WORDS'03 (Turku, September 11, 2003)

Reflexive relations, extensive transformations and piecewise testable languages of a given height

Dedicated to Imre Simon on the occasion of his 60th birthday

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Simon’s Theorem

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- is easy to explain;
- is hard to prove;
- has many surprising connections, including those with combinatorics of words theory;
- has opened an area which still remains very vivid.

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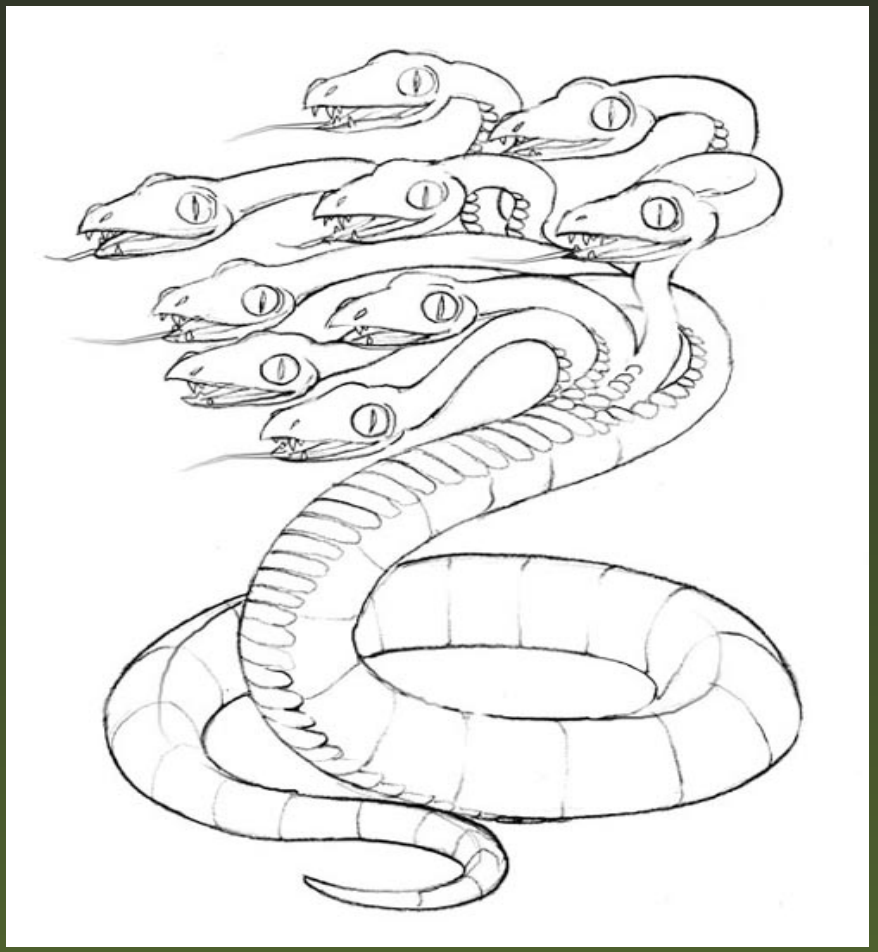

Figure 1: A 9-head hydra

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Figure 3: A 7-head hydra automaton

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Figure 2: A 7-head hydra automaton

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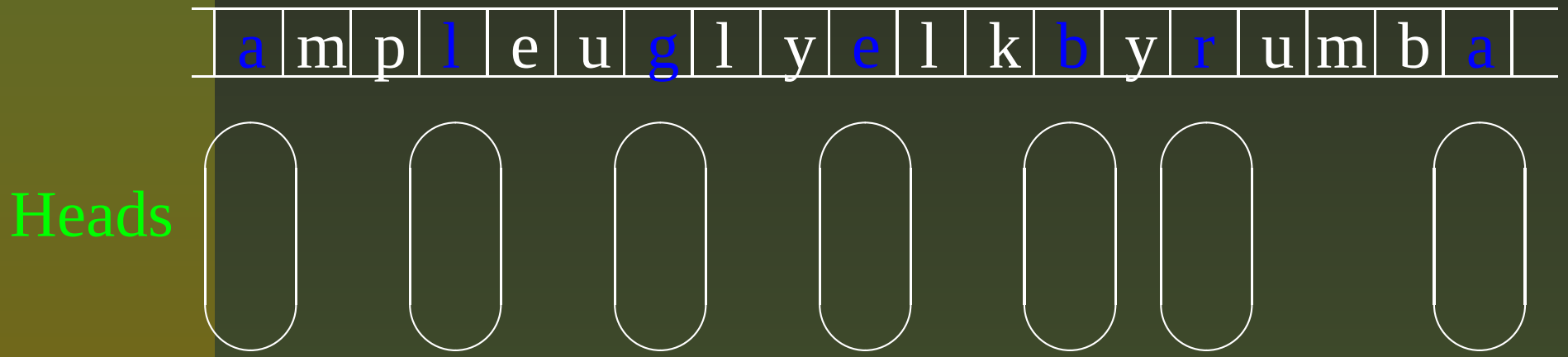

