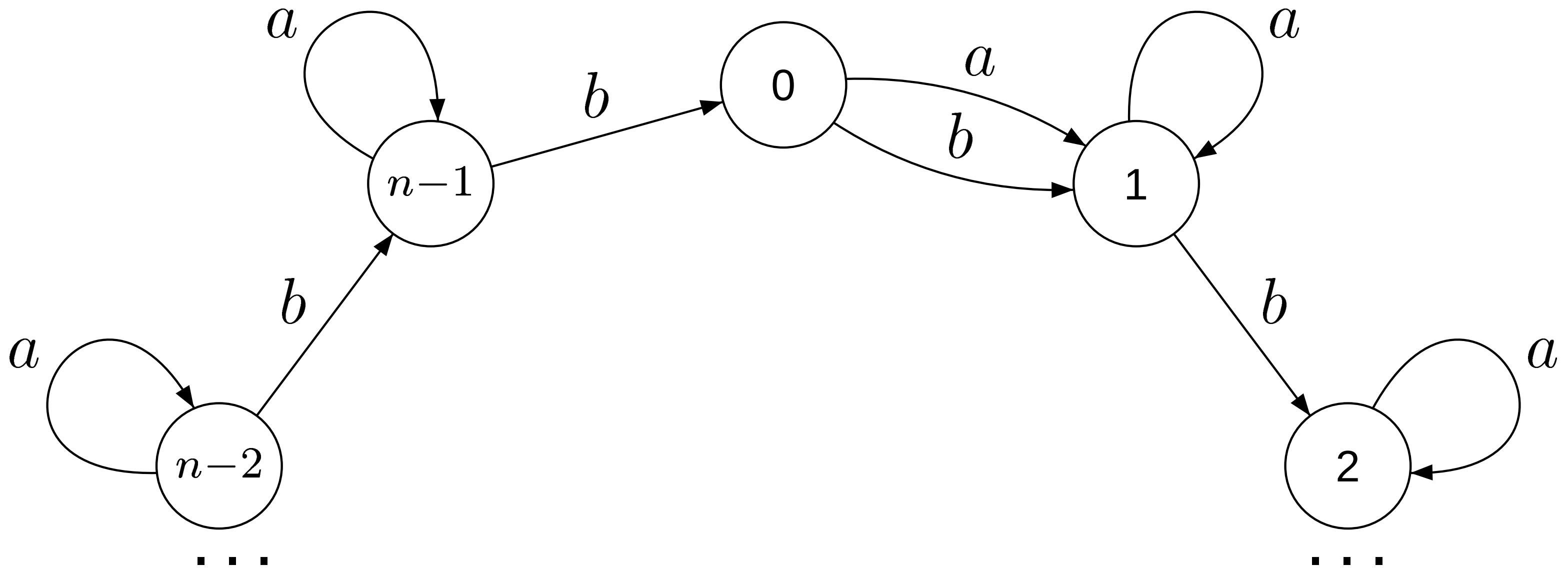

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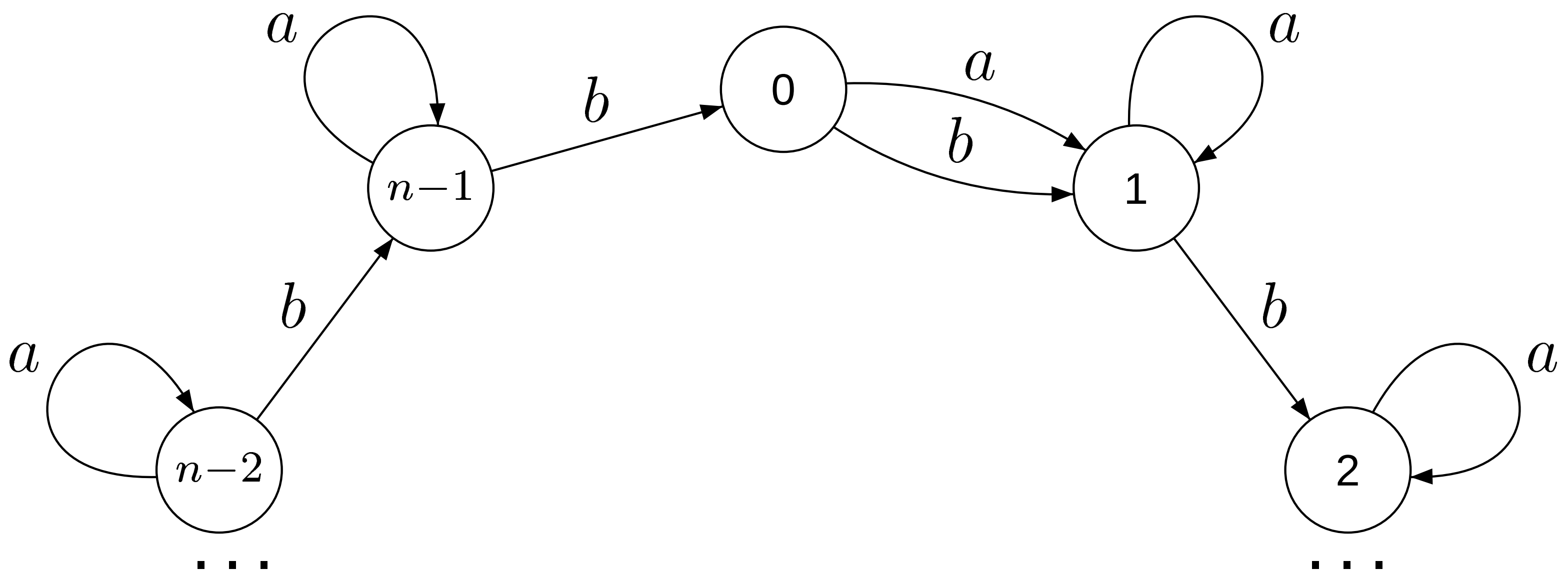

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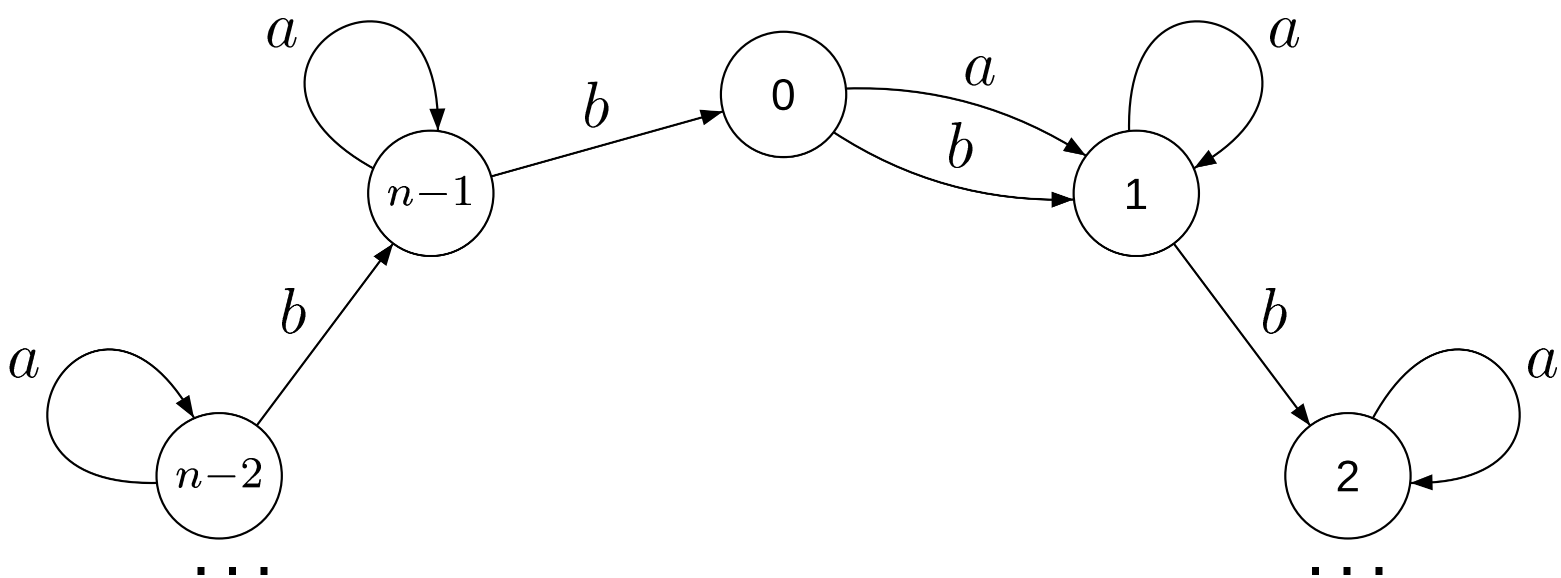

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$. As other results from Černý's paper of 1964, this nice series of automata has been rediscovered many times, see references in the pre-proceedings.

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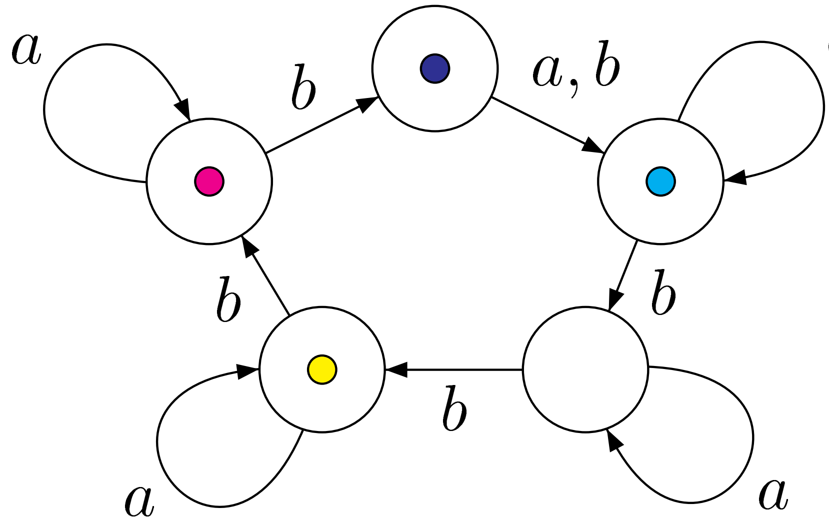
- The only coin that remains at the end of the game is the golden coin G.

The Černý automata



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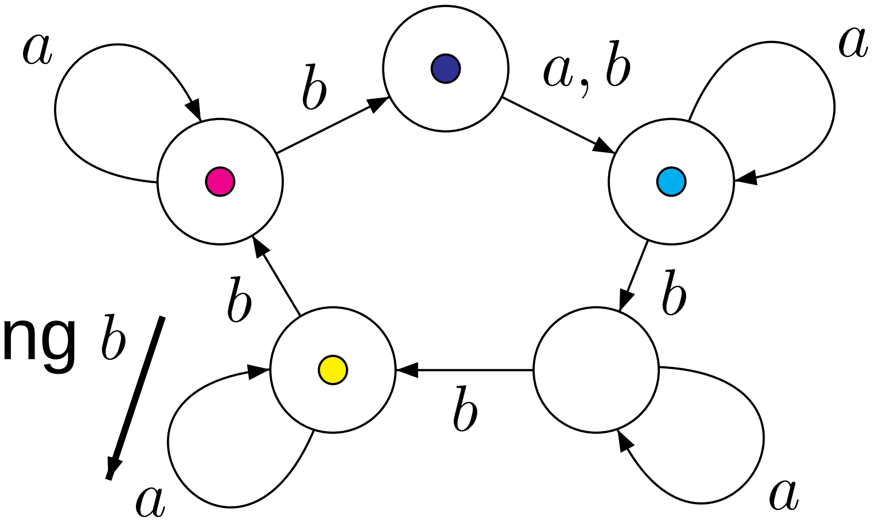
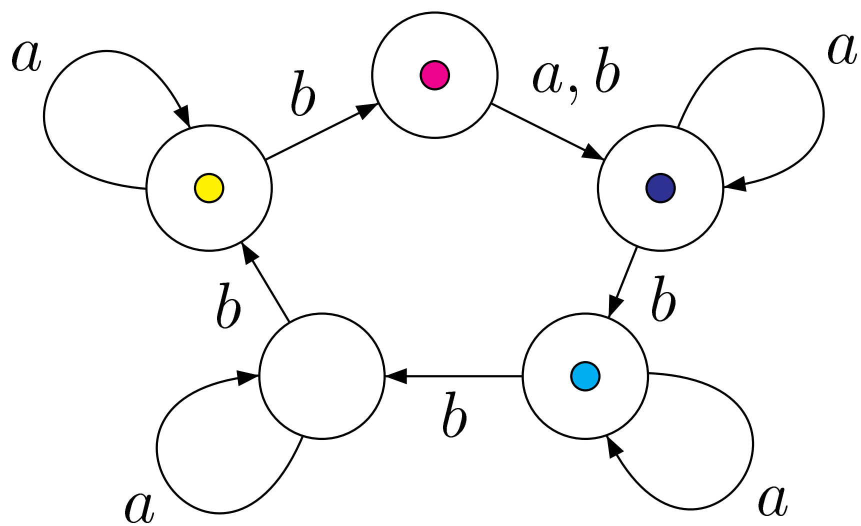


Applying b



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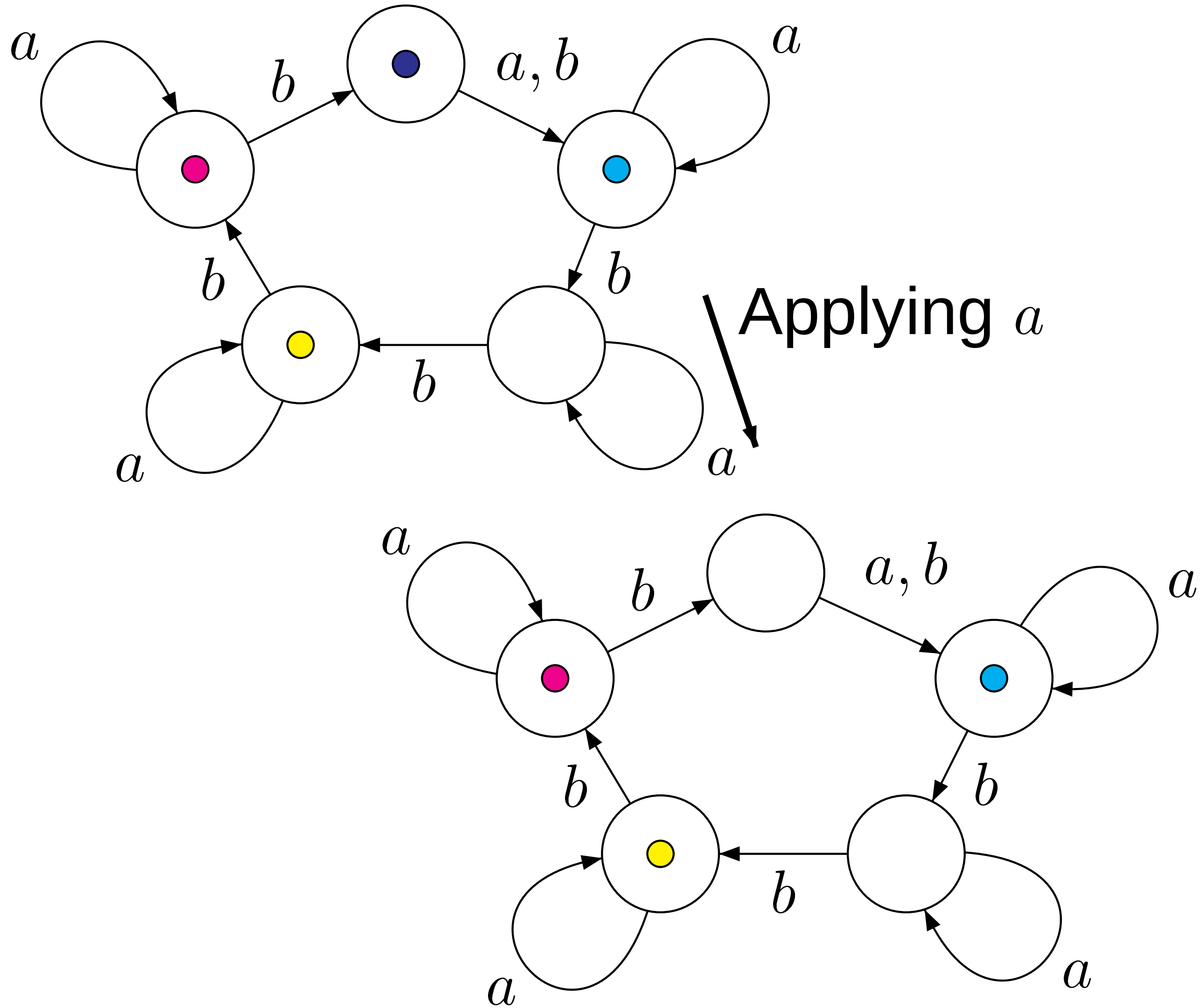

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Then $|w| = \sum_{i=1}^{|w|} 1 \geq \sum_{i=1}^{|w|} \big(\mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)\big) =$

$$\mathrm{wg}(P_0) - \mathrm{wg}(P_{|w|}) \geq n(n-1) - (n-1) = (n-1)^2.$$

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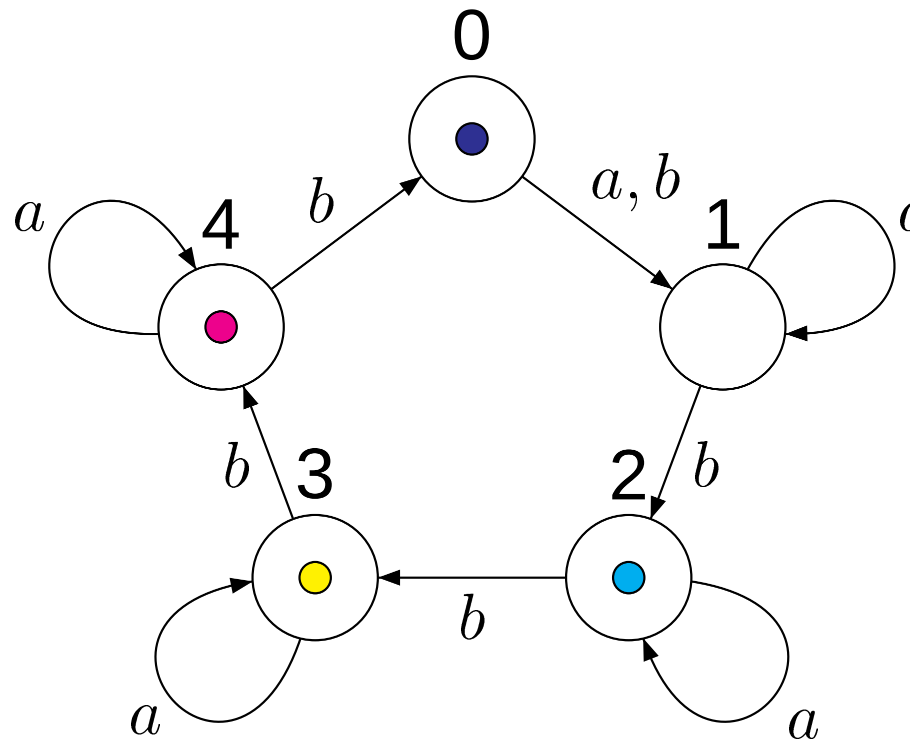
The Černý automata



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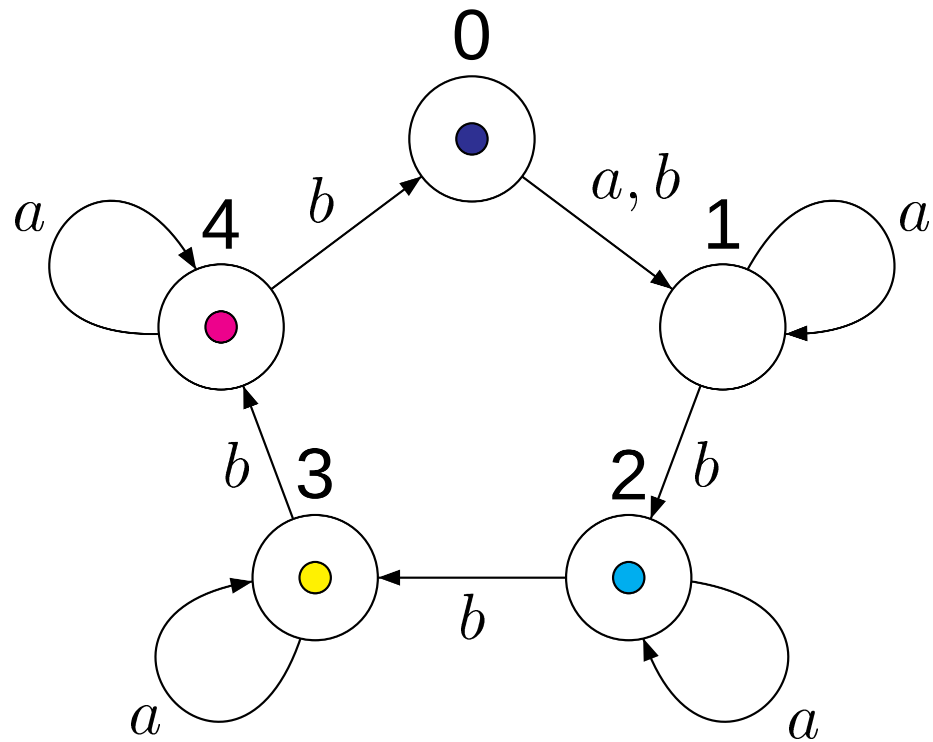