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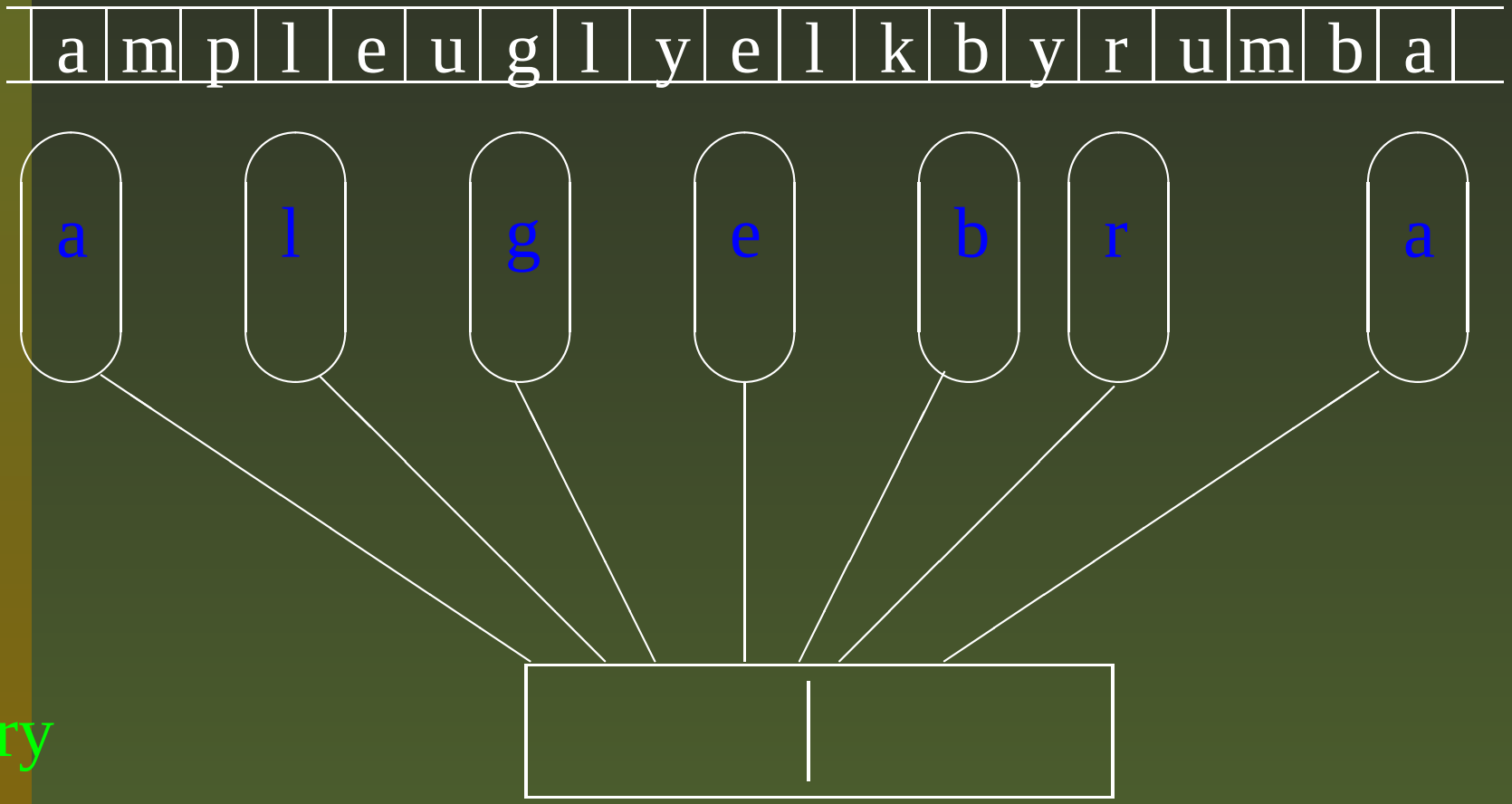

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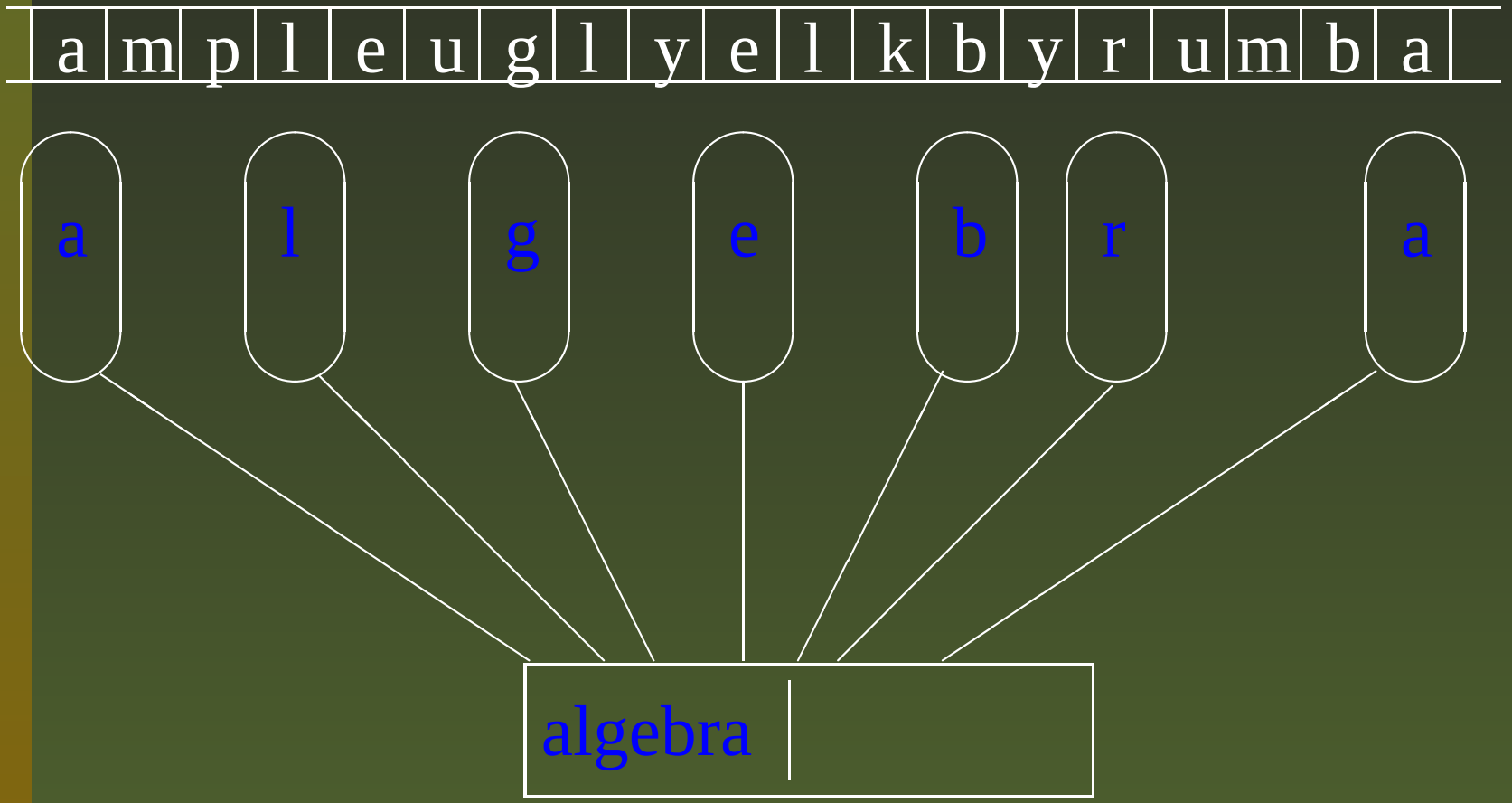

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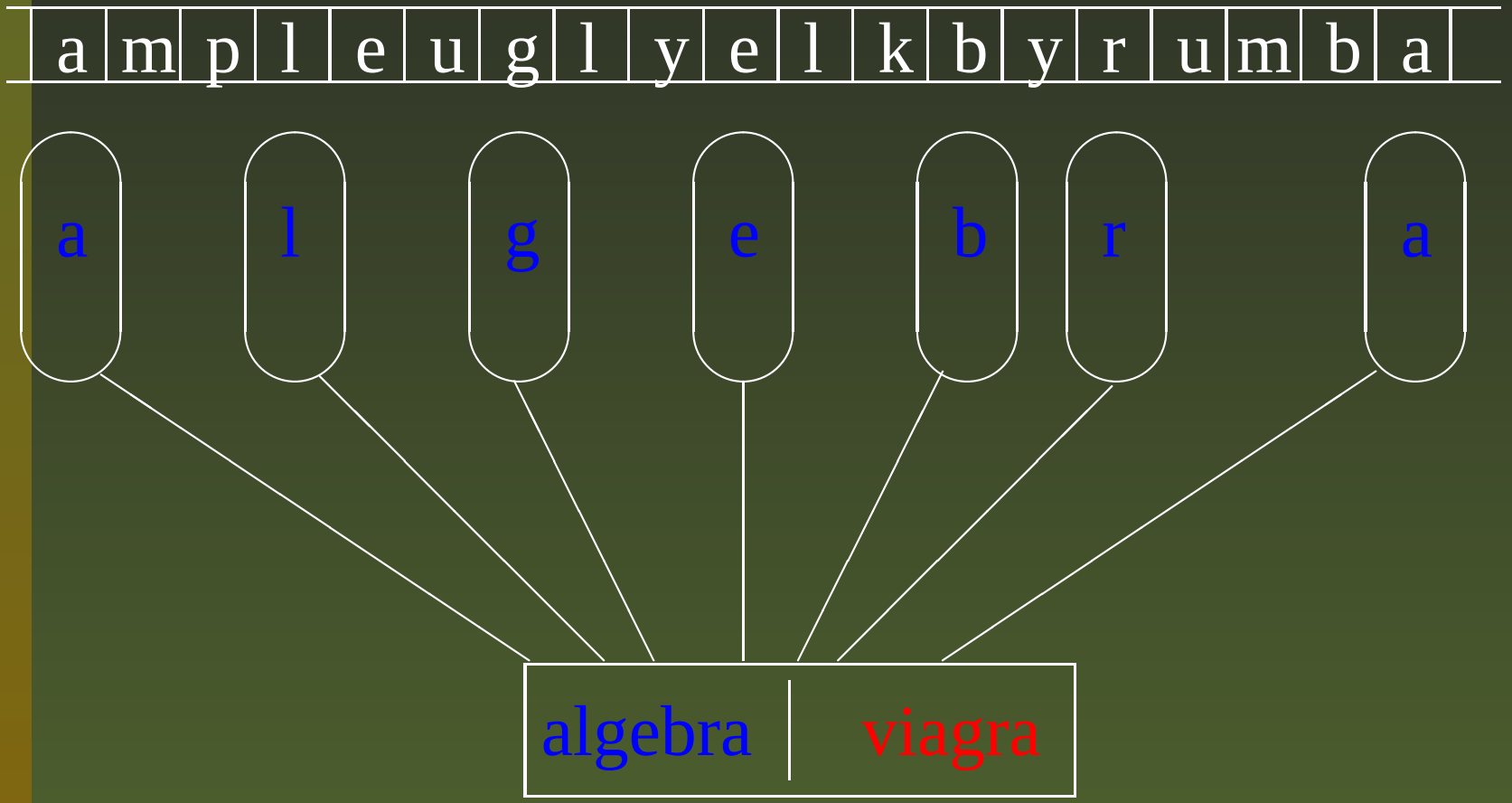

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A language $L \subseteq \Sigma^*$ is said to be *recognized* by a hydra automaton $\mathscr{H}$ if $\mathscr{H}$ accepts exactly words that are members of L. Such languages are called *piecewise testable*.

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Simon's hierarchy of piecewise testable languages:

$$PTL_1(\Sigma) \subset PTL_2(\Sigma) \subset PTL_3(\Sigma) \subset \ldots$$

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For a regular language L, the syntactic monoid $M(L)$ can be also defined as the *transition monoid* of the *minimal automaton* of L.

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Thus, whenever L is "given", so is $M(L)$.

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Theorem 1. (Imre Simon, 1972) *A language L is piecewise testable if and only if its syntactic monoid $M(L)$ is $\mathscr{J}$-trivial.*

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- Very efficient: There are polynomial time algorithms to verify if the syntactic monoid $M(L)$ is $\mathscr{J}$-trivial when presented the minimal automaton of L. Such a description of $M(L)$ is much more compact than the Cayley table — recall that the transition monoid of an automaton with n states may consist of as many as n^n elements!

Simon vs. Schützenberger

Compare with Schützenberger's theorem (1966) that provides an algebraic characterization of star-free languages: *a language L can be defined by a star-free expression* (that is, involving only Boolean operations and products but not Kleene's star) *if and only if the syntactic monoid $M(L)$ has only trivial subgroups*.

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Again a very natural language property is related to a natural semigroup property that can be verified in time $O(|M(L)|^2)$. On the other hand, the problem of deciding whether or not $M(L)$ has only trivial subgroups from the minimal automaton of L is PSPACE-complete!

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- Endomorphisms of linear orders — Higgins, 1997.

Simon vs. Eilenberg

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$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler) vs. $\sum_{n=1}^{\infty} \frac{1}{n^z} = \zeta(z)$ (Riemann)

Euler's result can be now written as $\zeta(2) = \frac{\pi^2}{6}$ but this is not a *consequence* of Riemann's considerations.

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Let $\mathbf{J}_h$ denote the pseudovariety of finite monoids that corresponds to the class of piecewise testable languages of height $\leq h$. We have

$$\mathbf{J}_1 \subset \mathbf{J}_2 \subset \mathbf{J}_3 \subset \cdots \subset \mathbf{J} = \bigcup_{h=1}^{\infty} \mathbf{J}_h$$

— Simon's hierarchy of $\mathscr{J}$-trivial monoids.

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Recall that by the definition $\mathbf{J}_h$ is the pseudovariety generated by the syntactic monoids of languages from $PLT_h(\Sigma)$ for all finite alphabets Σ. Thus, the algebraic counterpart of Question 2 is the following:

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This is a typical instance of the **PMP** (Pseudovariety Membership Problem). The PMP has proved to systematically arise whenever one translates a "real world" (computer science) question into algebra.

Straubing's Theorem

- $\mathscr{R}_n$ — the monoid of all reflexive binary relations on a set with n elements. It can be thought of as the monoid of all $n \times n$ matrices whose diagonal entries are 1 over the boolean semiring $\mathscr{B} = \langle \{0, 1\}; +, \cdot \rangle$.

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  A transformation α of a chain $\langle Q, \leq \rangle$ is *order preserving* if $q \leq q'$ implies $q.\alpha \leq q'.\alpha$ for all $q, q' \in Q$ and *extensive* if $q \leq q.\alpha$ for every $q \in Q$.