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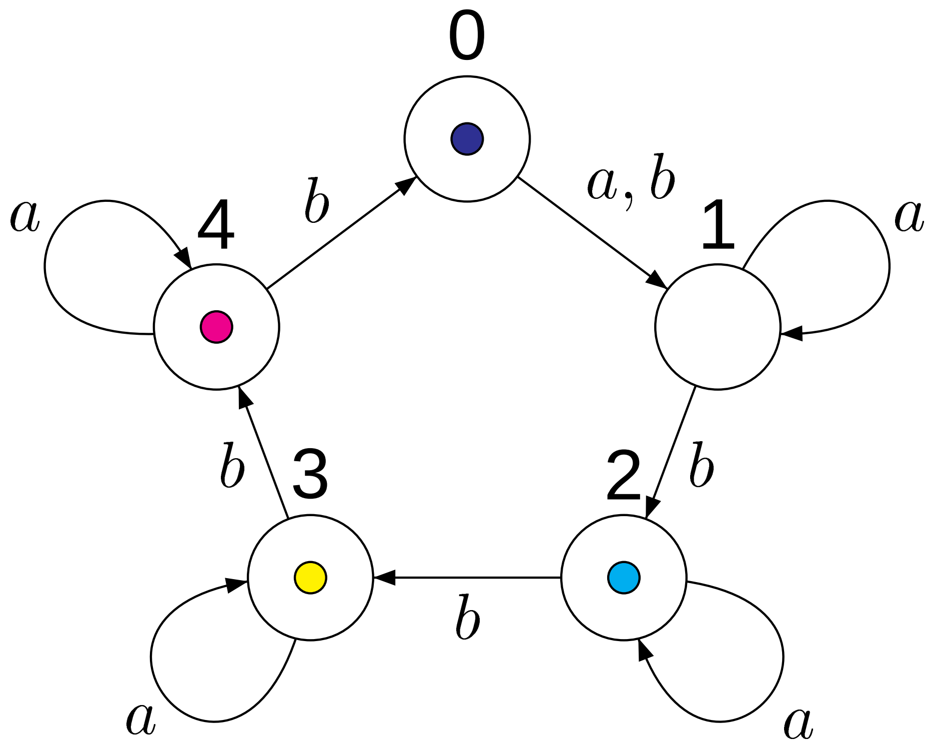

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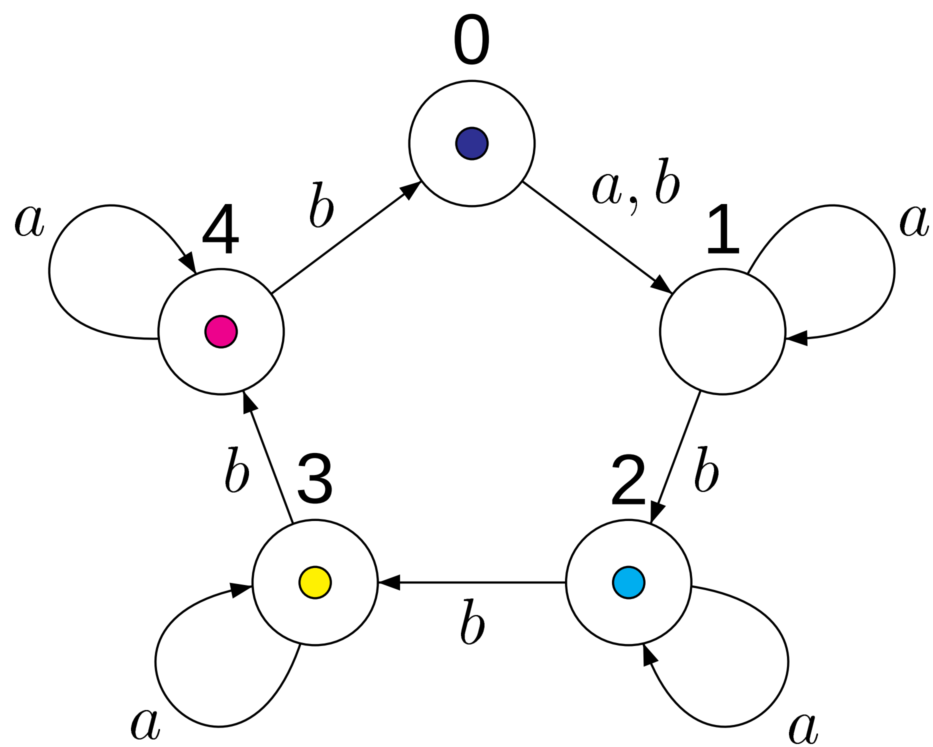

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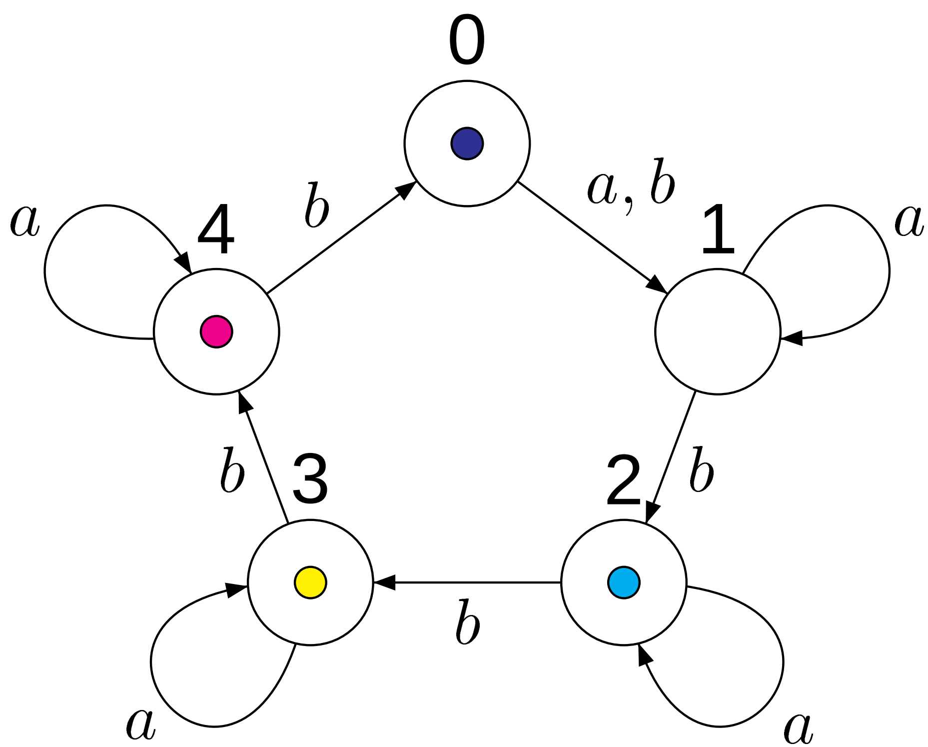

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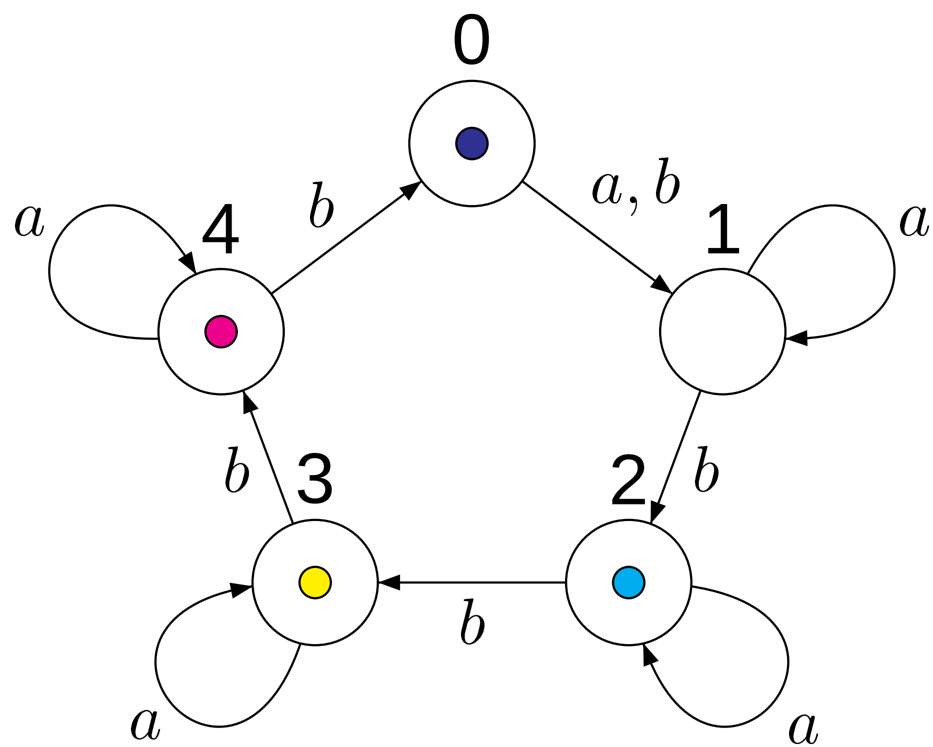

Assume that the yellow coin is the golden one. Then its weight is $5 - 3 = 2$. The weight of the cyan coin is $5 \cdot 1 + (5 - 2) = 8$. The weight of the blue coin is $5 \cdot 3 + 0 = 15$. The weight of the magenta coin is $5 \cdot 4 + (5 - 4) = 21$, and this is the weight of the position.

The Černý automata

We have to check that our weight function satisfies the conditions

(i) $\mathrm{wg}(P_0) \geq n(n-1)$ and $\mathrm{wg}(P_{|w|}) \leq n-1$;

(ii) $1 \geq \mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)$ for each $i = 1, \ldots, |w|$.

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In the initial position all states are covered with coins. Consider the coin C that covers the state $s_0(G) + 1 \pmod{n}$, that is the state in one step clockwise after the state covered with the golden coin. Then $d_0(C) = n - 1$ whence $\mathrm{wg}(C, P_0) = n \cdot (n-1) + m_0(C) \geq n(n-1)$. Since the weight of a position is not less that the weight of any coin in this position, we have $\mathrm{wg}(P_0) \geq n(n-1)$.

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In the final position only the golden coin G remains, whence the weight of $P_{|w|}$ is the weight of G. Clearly, $\mathrm{wg}(G, P_i) = m_i(G) \leq n - 1$ for any position P_i.

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Let C be a coin of maximum weight in P_{i-1}. If the transition from P_{i-1} to P_i is caused by b, then $d_i(C) = d_{i-1}(C)$ (because the relative location of the coins does not change) and $m_i(C) = m_{i-1}(C) - 1$ if $m_{i-1}(C) > 0$, otherwise $m_i(C) = n - 1$.

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$$\mathrm{wg}(P_i) \geq \mathrm{wg}(C, P_i) = n \cdot d_i(C) + m_i(C) \geq$$

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Suppose the transition from P_{i-1} to P_i is caused by a. If $s_{i-1}(C) \neq 0$, then $m_i(C) = m_{i-1}(C)$ and $d_i(C) = d_{i-1}(C)$ if $s_{i-1}(G) \neq 0$, otherwise $d_i(C) = d_{i-1}(C) + 1$.

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$$\mathrm{wg}(P_i) \geq \mathrm{wg}(C', P_i) = n \cdot d_i(C') + n - 1 = n \cdot (d_{i-1}(C) - 1) + n - 1$$
$$= n \cdot d_{i-1}(C) - 1 = \mathrm{wg}(C, P_{i-1}) - 1 = \mathrm{wg}(P_{i-1}) - 1.$$

The Černý conjecture

Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, $n = 2, 3, \ldots$, yields the inequality $C(n) \geq (n-1)^2$.

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$$(n-1)^2 \leq C(n) \leq \frac{n^3 - n}{6}.$$

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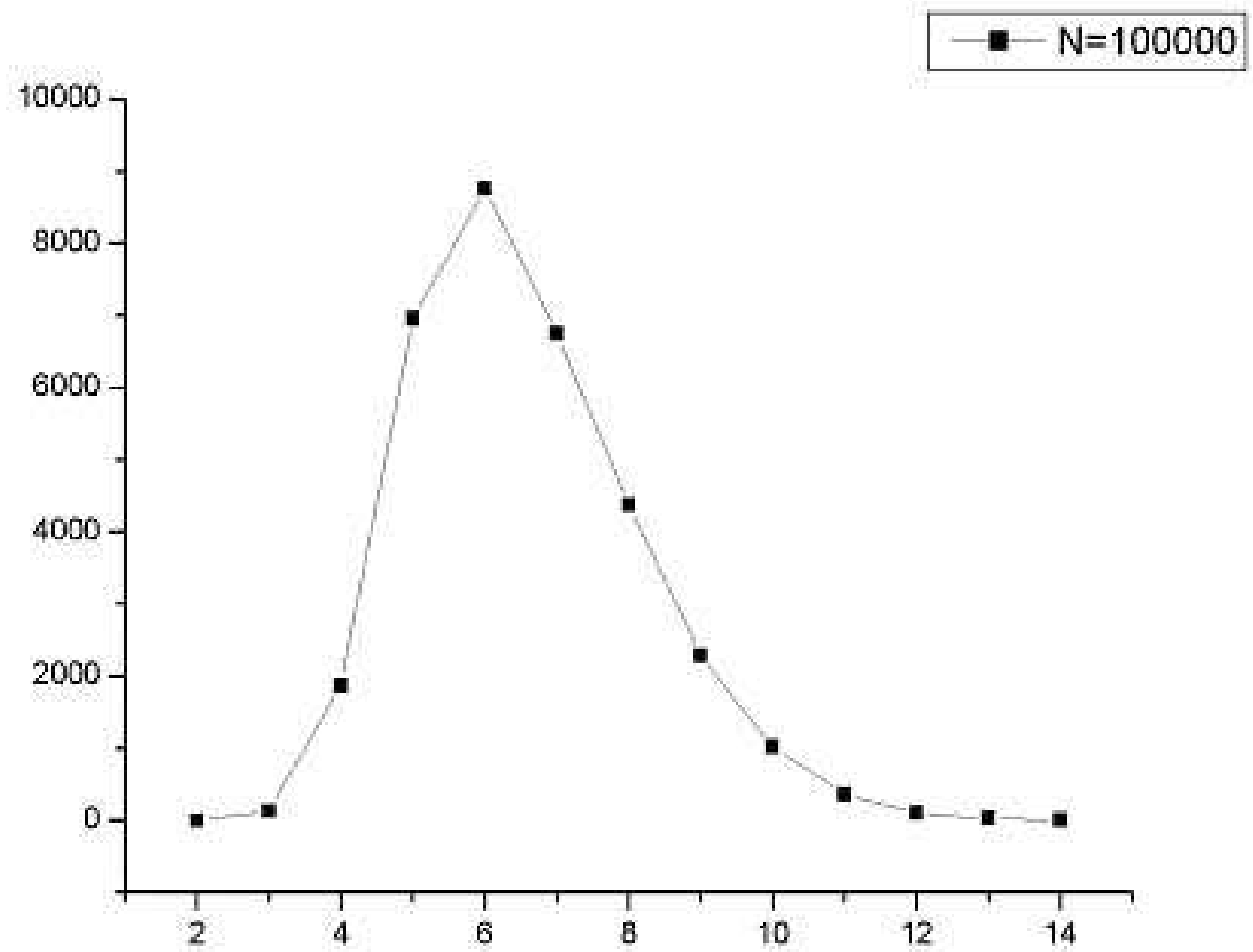
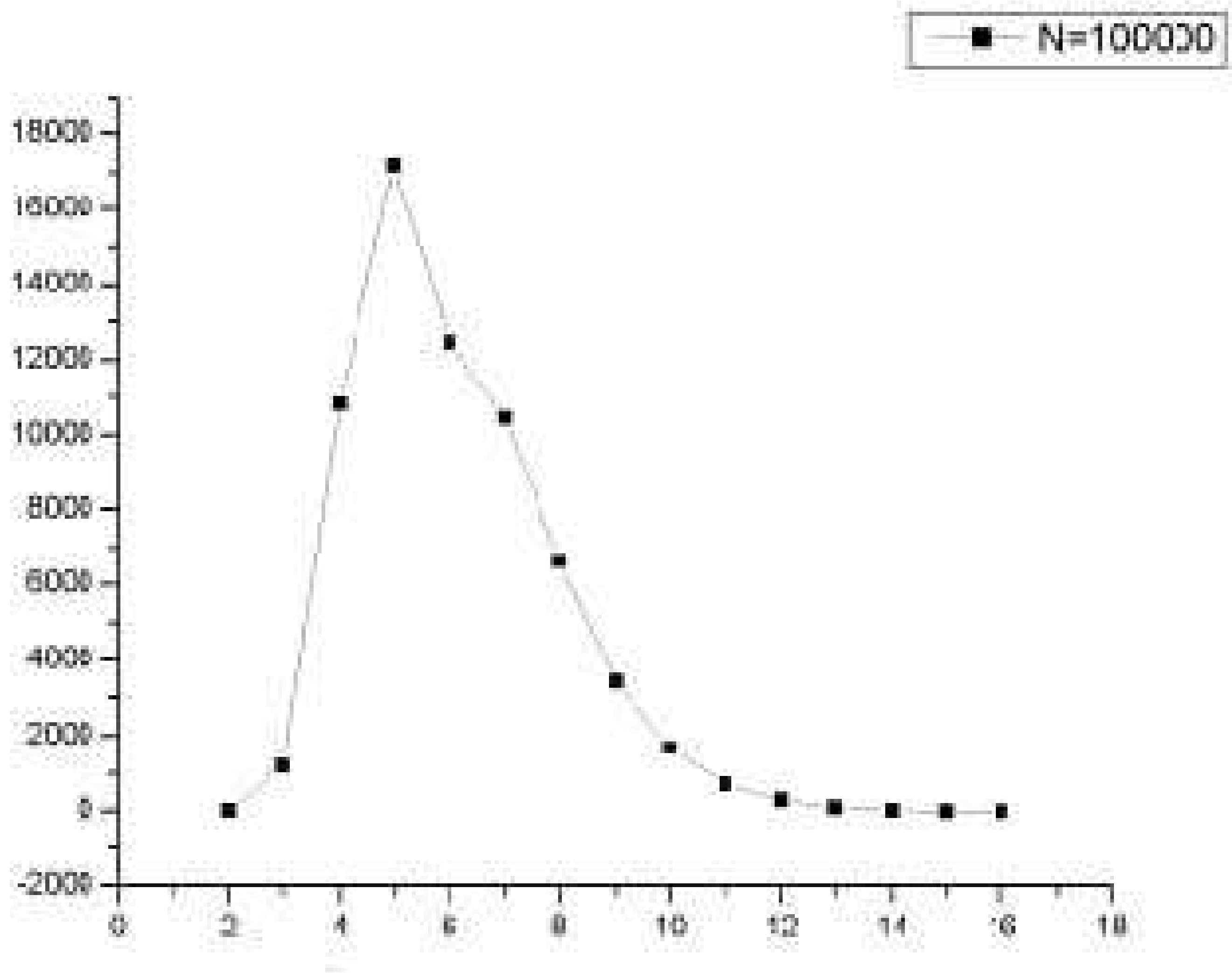
Yet another reason: "slowly" synchronizing automata turn out to be extremely rare. The only known infinite series of n-state synchronizing automata with shortest reset words of length $(n-1)^2$ is the Černý series $\mathscr{C}_n$, $n = 2, 3, \dots$, with a few sporadic examples for $n \leq 6$.

20-State Experiment



30-State Experiment



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Thus, “slowly” synchronizing automata cannot be discovered via a random sampling.