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This looks as a quite innocent Cayley-type theorem but in fact the proof heavily depends on Simon's theorem, and moreover, it can be shown relatively easily that the two theorems are equivalent.

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Corollary. *Each of the three sequences* $\{\mathscr{R}_n\}$, $\{\mathscr{U}_n\}$ *and* $\{\mathscr{C}_n\}$ $(n = 2, 3, \ldots)$ *generates the pseudovariety* **J** *of all finite* $\mathscr{J}$*-trivial monoids.*

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We thus have four stratifications for $\mathbf{J}$:

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Corollary. (Jean-Eric Pin, 1984) *For each* $h = 1, 2, \ldots,$ *the membership problem for the pseudovariety* $\mathbf{J}_h$ *is decidable, and hence, given a piecewise testable language, its height can be algorithmically determined.*

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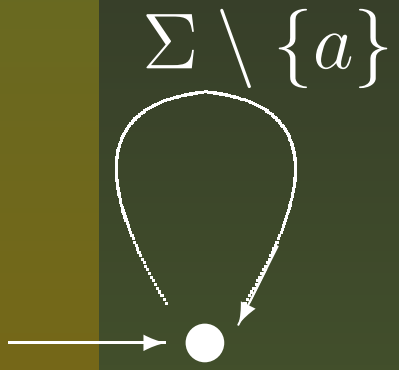

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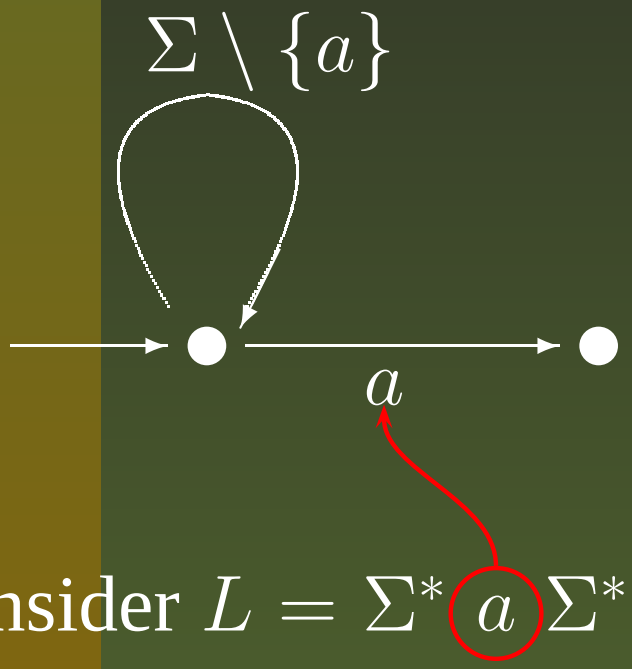

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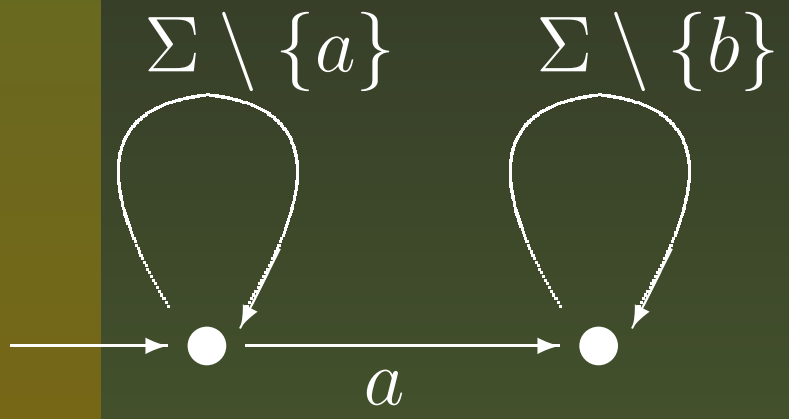

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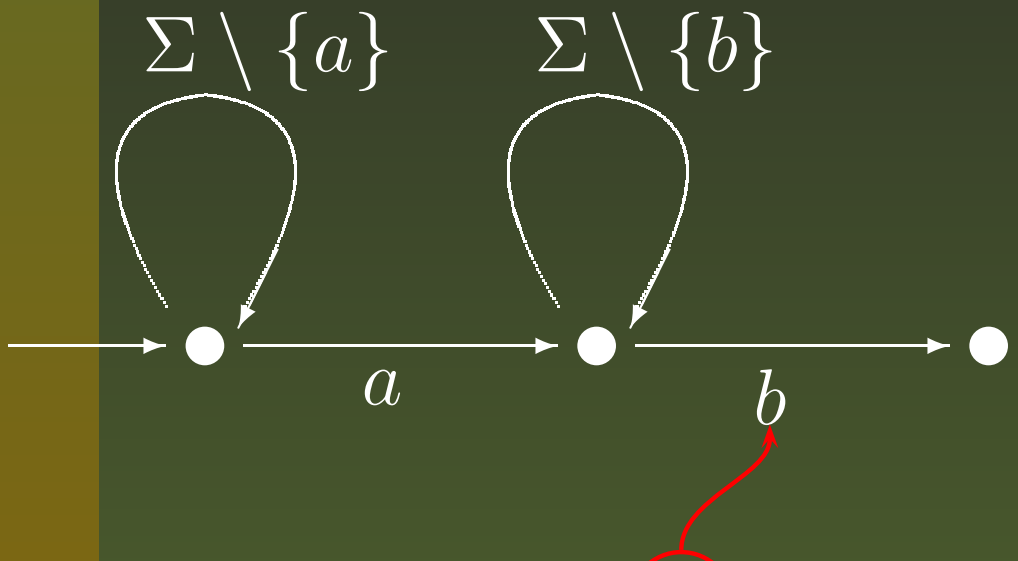

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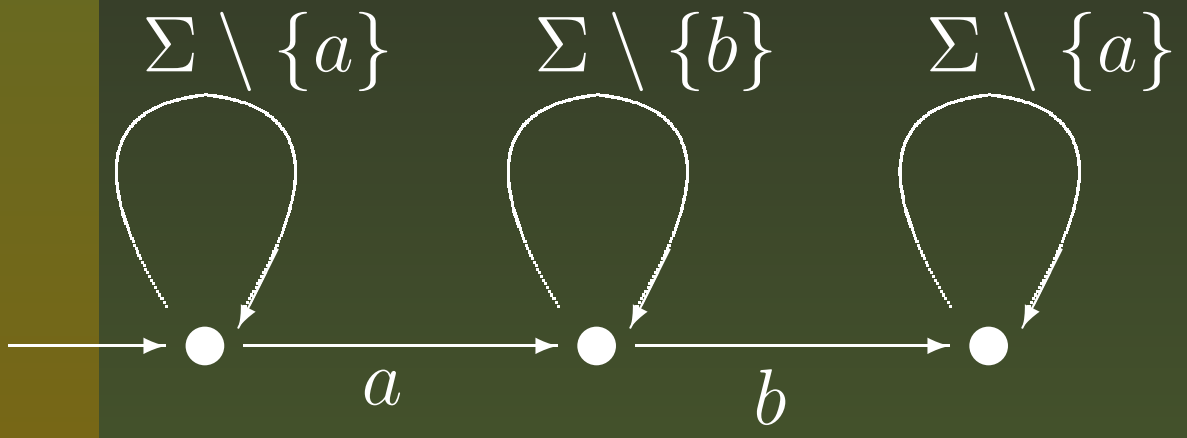