Sporadic Examples

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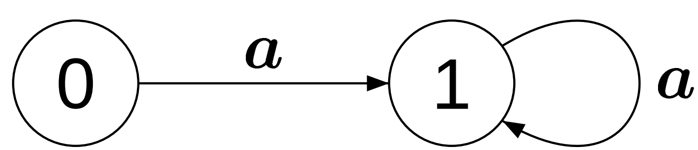
For the sake of completeness, we start with $n = 2$:



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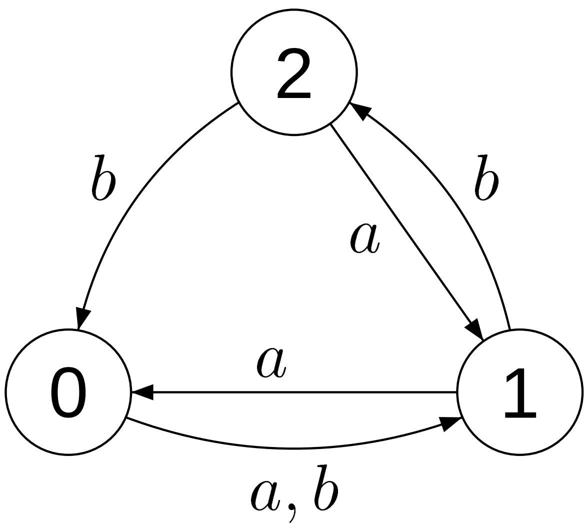

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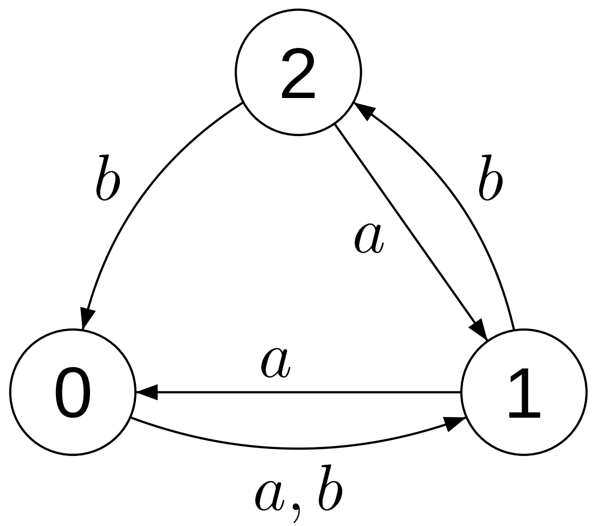

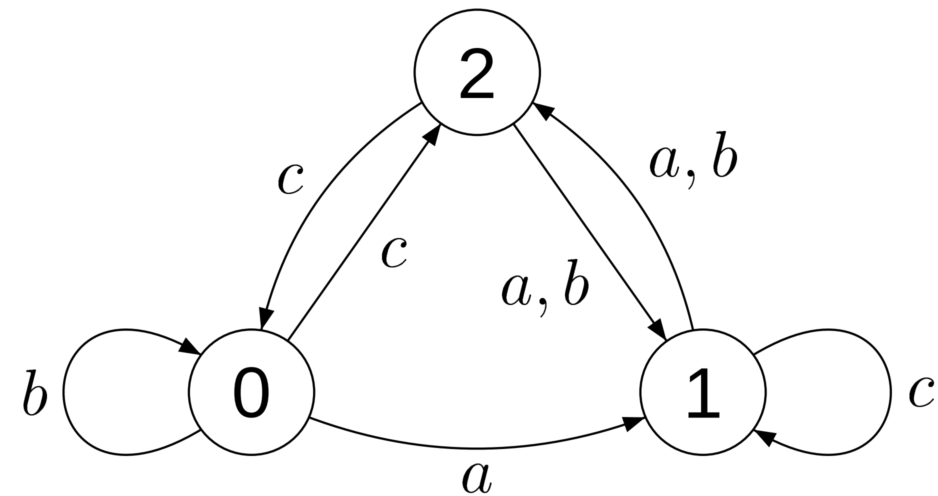

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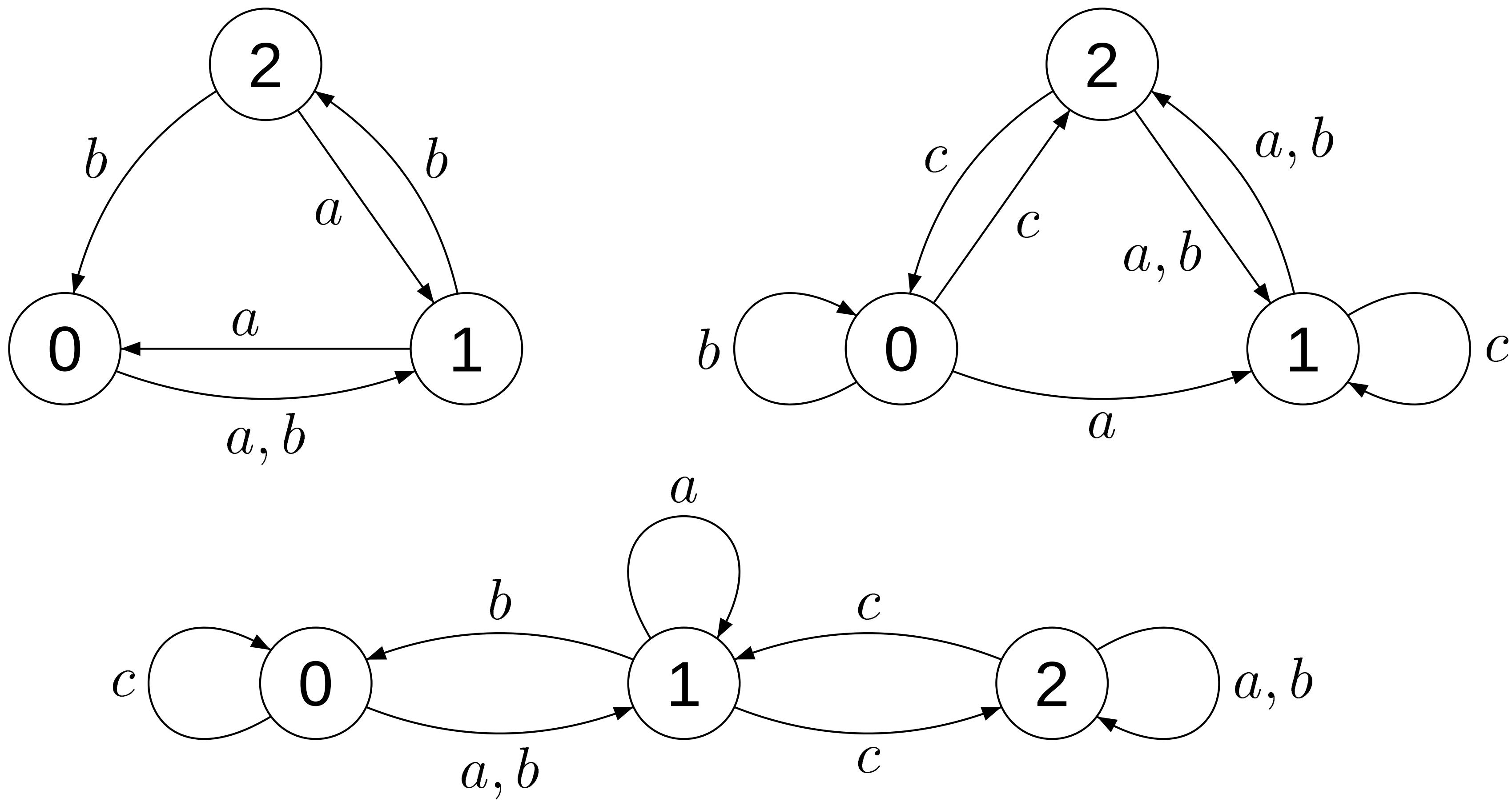

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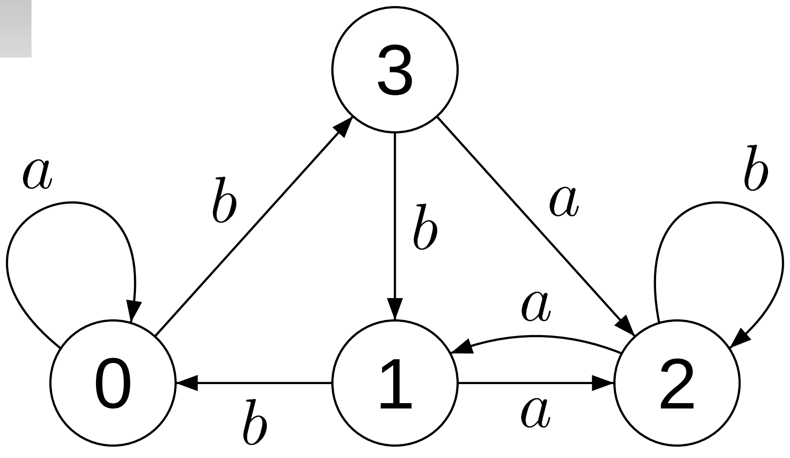

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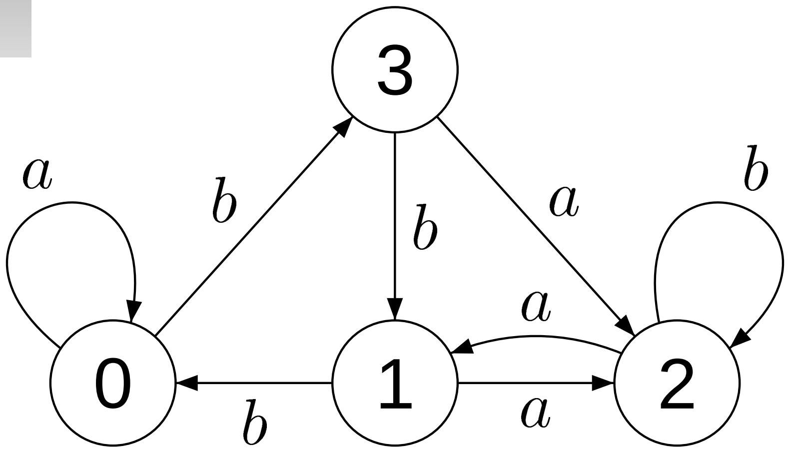

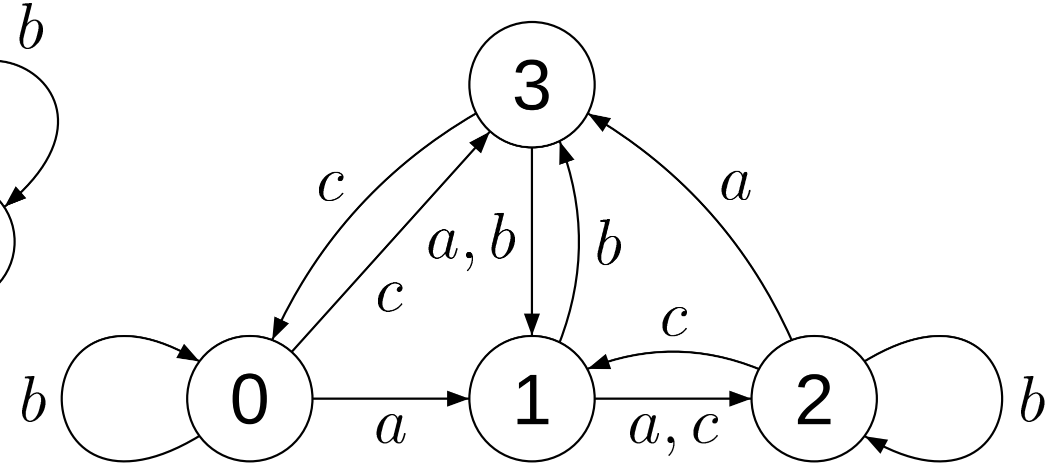

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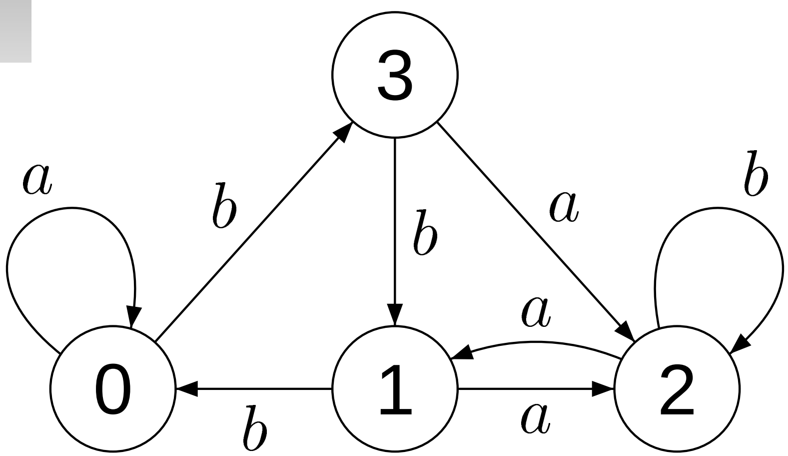

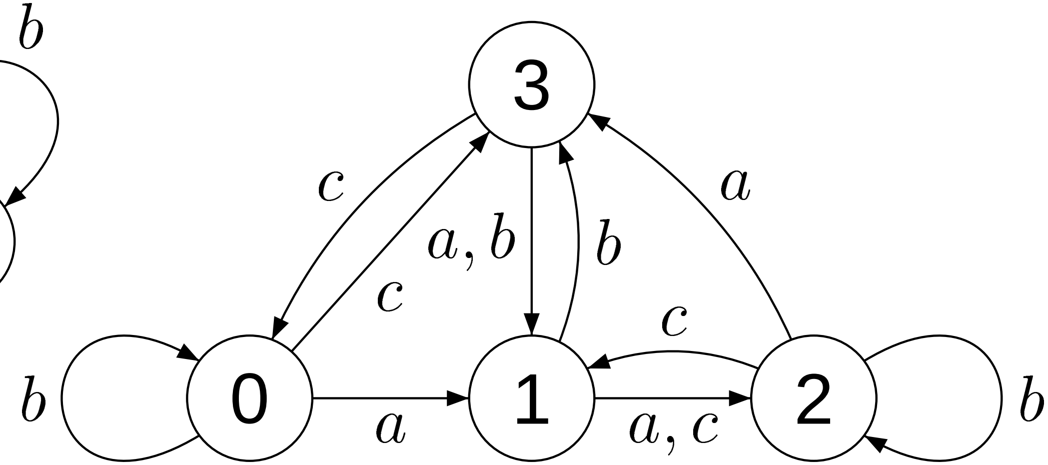

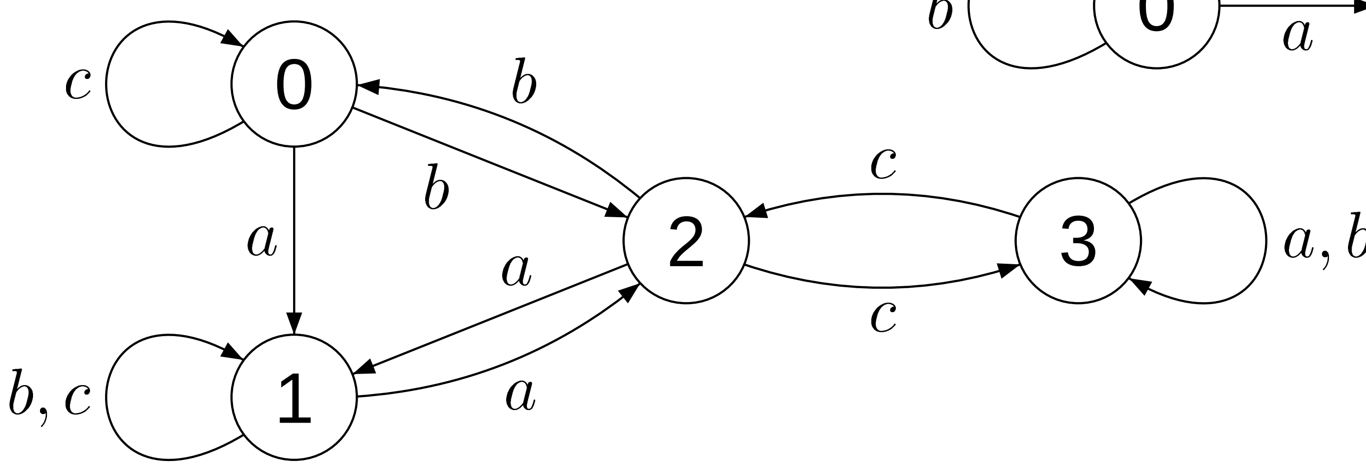

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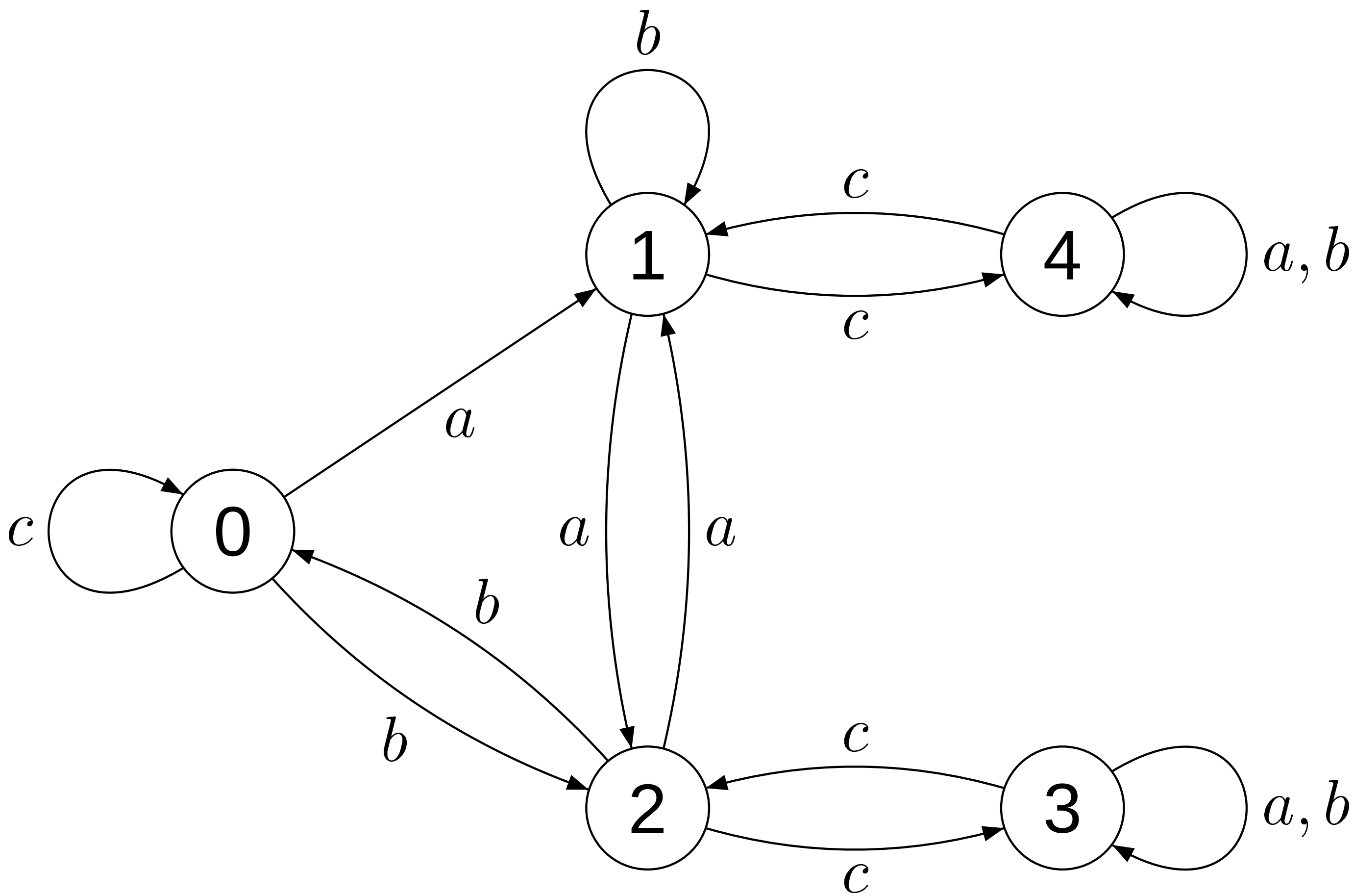

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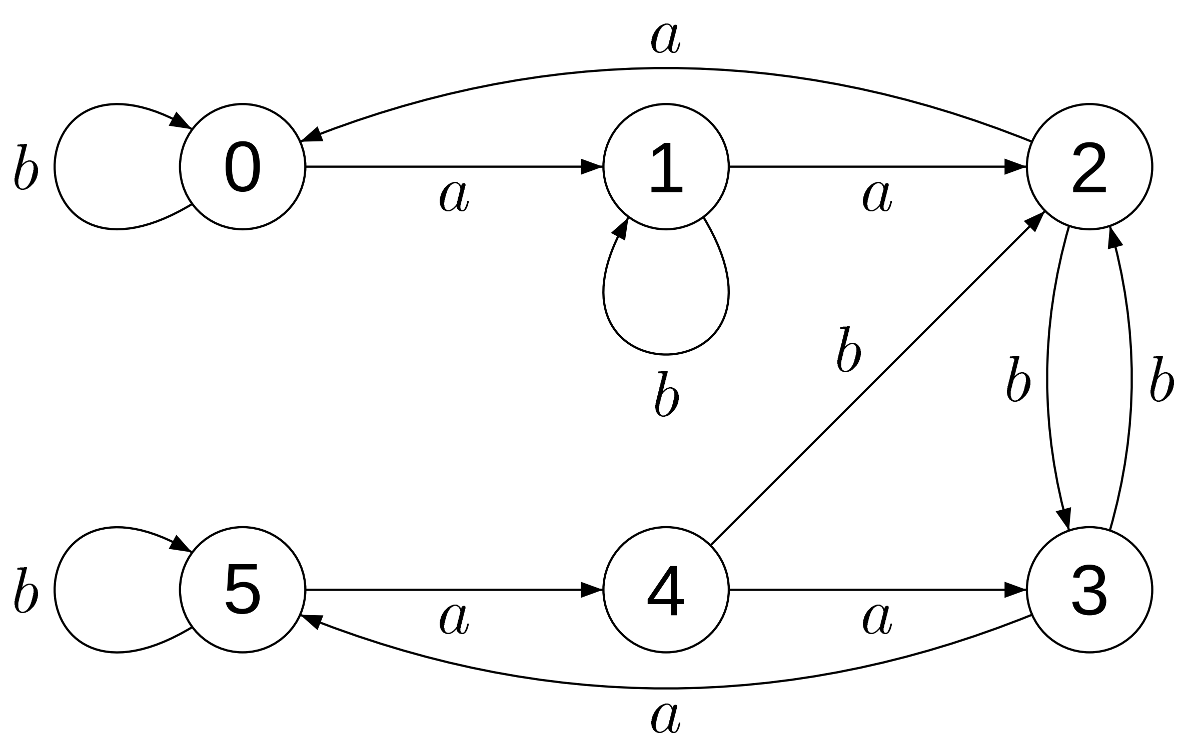

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However, in $\mathscr{K}_6$ there is no word w of length $16 = (6-2)^2$ such that $|Q \,.\, w| = 2$.

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The *rank* of a DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is the minimum cardinality of the sets $\boldsymbol{Q} \,.\, \boldsymbol{w}$ where $\boldsymbol{w}$ runs over Σ^*.

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Yet another hope killed by Kari's example is the extensibility conjecture.

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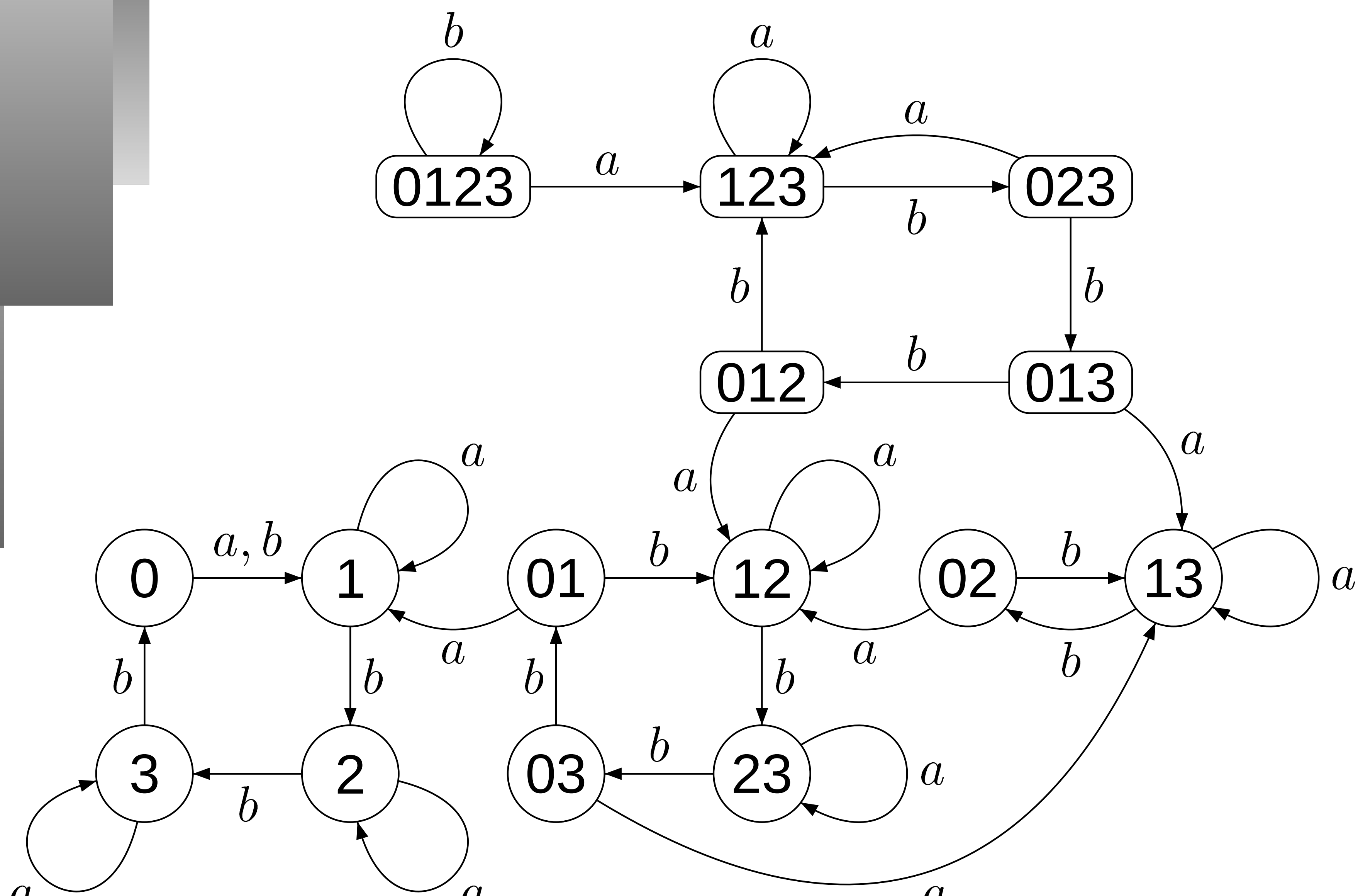
Extensibility



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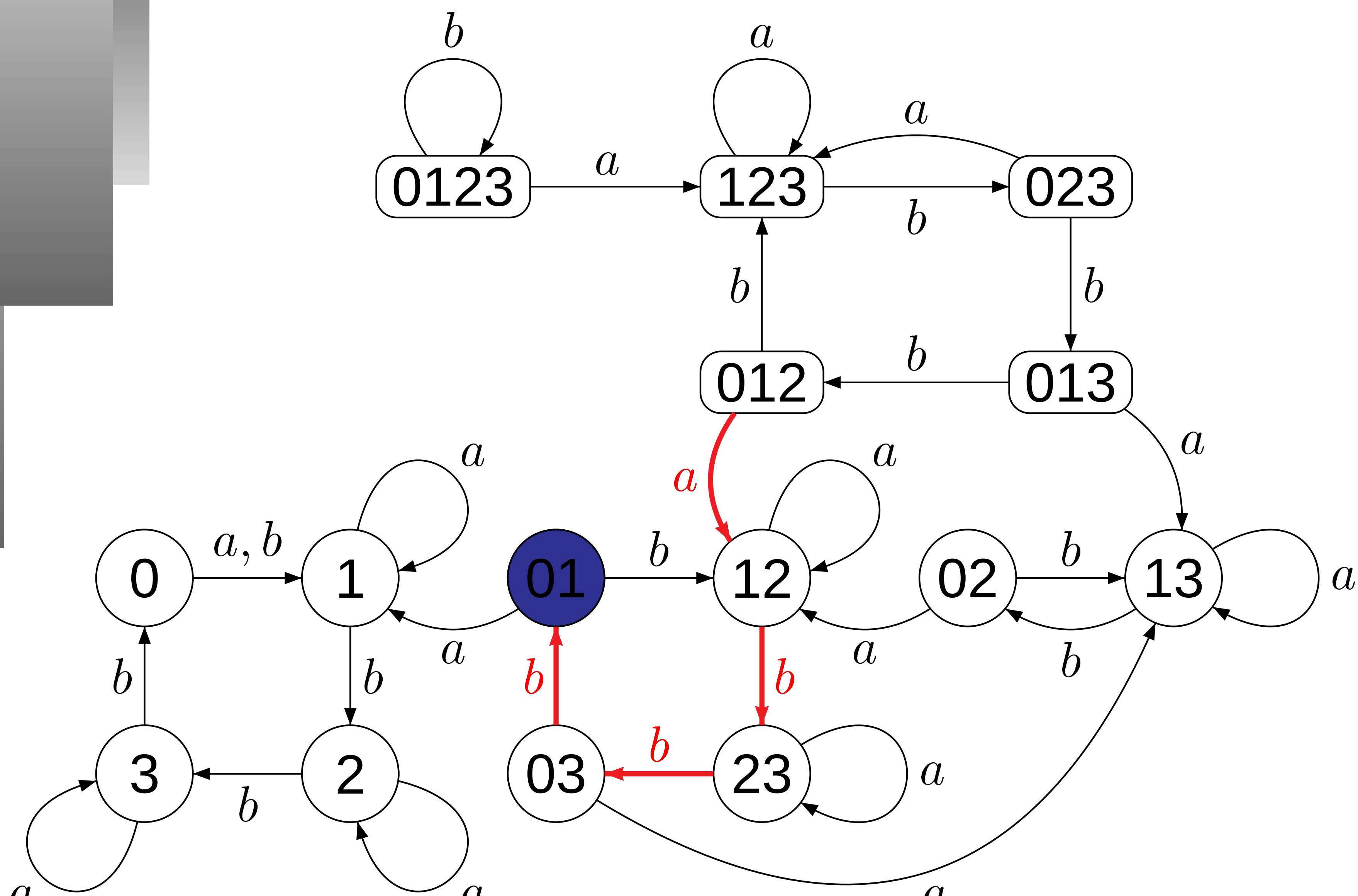

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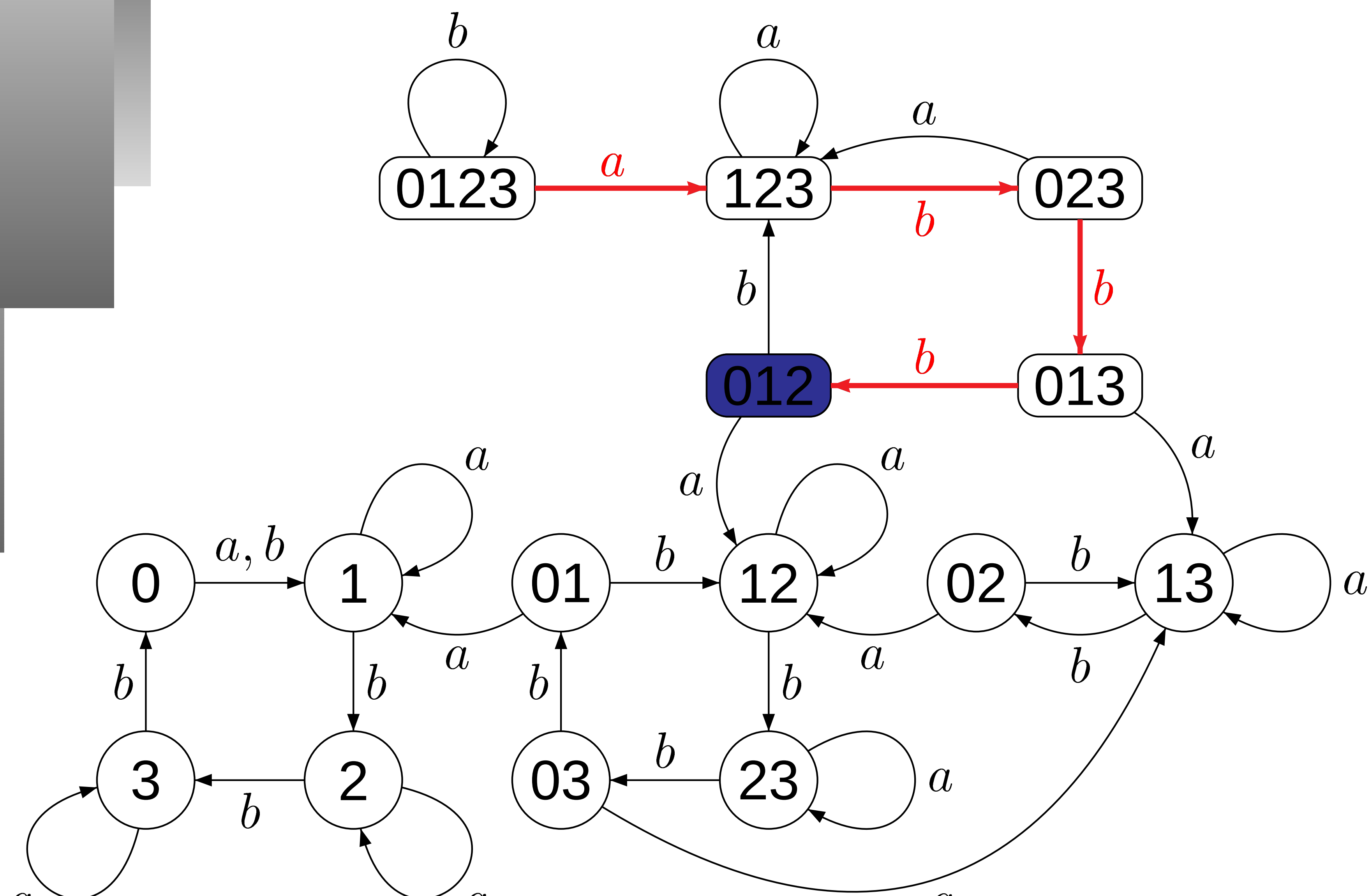

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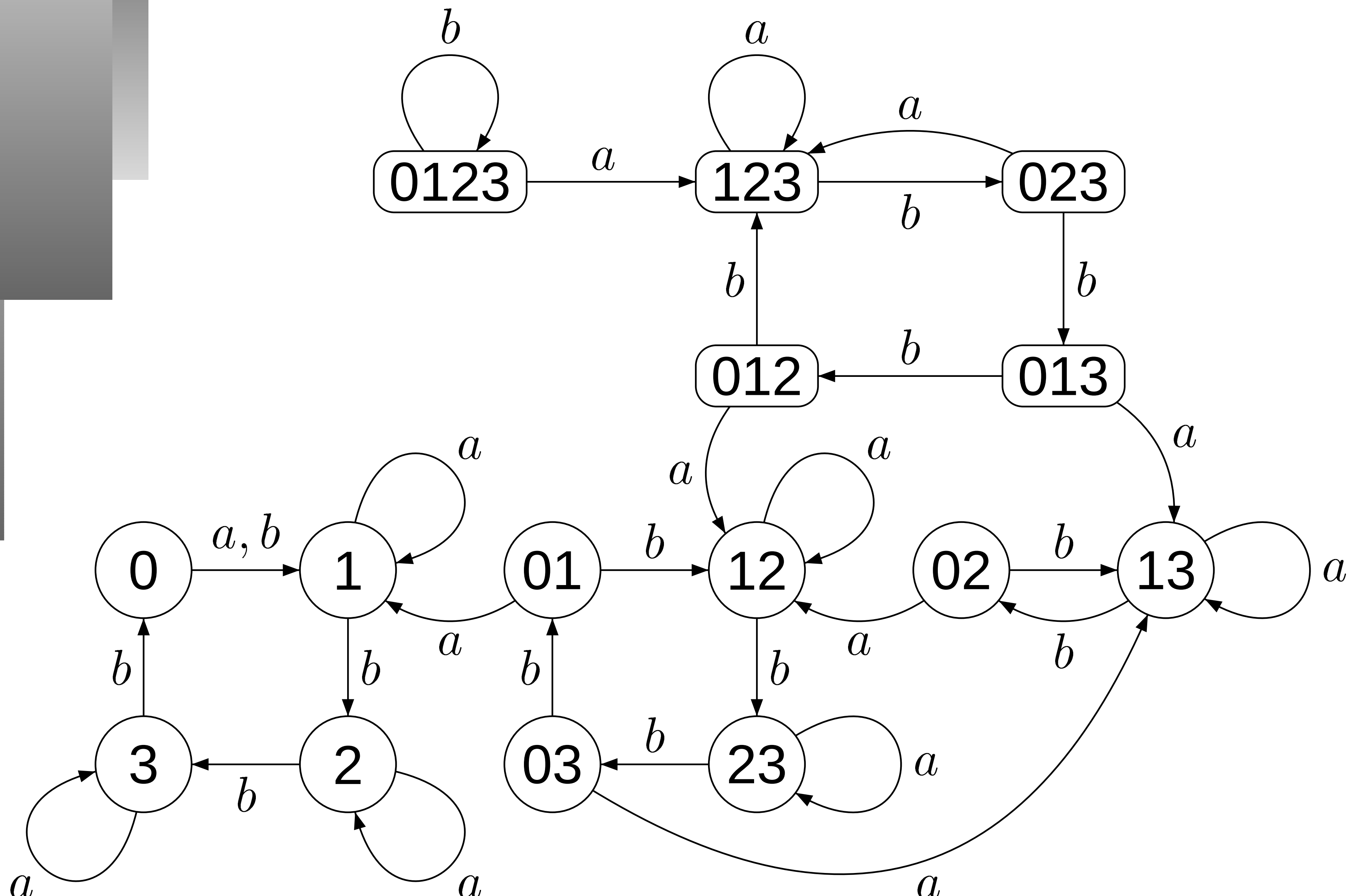

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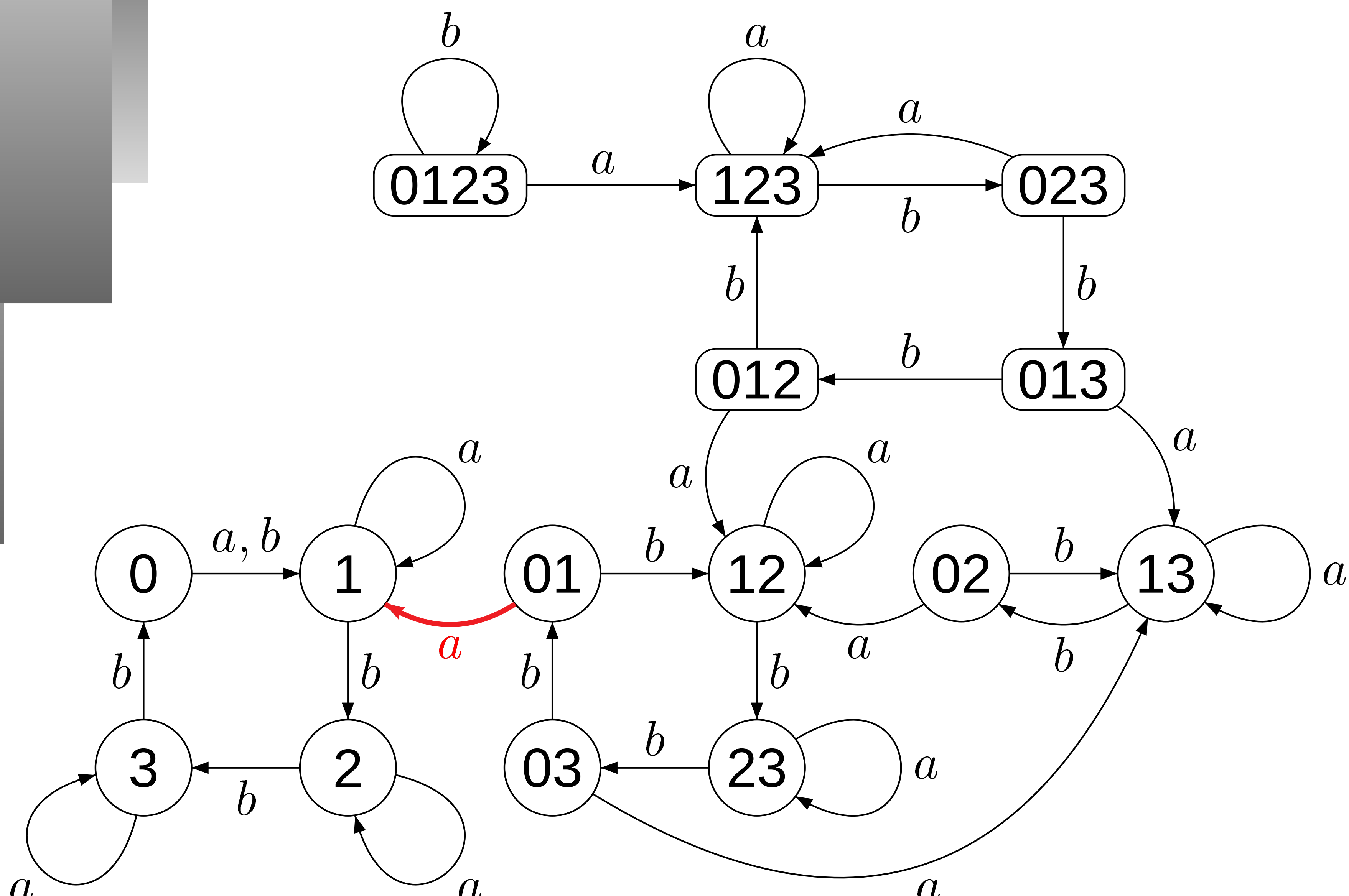