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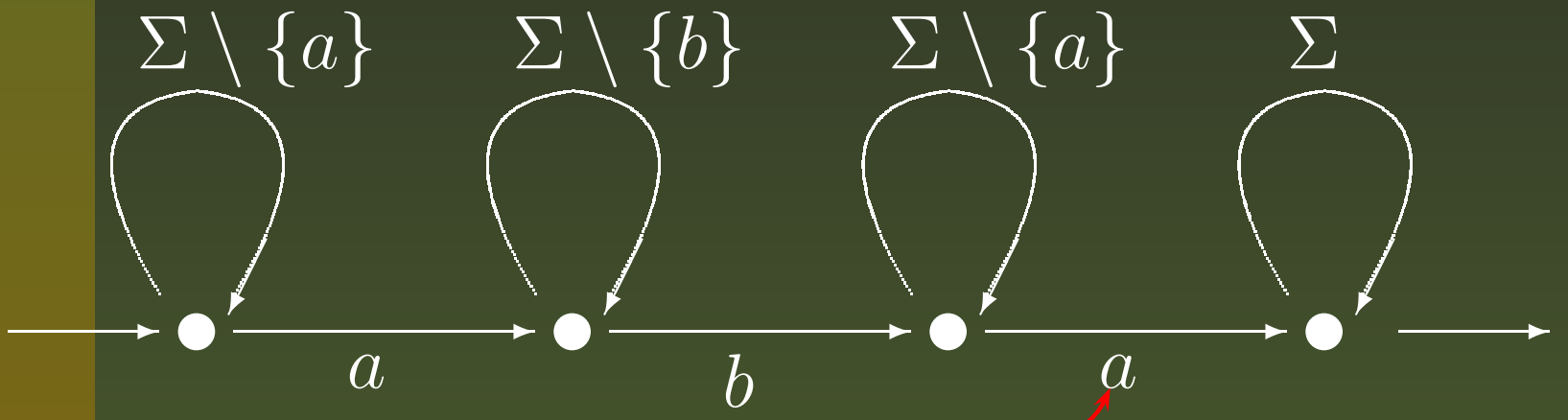

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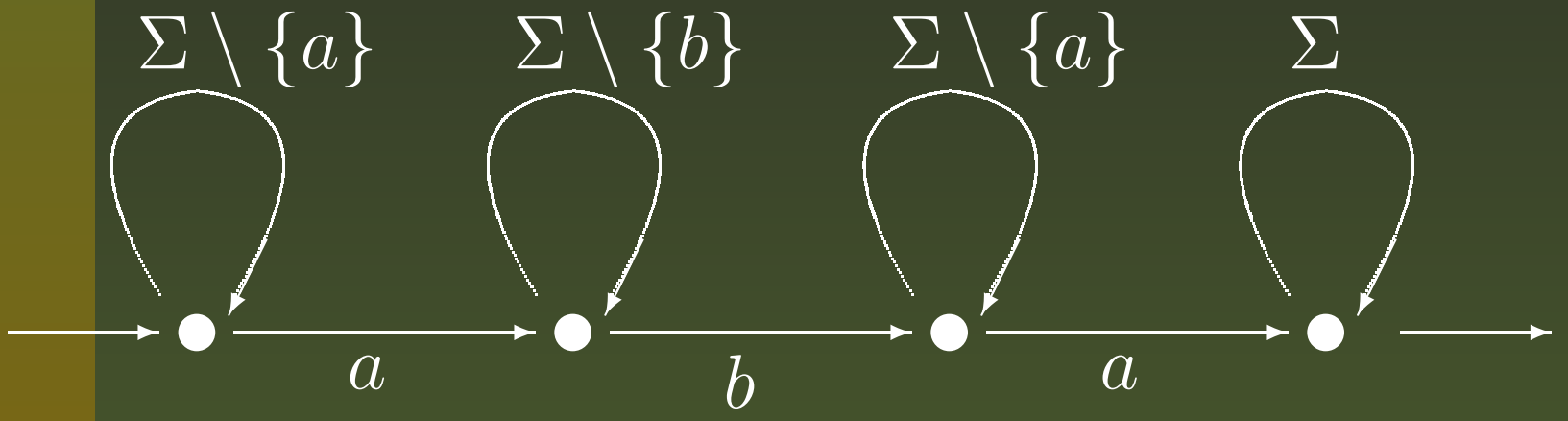

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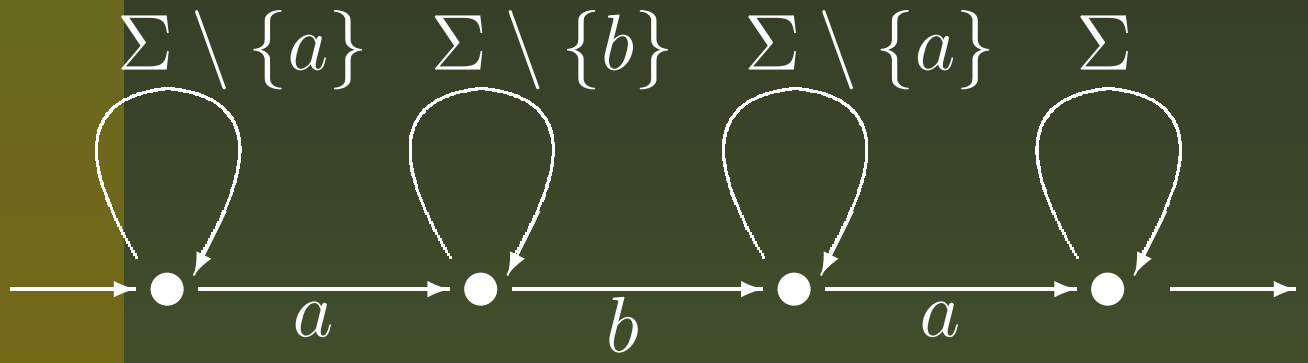