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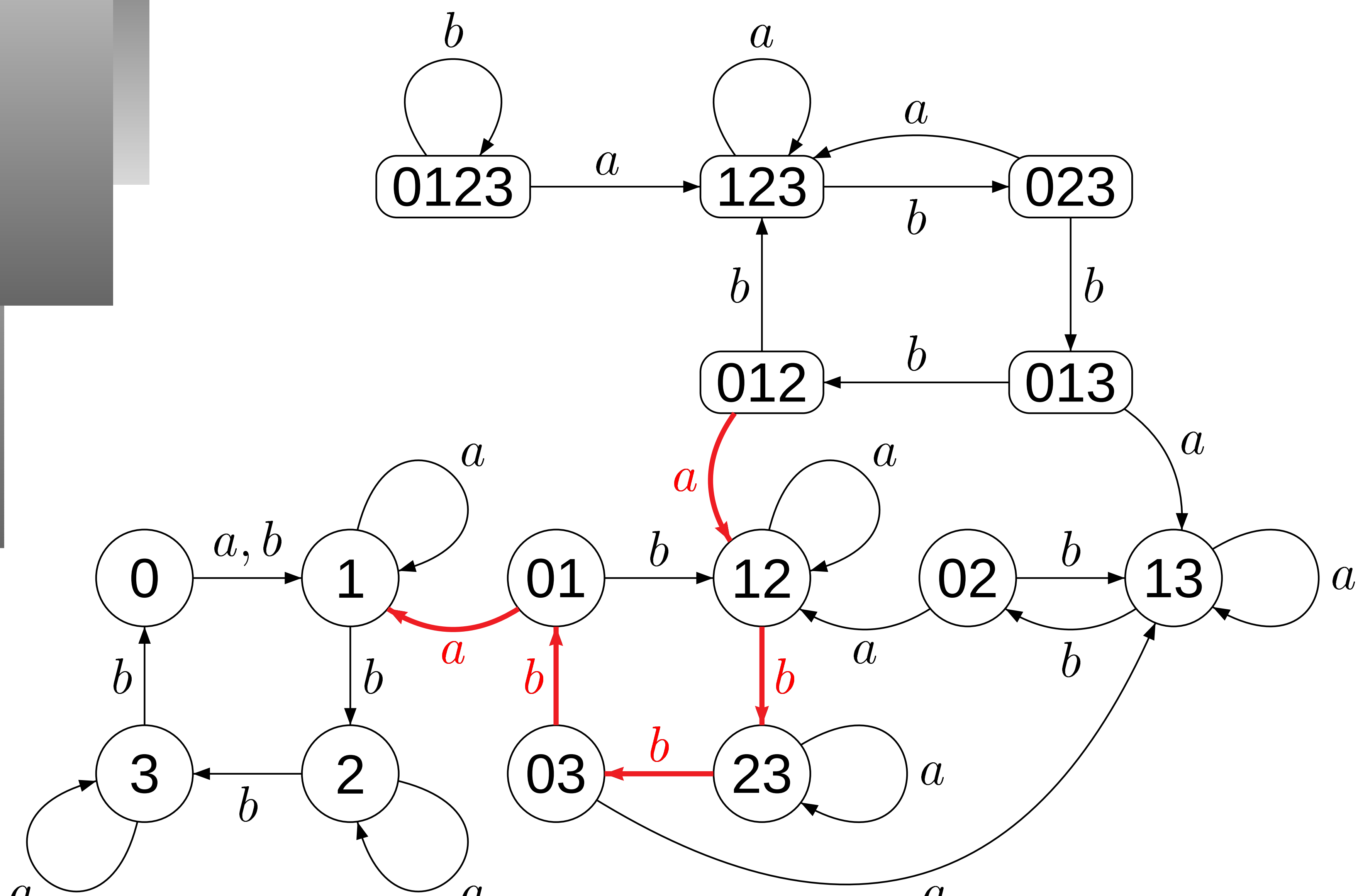

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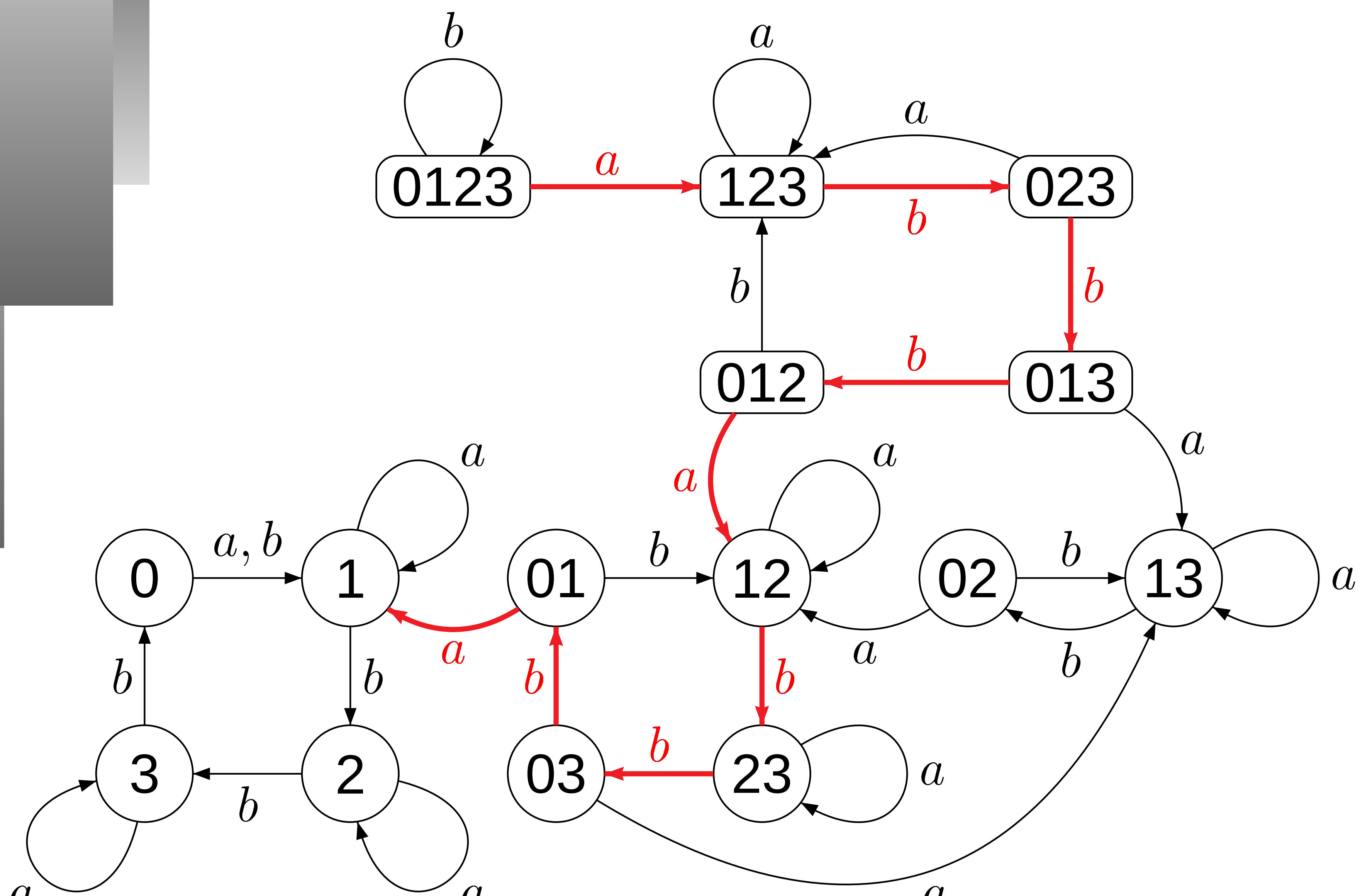

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• Jarkko Kari's result for automata with Eulerian digraphs (Synchronizing finite automata on Eulerian digraphs, Theoret. Comput. Sci., 295 (2003) 223–232.)

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Thus, the extensibility conjecture fails, and the approach based on it cannot prove the Černý conjecture in general.

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We recall the notion of a congruence and the related notion of the quotient automaton w.r.t. a congruence in the next slide. They will be essentially used in this lecture!

Congruences and Quotient Automata

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle \boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{\delta} \rangle$ is called a *congruence* if $(\boldsymbol{p}, \boldsymbol{q}) \in \boldsymbol{\pi}$ implies $\big(\boldsymbol{\delta}(\boldsymbol{p}, \boldsymbol{a}), \boldsymbol{\delta}(\boldsymbol{q}, \boldsymbol{a})\big) \in \boldsymbol{\pi}$ for all $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{Q}$ and all $\boldsymbol{a} \in \boldsymbol{\Sigma}$.

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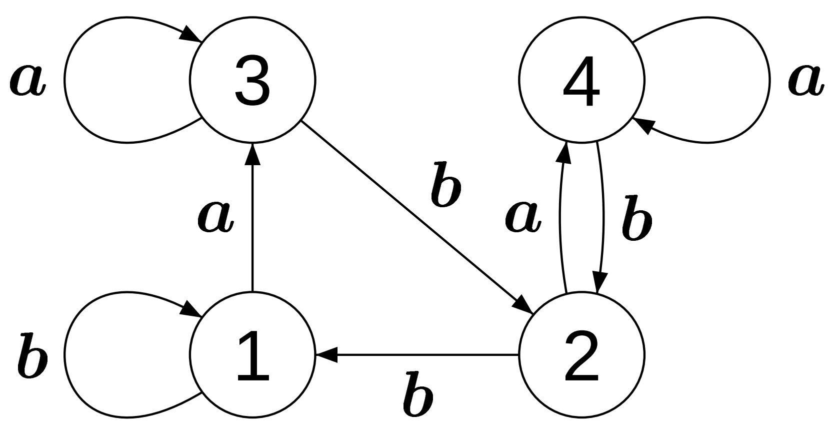

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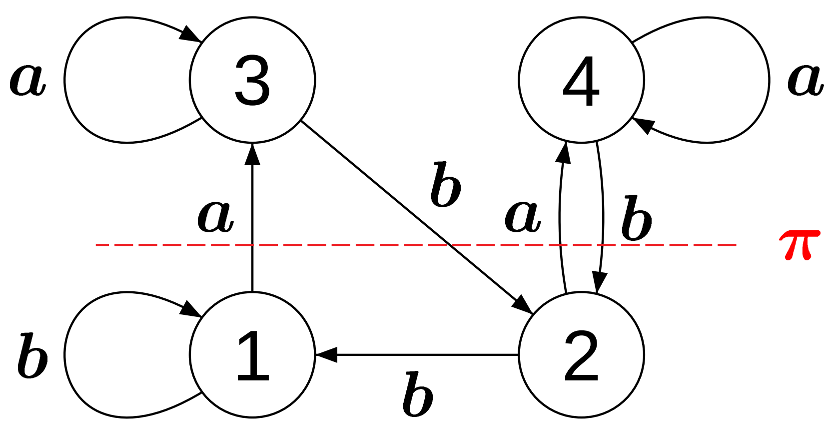

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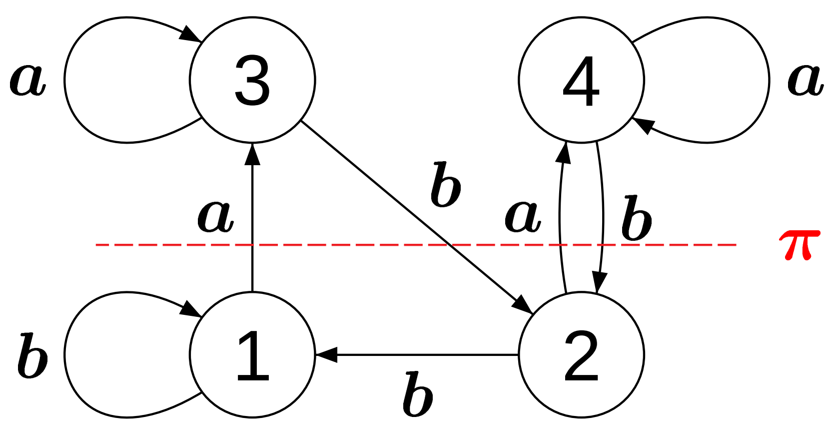

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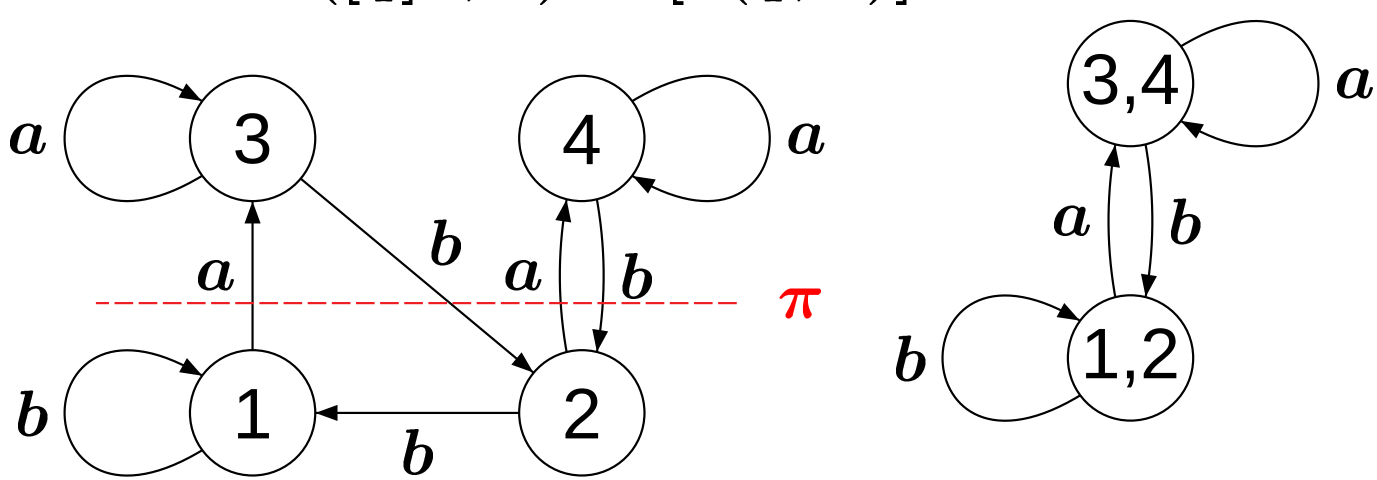

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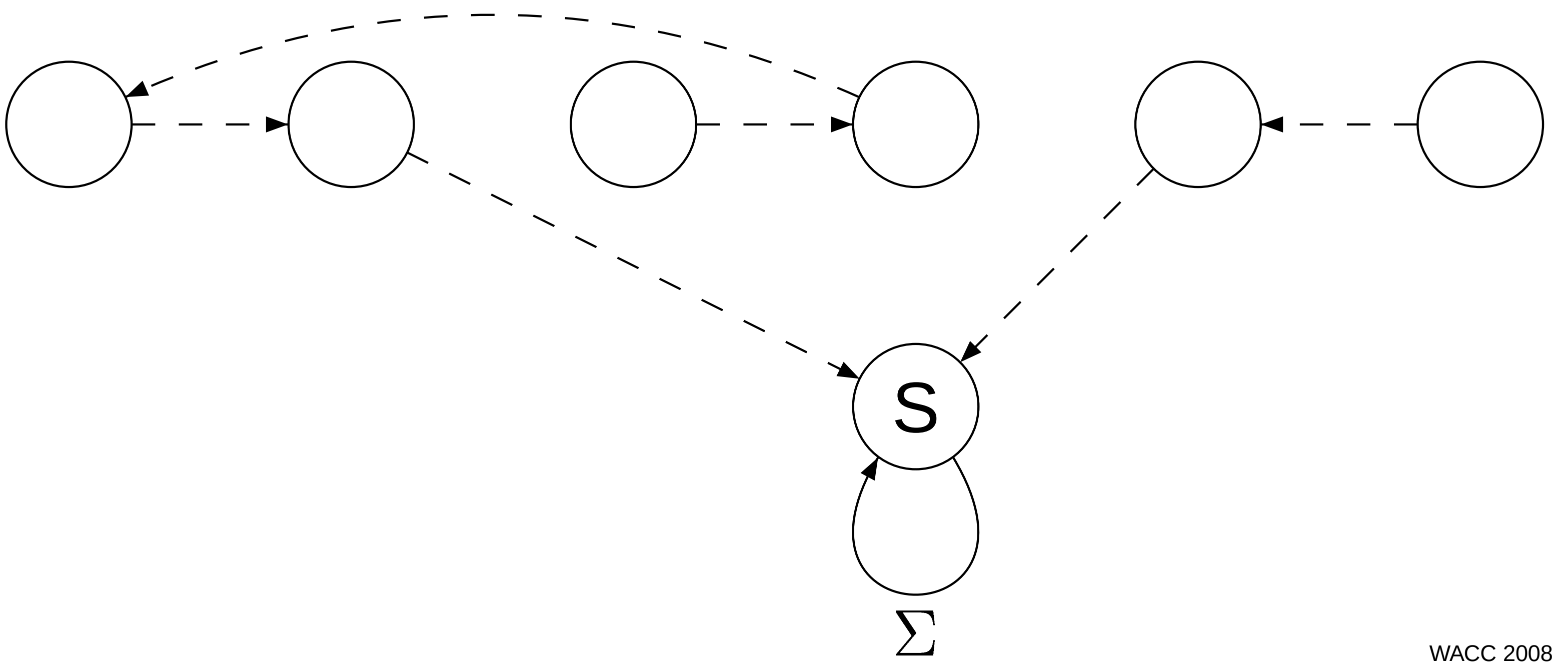

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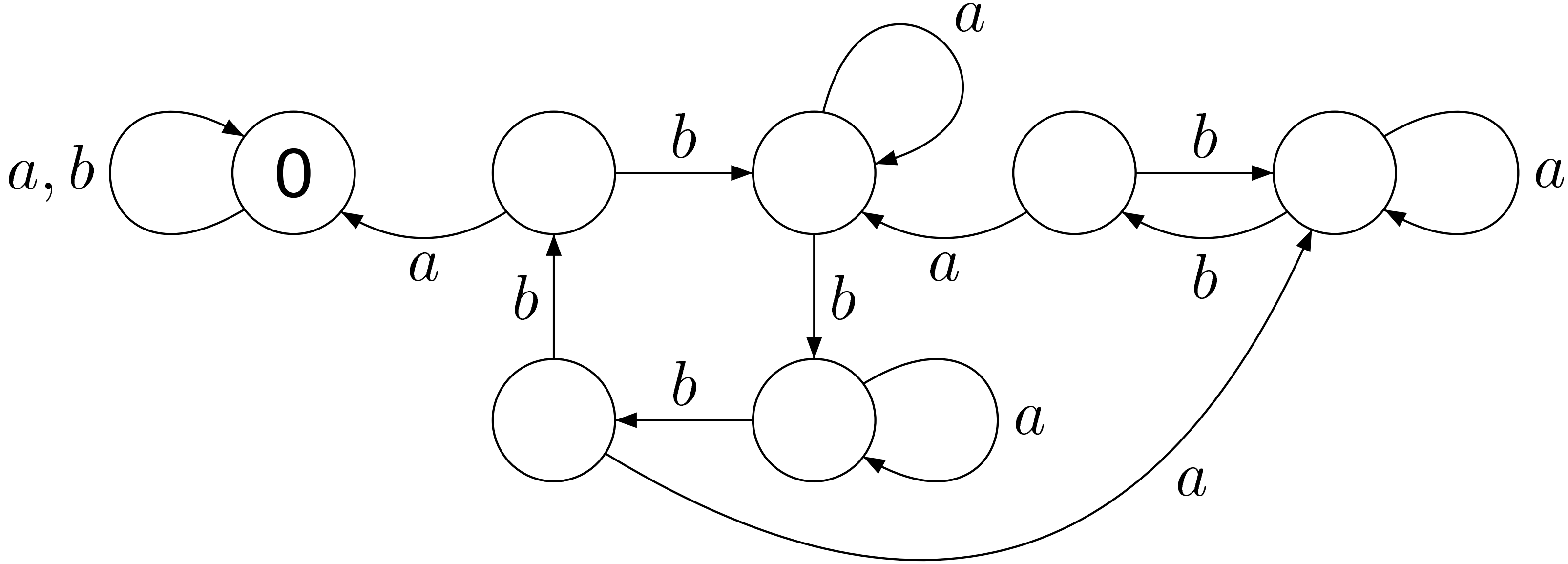