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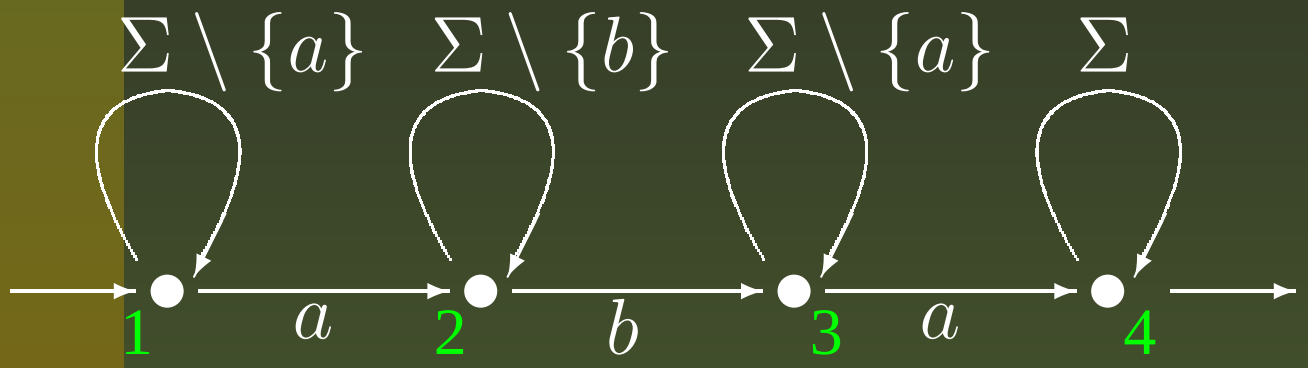

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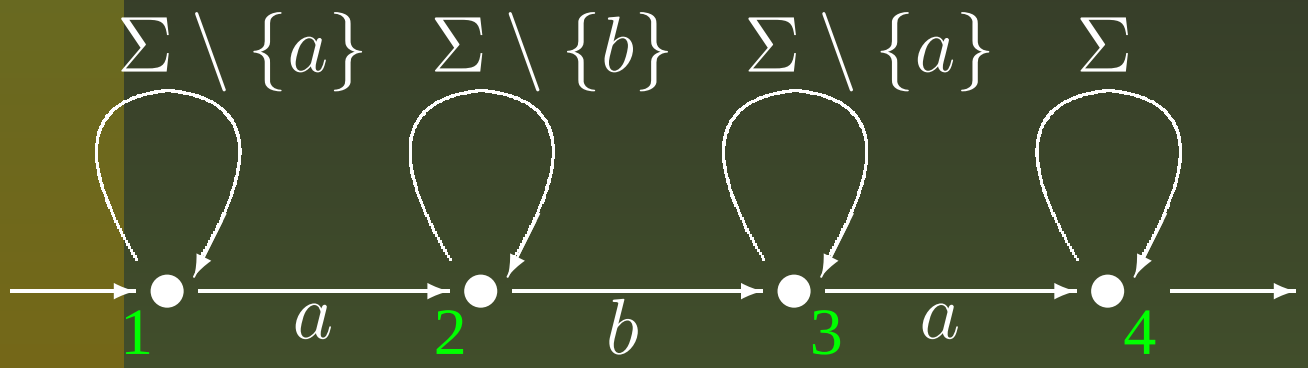

It is easy to see that with respect to this order the transformation induced by the letters are order preserving and extensive.

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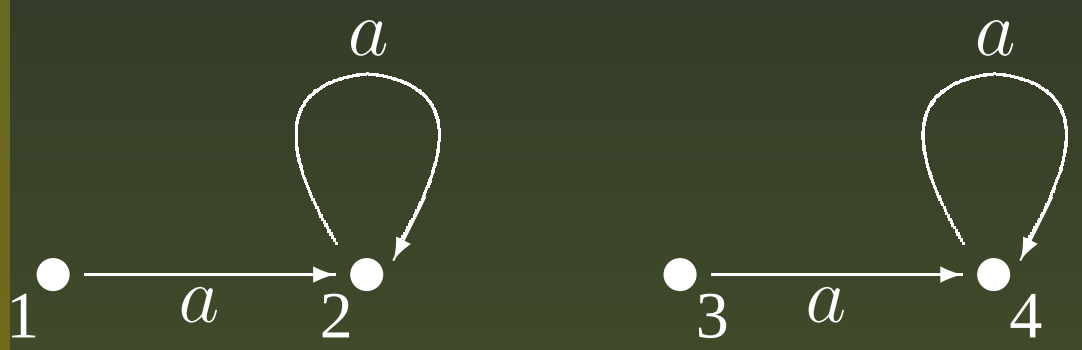

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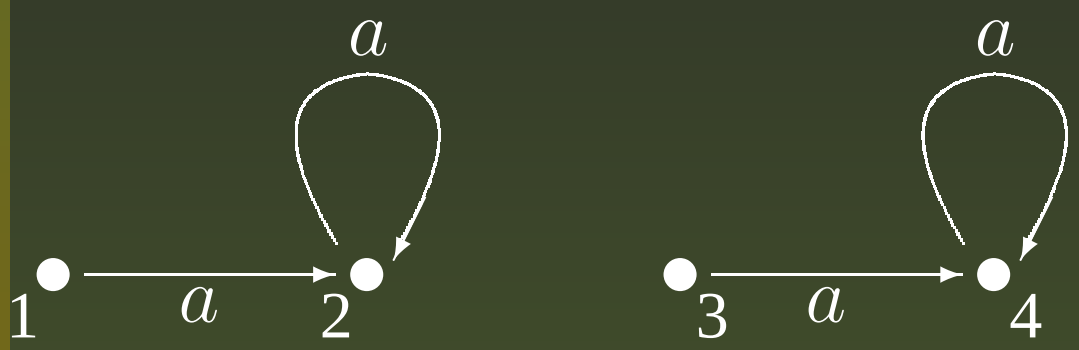

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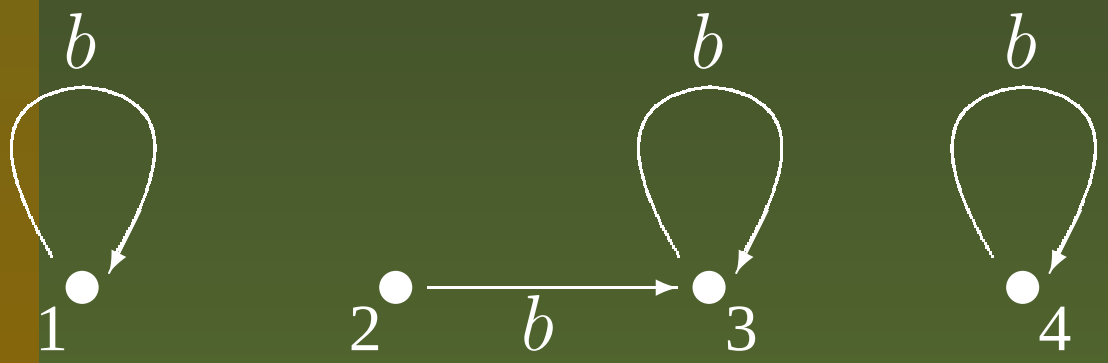

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We see that the transition monoid of the deterministic automaton recognizing our language $L = \Sigma^* a \Sigma^* b \Sigma^* a \Sigma^*$ consists of order preserving and extensive transformations of the chain $1 < 2 < 3 < 4$, i.e. it is a submonoid in $\mathscr{C}_4$.