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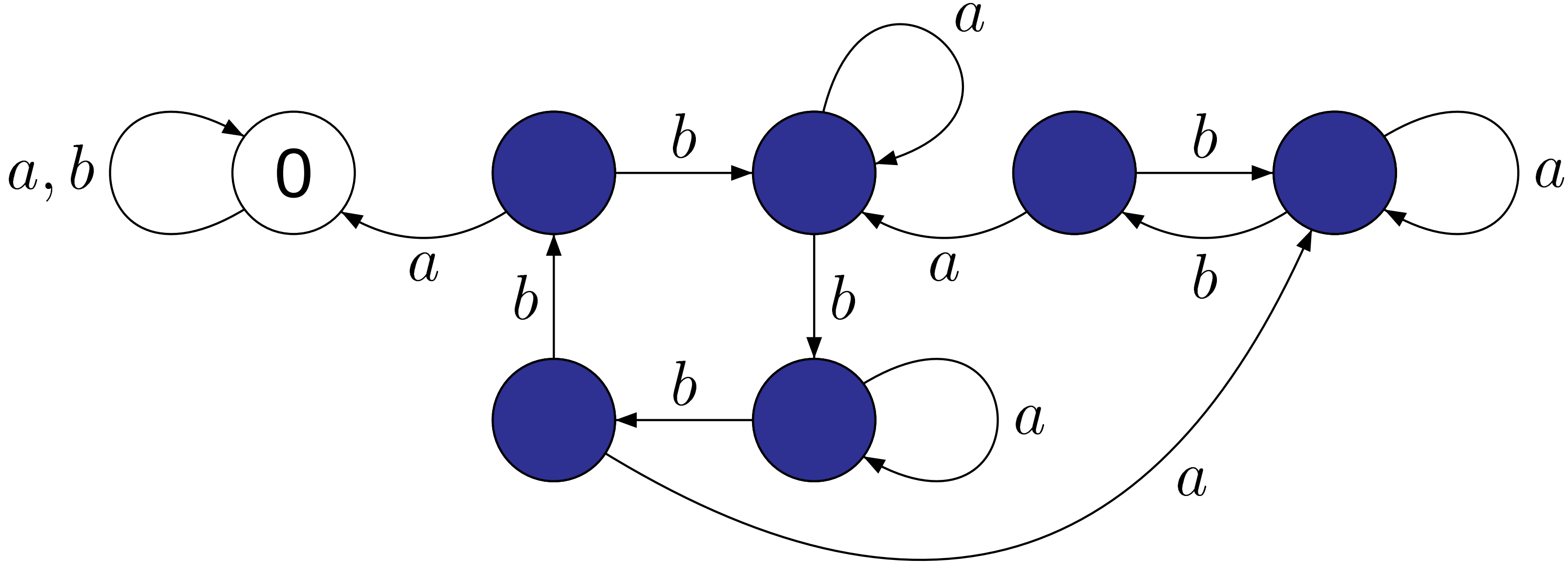

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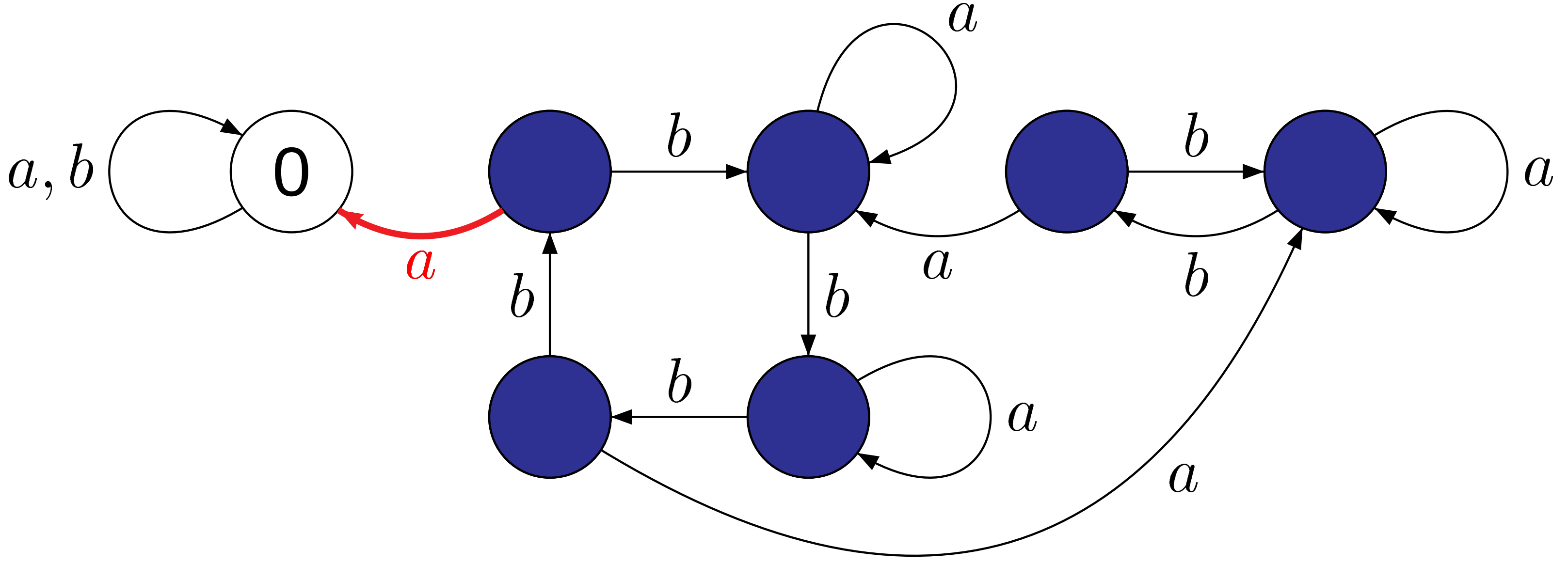

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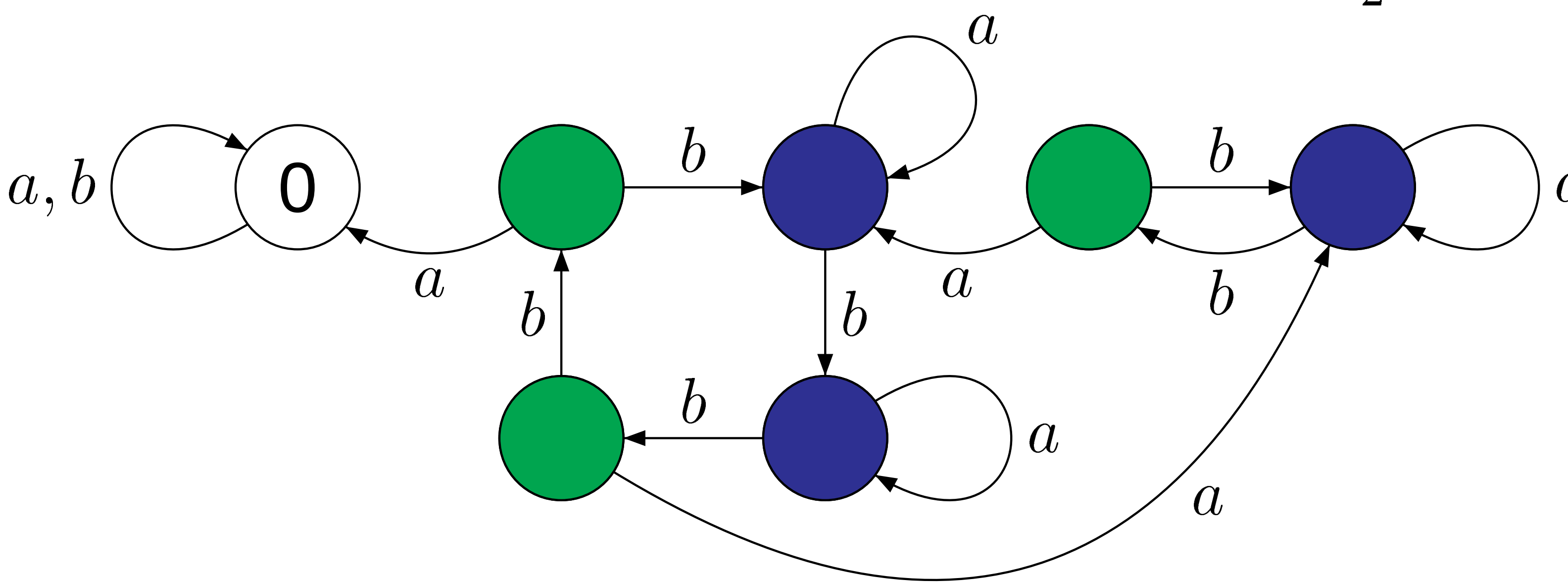

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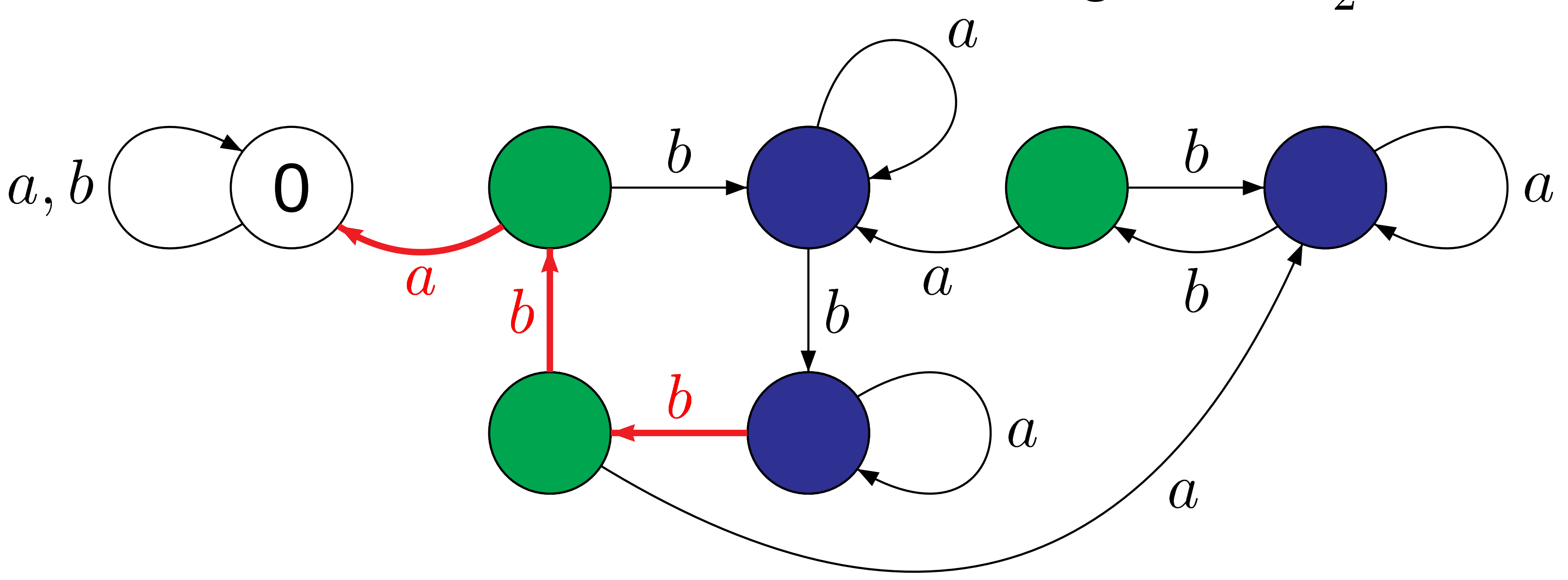

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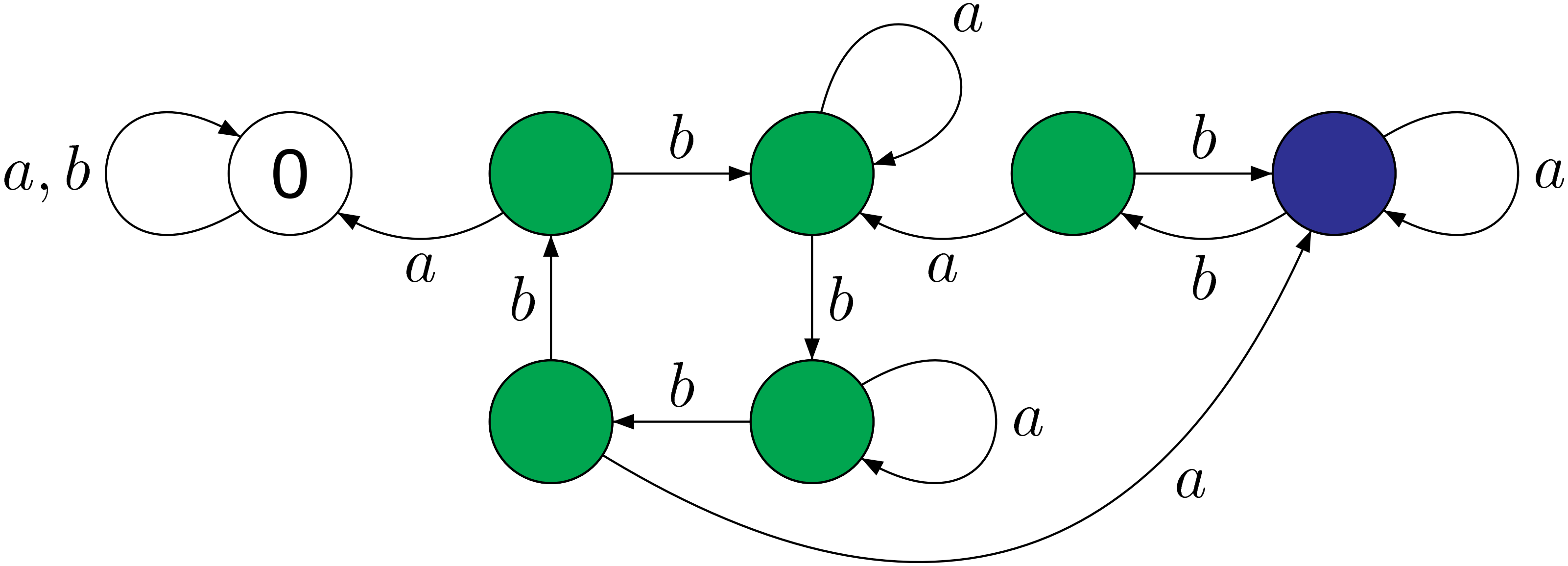

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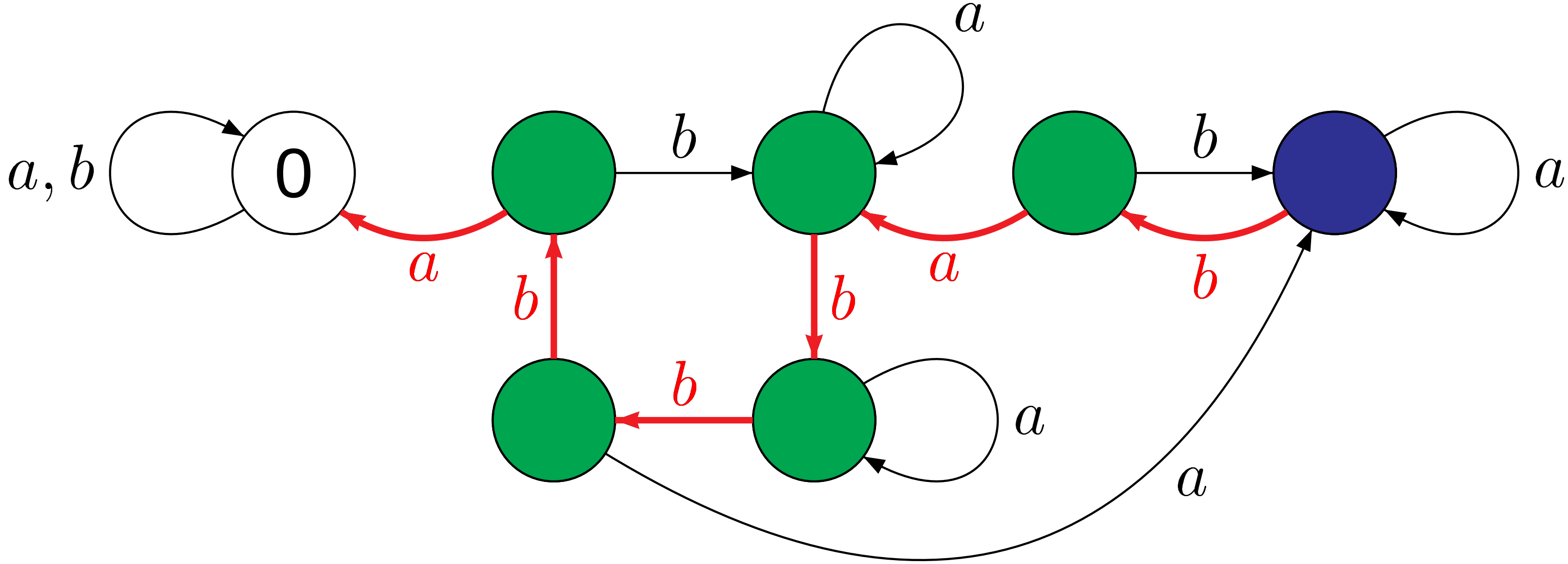

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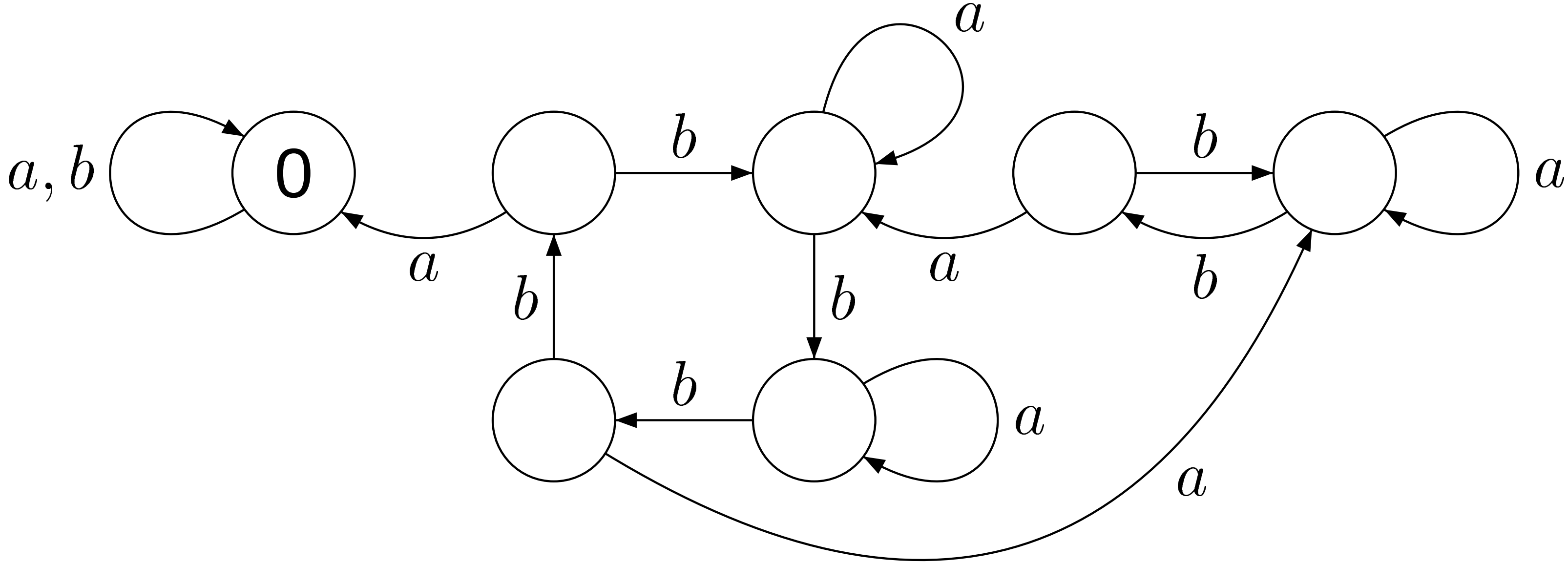

The algorithm makes at most $k-1$ steps and the length of the segment added in the step when t states still holds coins ($k-1 \geq t \geq 1$) is at most $k-t$. The total length is $\leq 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$.