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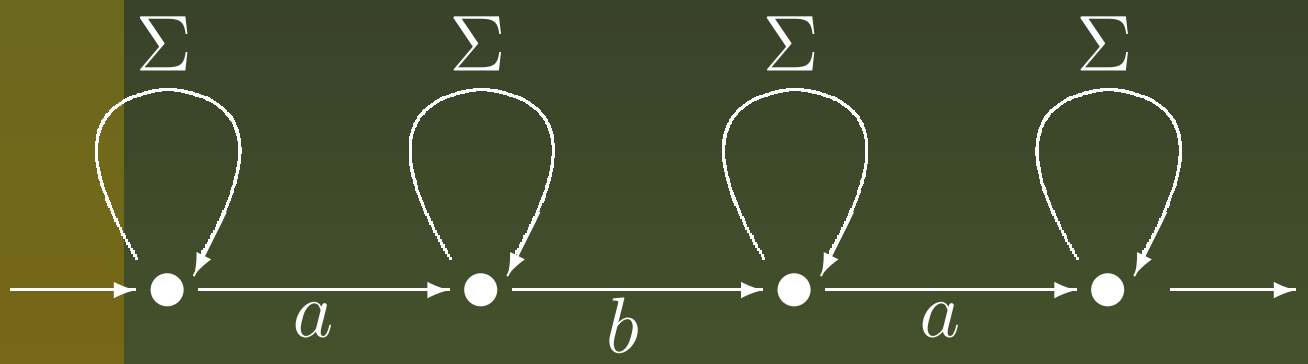

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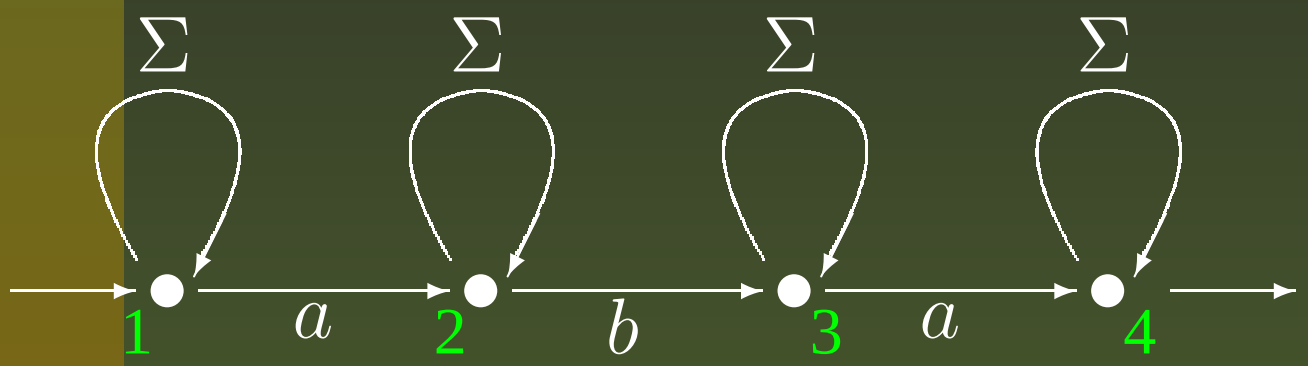

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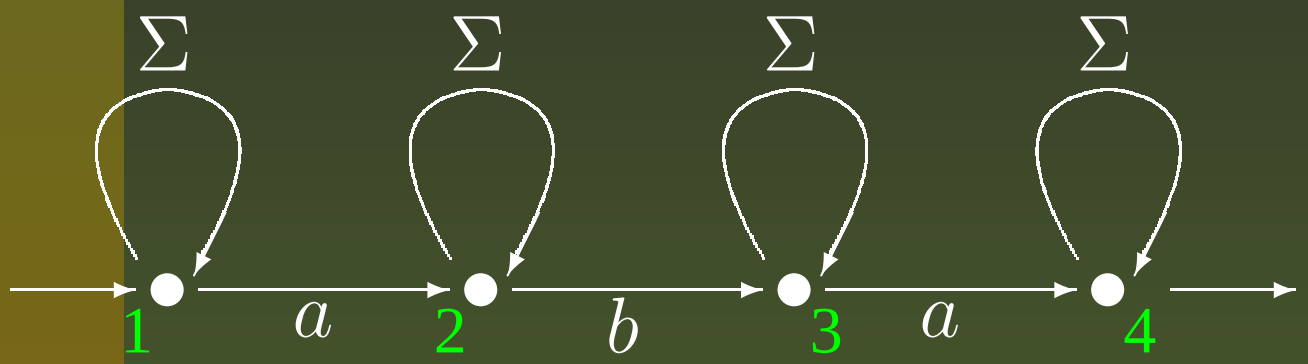

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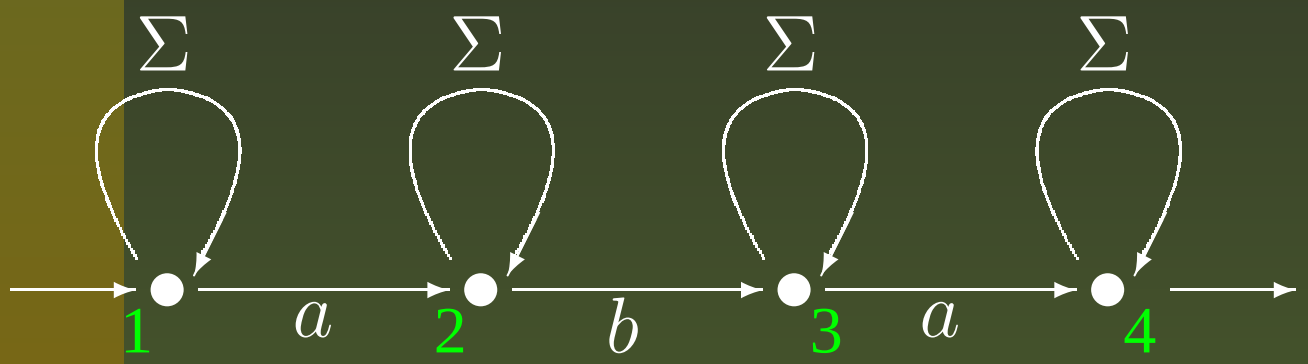

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d) *The monoids* $\mathscr{C}_n$ *with* $n > 4$ *are nonfinitely based.*

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Thus, there is an efficient algorithm to check if a given piecewise testable language can be recognized by a hydra automaton with 1, 2 or 3 heads, but this approach fails for larger numbers of heads.