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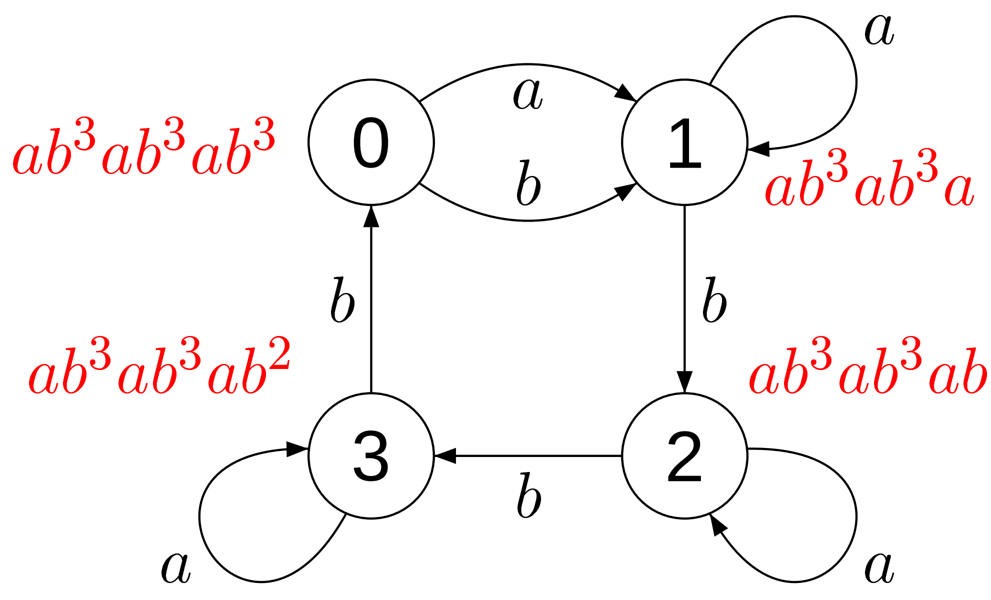

Road Coloring

Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.

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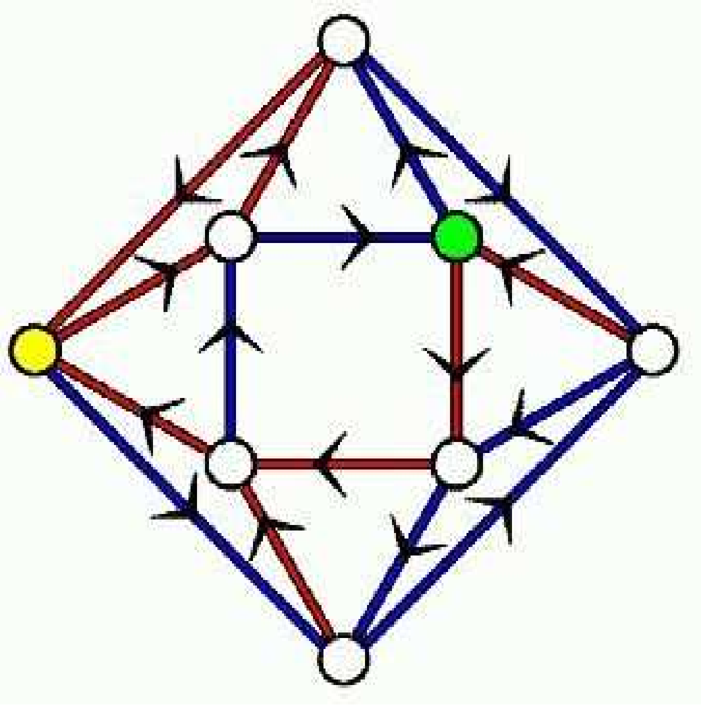

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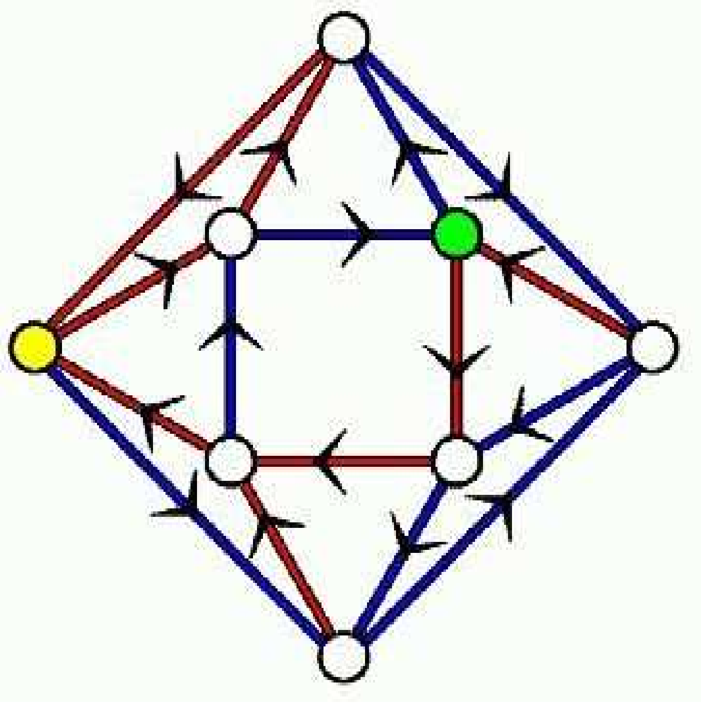

Then for each node there is a sequence of colors that brings one to the chosen node from anywhere.

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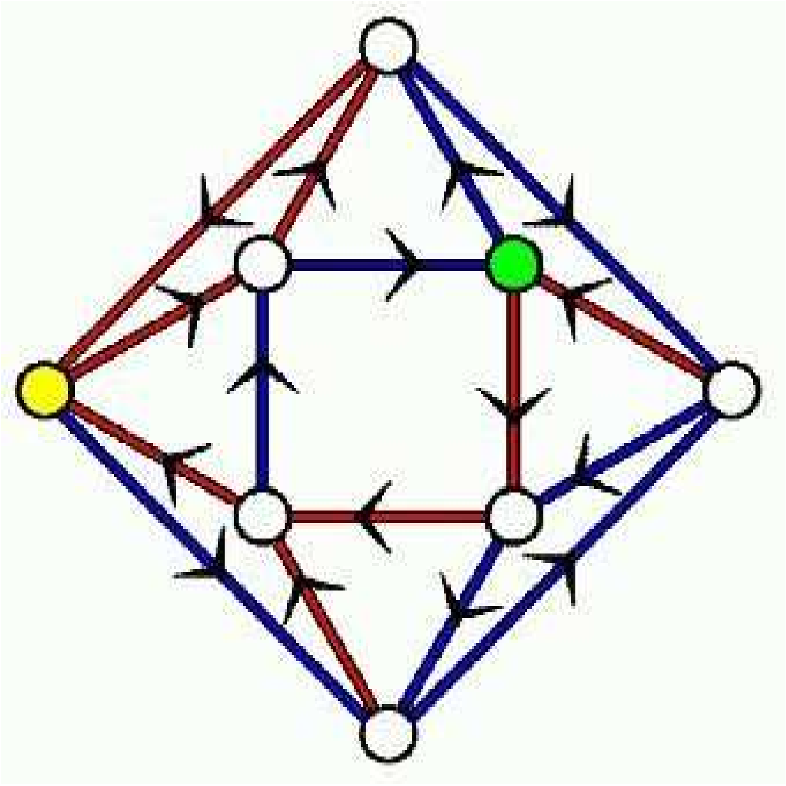

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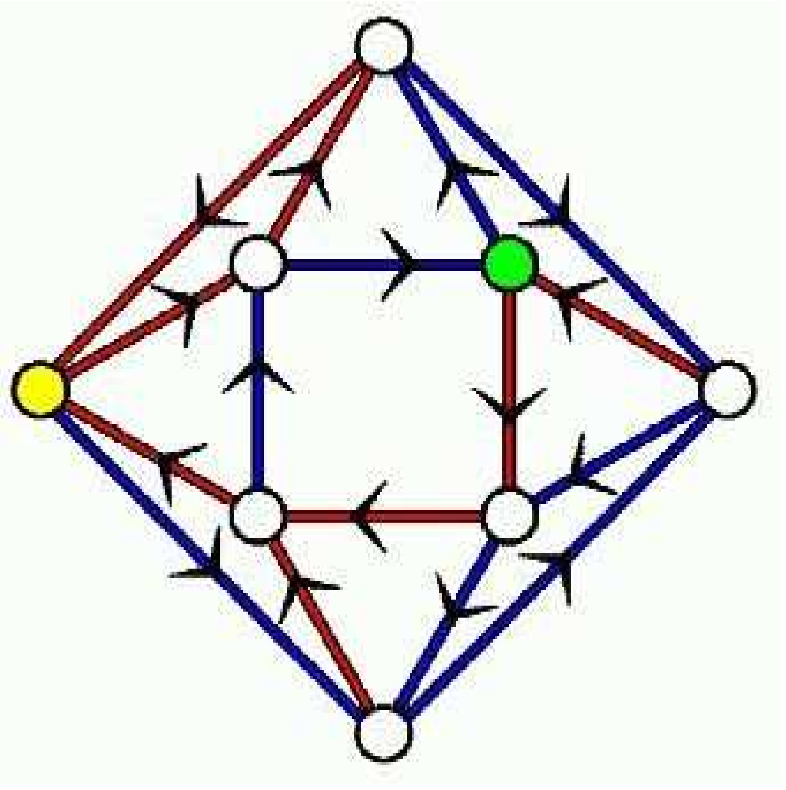

For the green node:
blue-blue-red-blue-blue-red-blue-blue-red.

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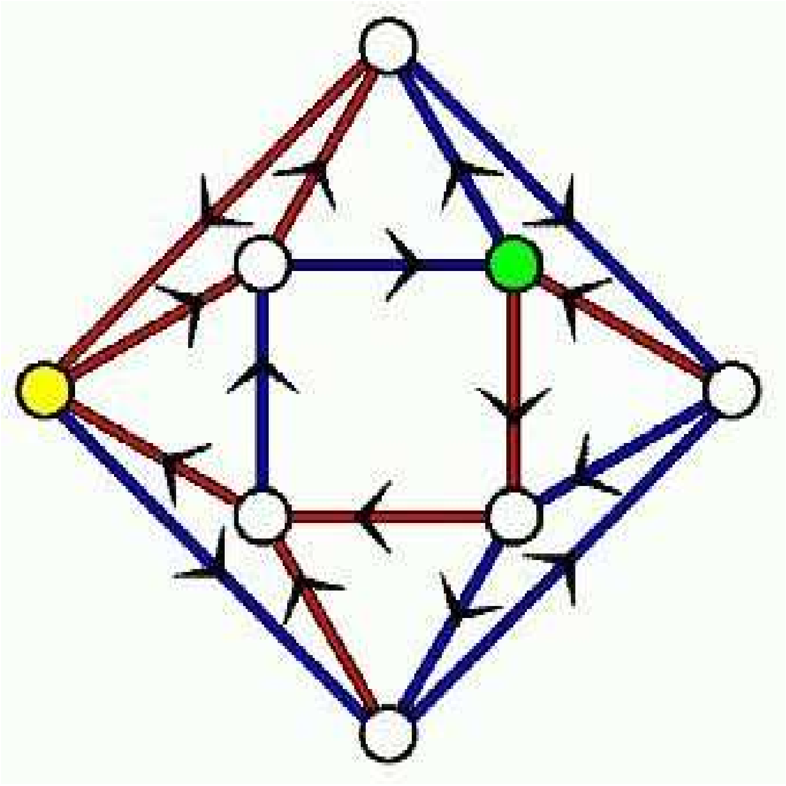

For the green node:
blue-blue-red-blue-blue-red-blue-blue-red.

For the yellow node:
blue-red-red-blue-red-red-blue-red-red.

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Now suppose that we have a transport network, that is, a strongly connected digraph.

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We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton.

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An obvious necessary condition:
all vertices should have the same out-degree.
In what follows we refer to this as to the constant out-degree condition.

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Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

Primitivity

Let $v \in V_i \cap V_j$ where $i \neq j$. This means that in Γ there are two paths from v_0 to v: of length $\ell \equiv i \pmod{k}$ and of length $m \equiv j \pmod{k}$.

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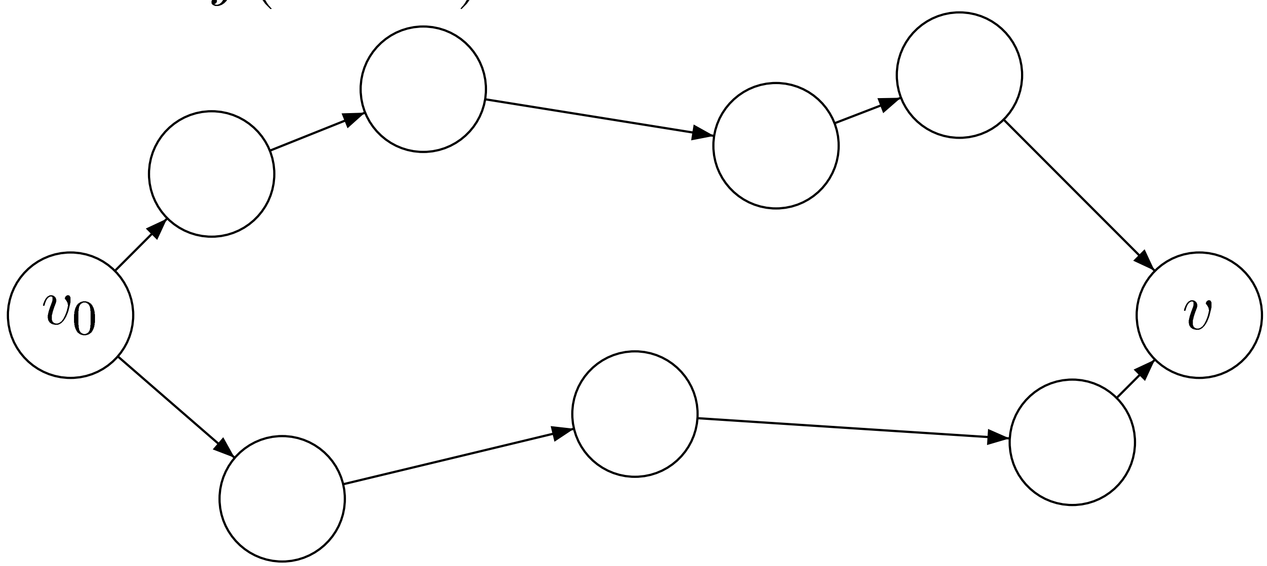

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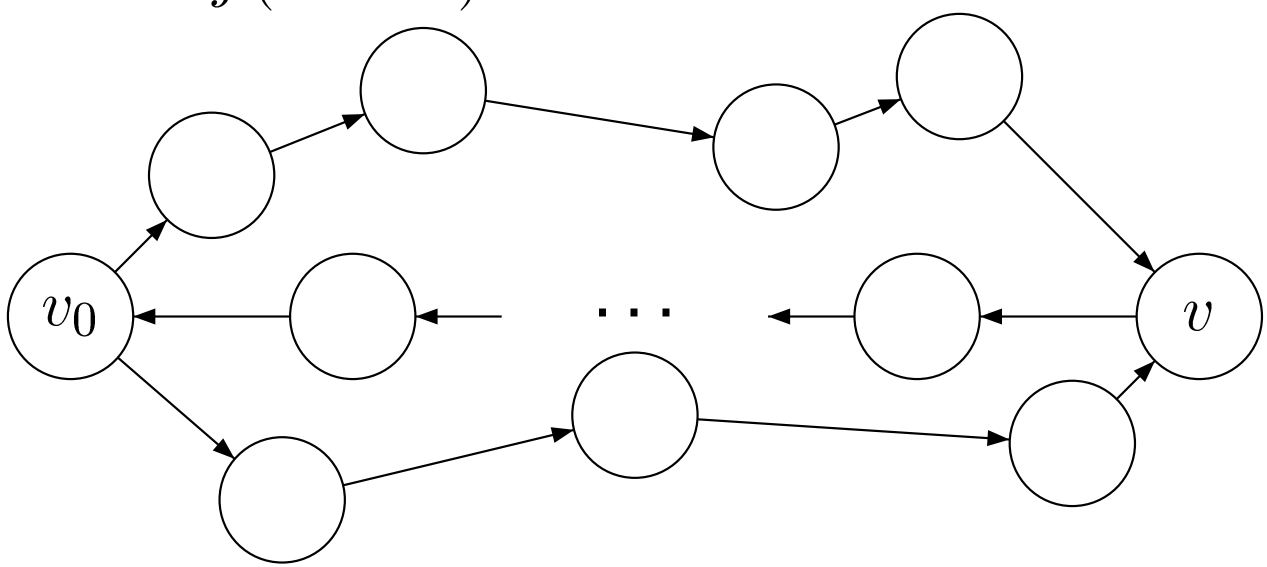

There is also a path v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

Primitivity

Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

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Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

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Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate in the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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The original motivation for the Road Coloring Conjecture comes from symbolic dynamics, see Marie-Pierre Béal and Dominiues Perrin's chapter "Symbolic Dynamics and Finite Automata" in Handbook of Formal Languages, Vol.I.

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Stability

Trahtman's proof heavily depends on a neat idea of stability which is due to Karel Culik II, Juhani Karhumäki and Jarkko Kari (A note on synchronized automata and Road Coloring Problem, Int. J. Found. Comput. Sci., 13 (2002) 459–471).

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The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/\sim$ admits a synchronizing recoloring by the induction assumption.

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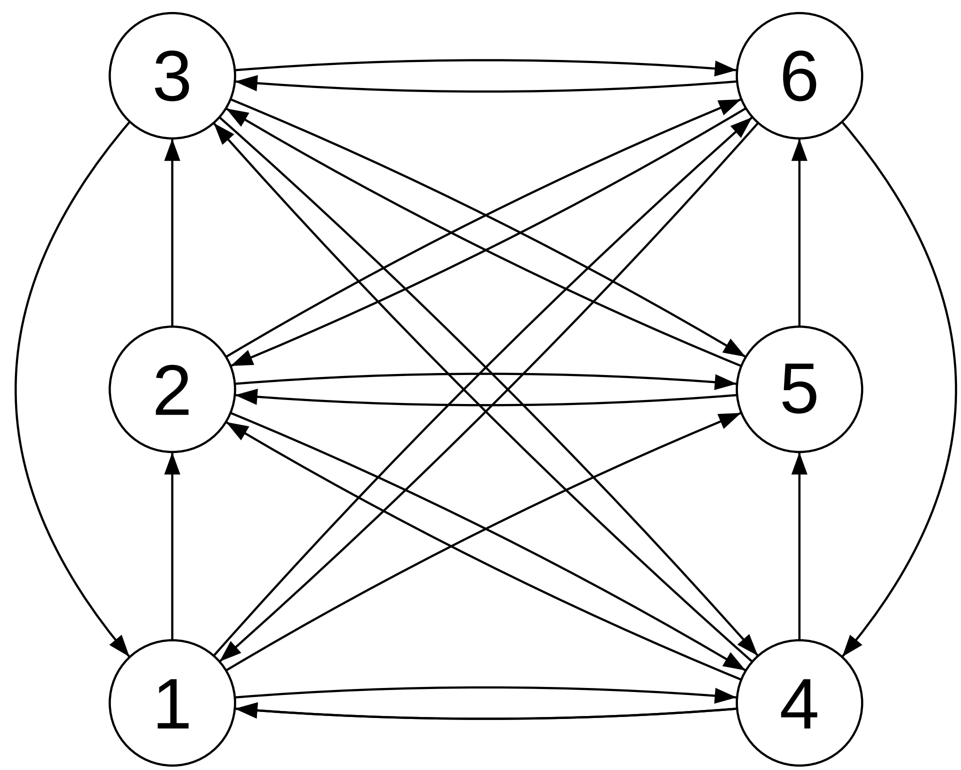

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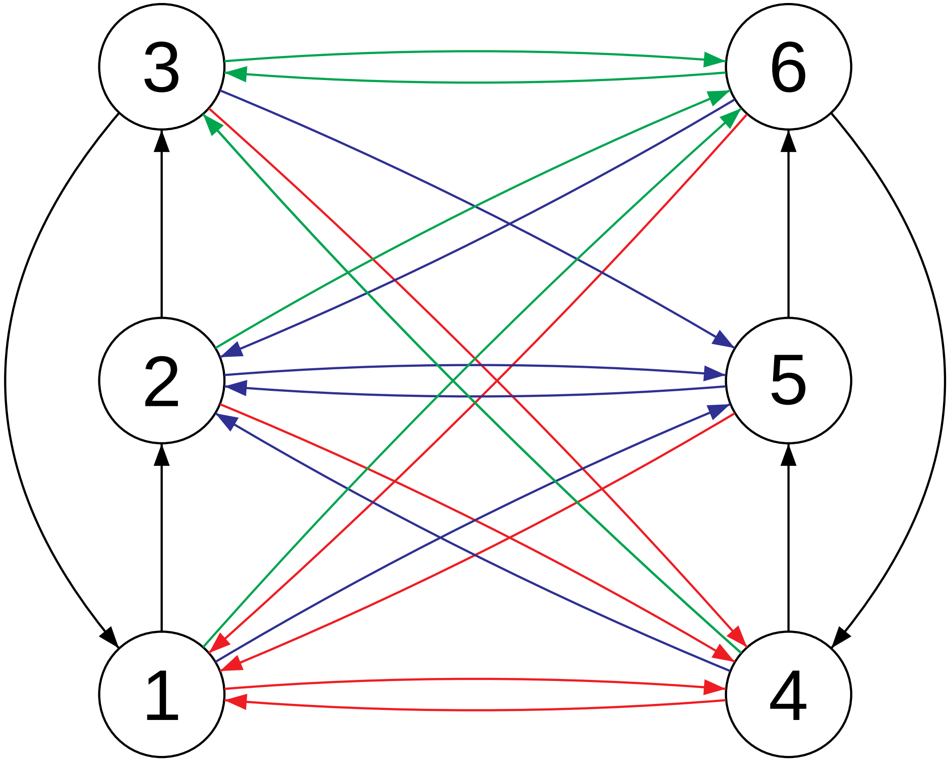

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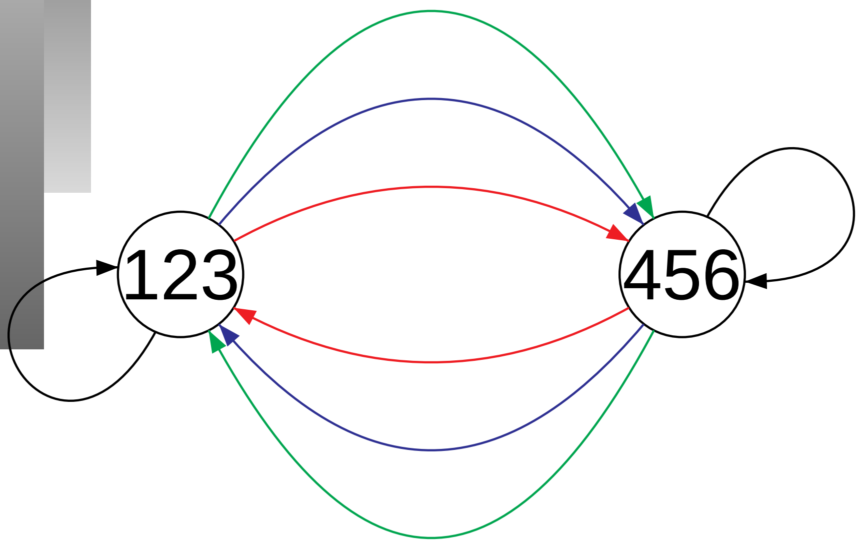

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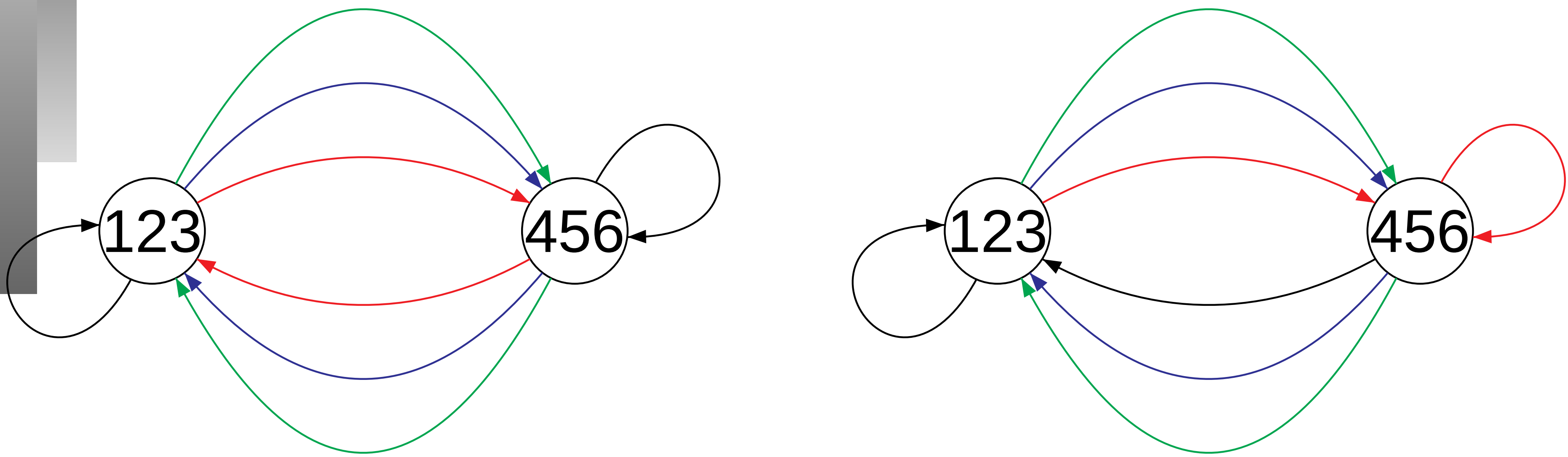

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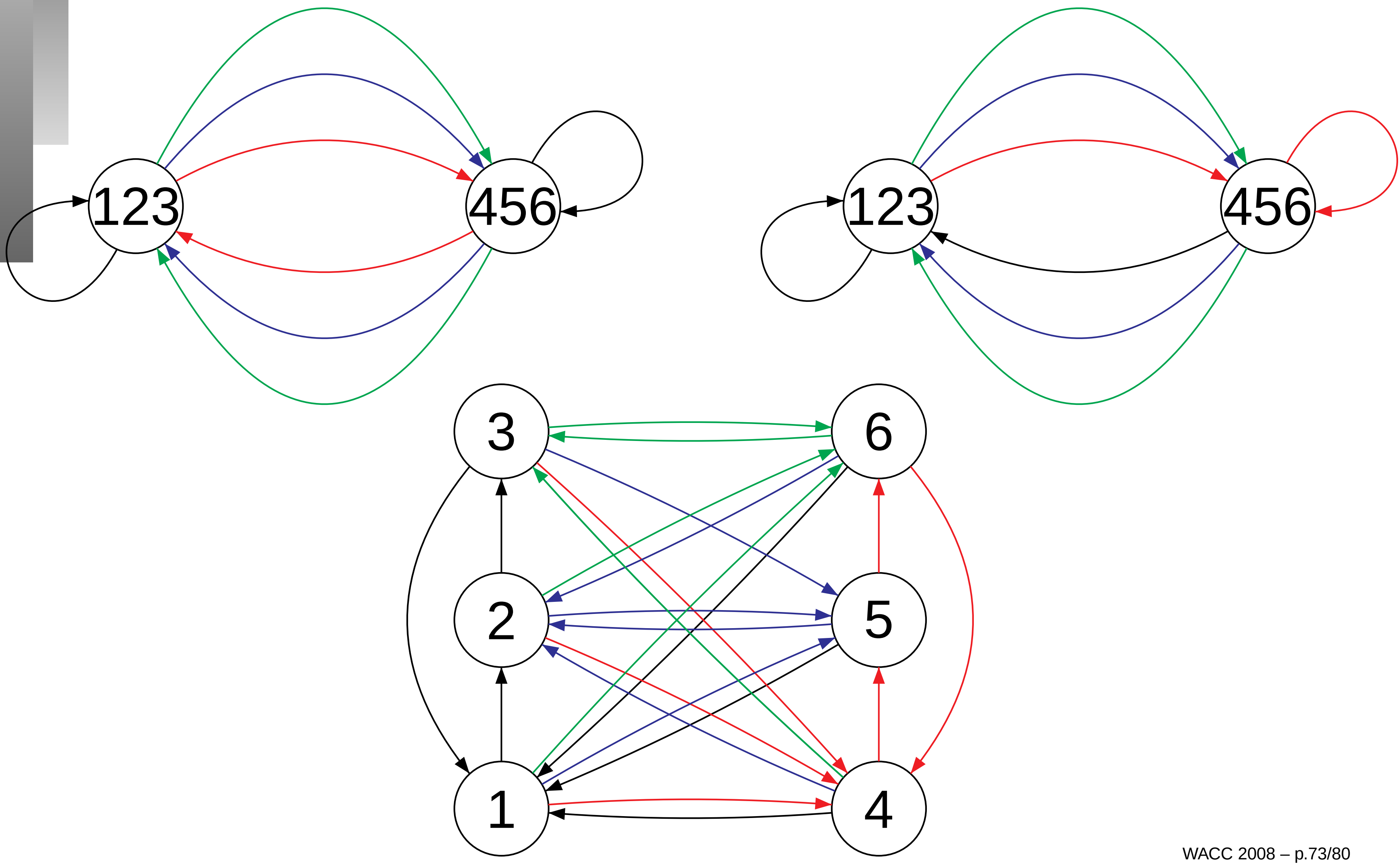

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The proof is not difficult but still a bit too technical for a presentation at the end of our conference.

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- The hybrid Černý/Road Coloring problem. Let Γ be a strongly connected primitive digraph with constant out-degree and n vertices. What is the minimum length of reset words for synchronizing colorings of Γ? For instance, the Černý automata admit synchronizing recolorings with pretty short reset words.

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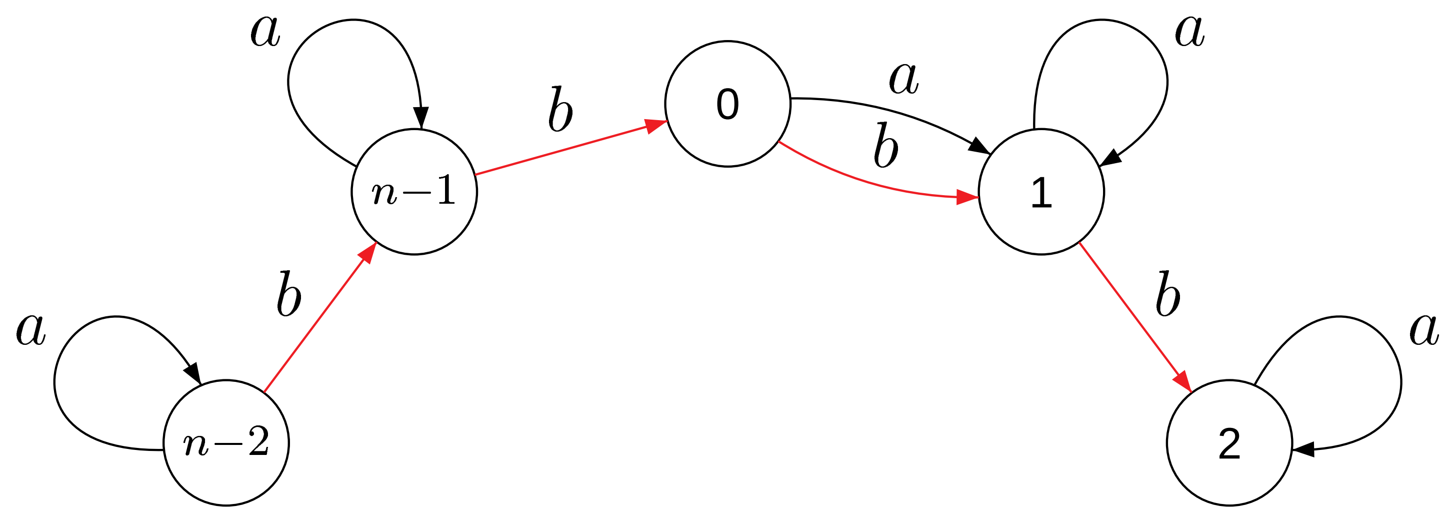

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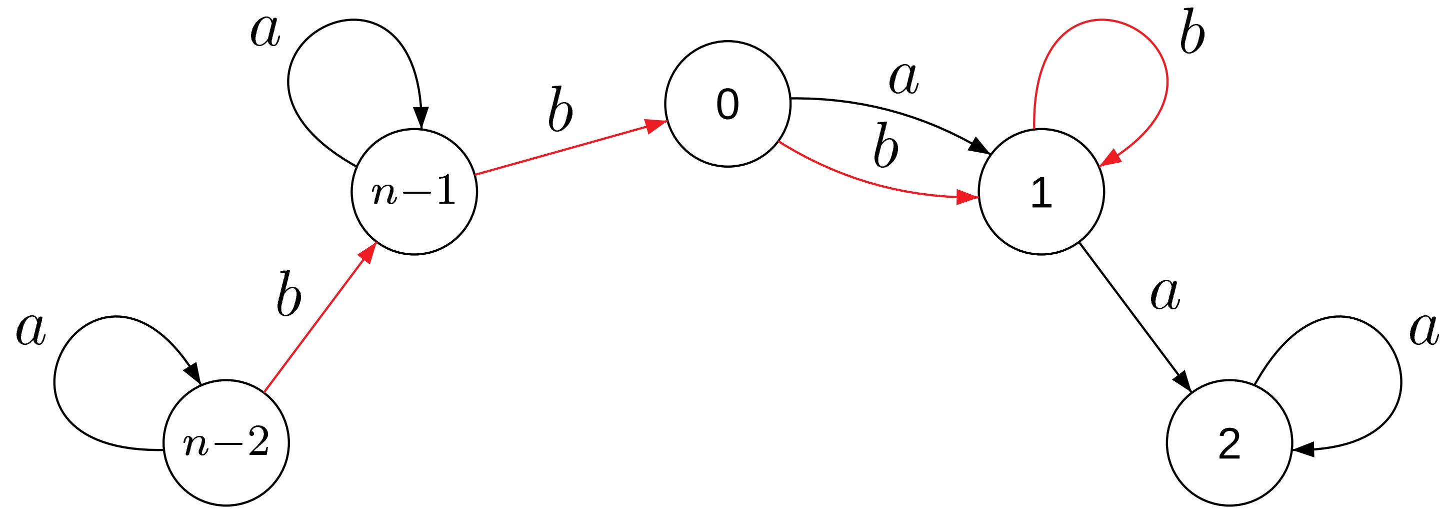

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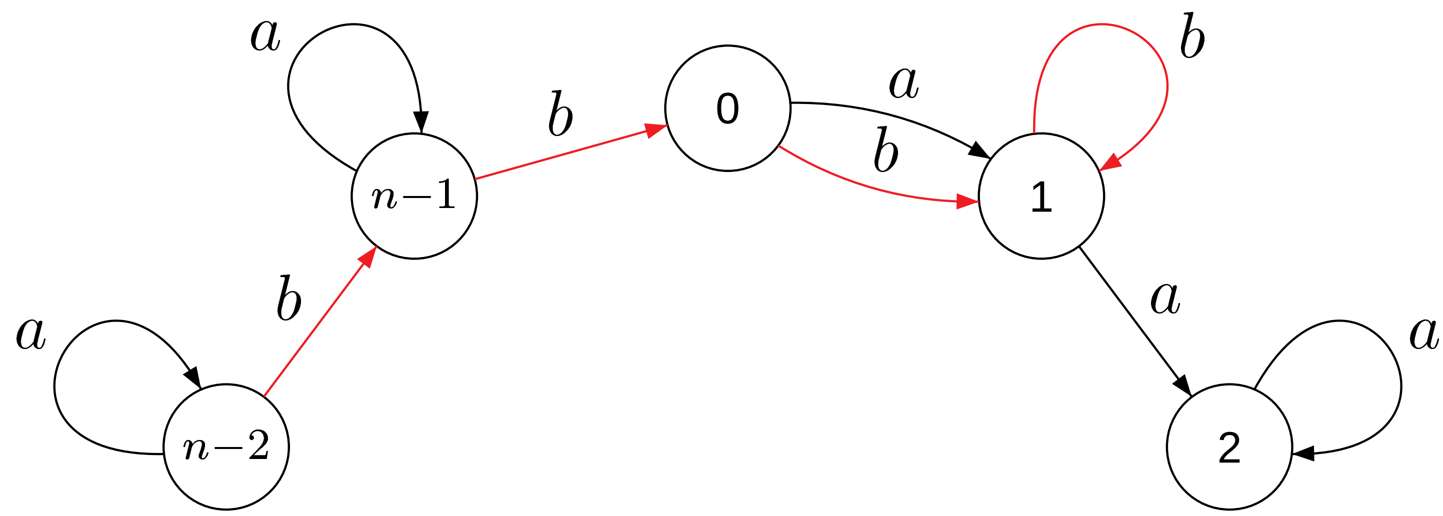

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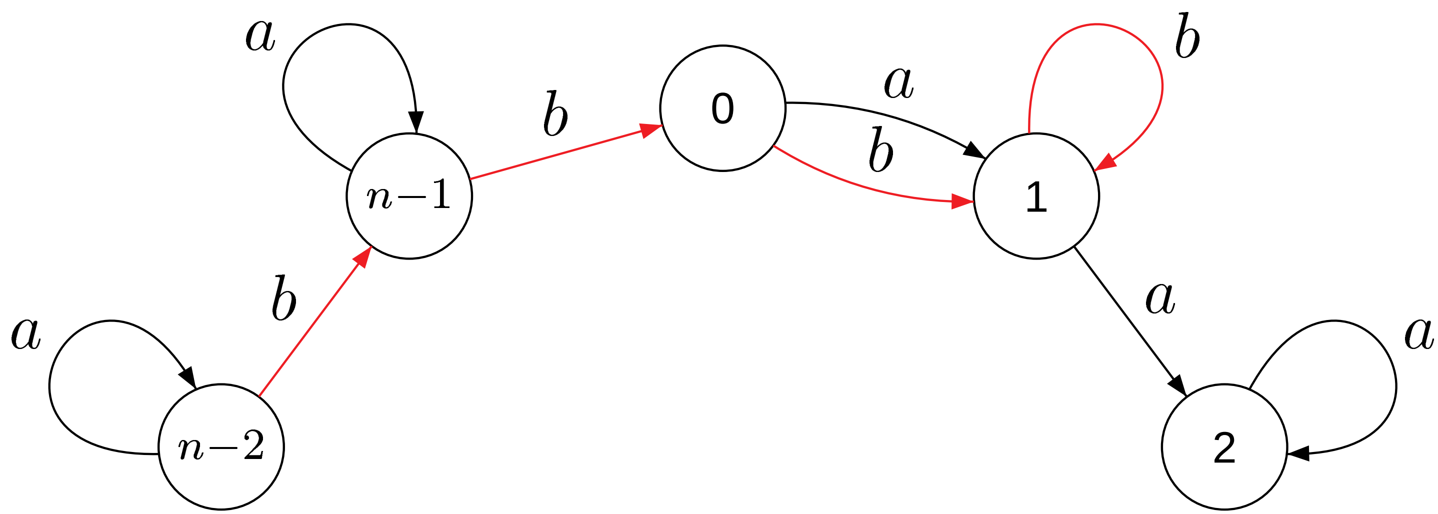

The recolored automaton is reset by the word b^{n-1}.

• Careful Road Coloring Problem. From the viewpoint of transportation network the constant out-degree condition does not seem to be natural. We rather want to find a synchronizing coloring for arbitrary strongly connected primitive digraph Γ, the number of colors being the maximal out-degree of Γ.

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In transport network terms this means that following the instruction w is always possible and brings one to the node which is independent of the initial node.

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The Careful Road Coloring Problem asks under which conditions strongly connected digraphs admit carefully synchronizing colorings. Is it true that every primitive strongly connected digraph has such a coloring? (The Careful Road Coloring Conjecture)

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