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One may conclude that the optimal number of heads is equal to 3!

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These properties define 4 monoids of partial transformations of $\{1, 2, \ldots, n\}$: $\mathscr{T}_n$, $\mathscr{I}_n$, $\mathscr{PO}_n$, $\mathscr{PE}_n$.

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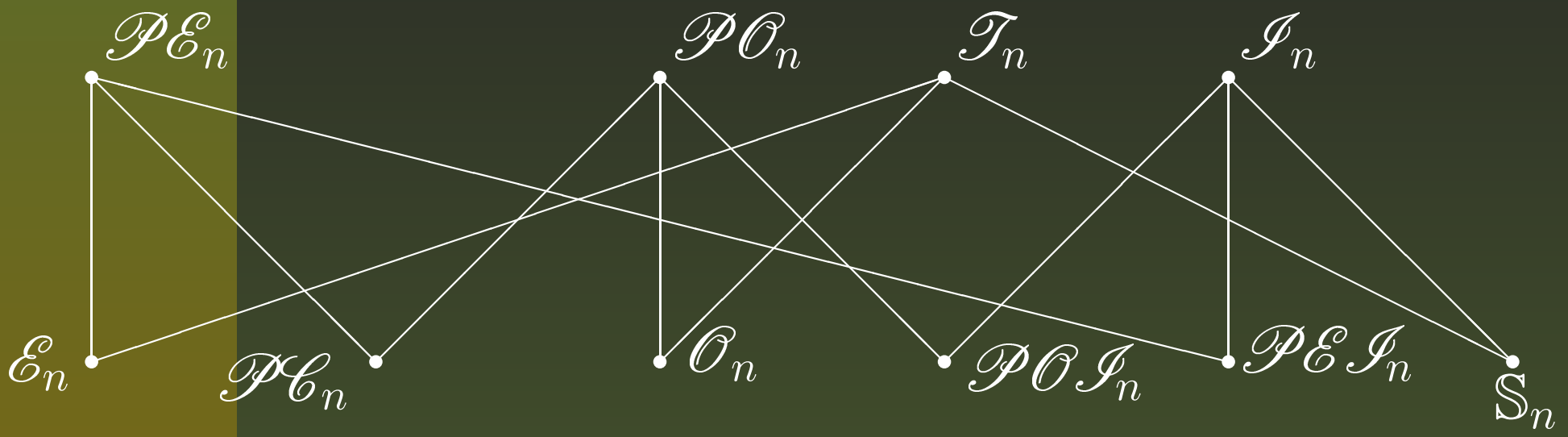
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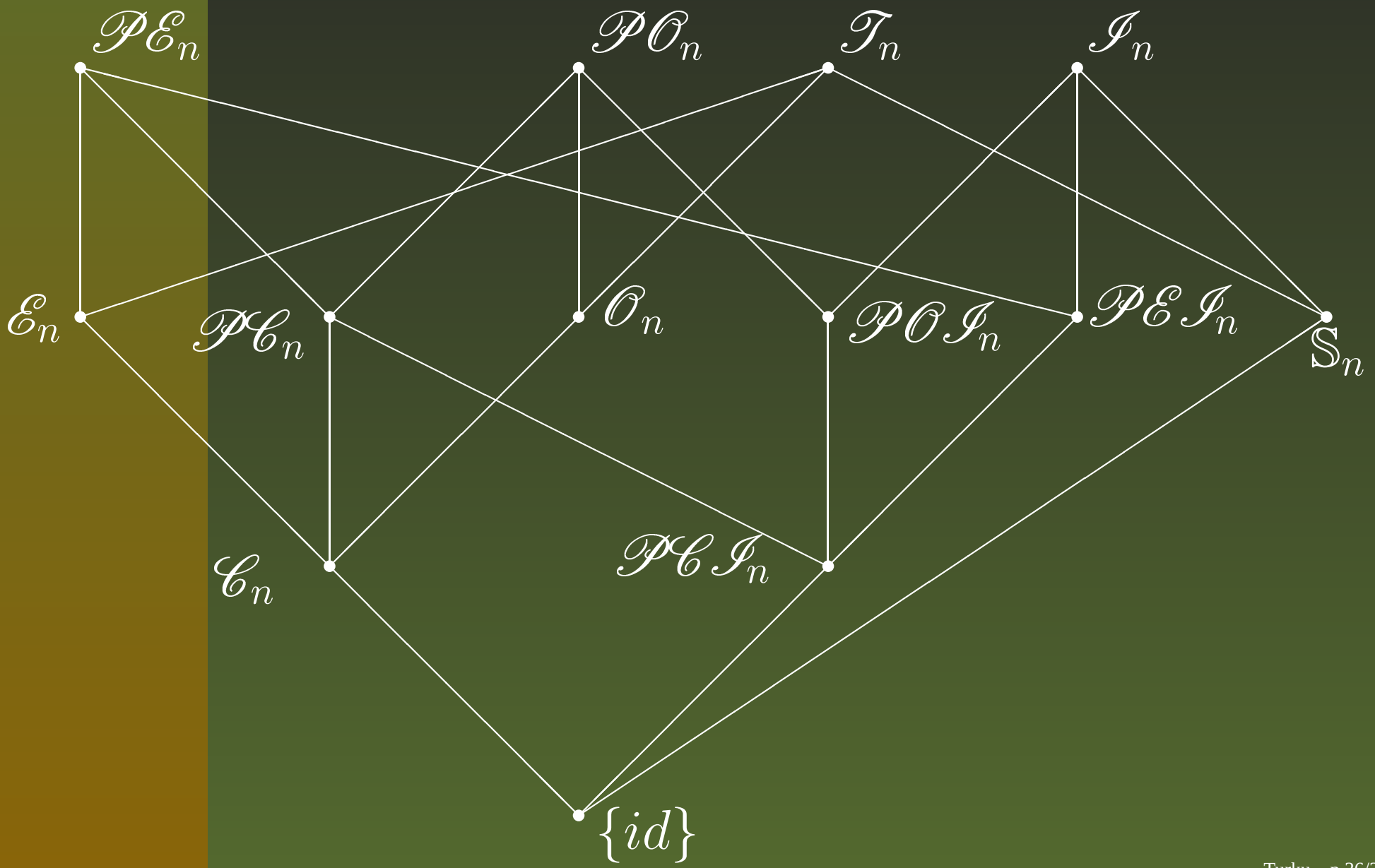
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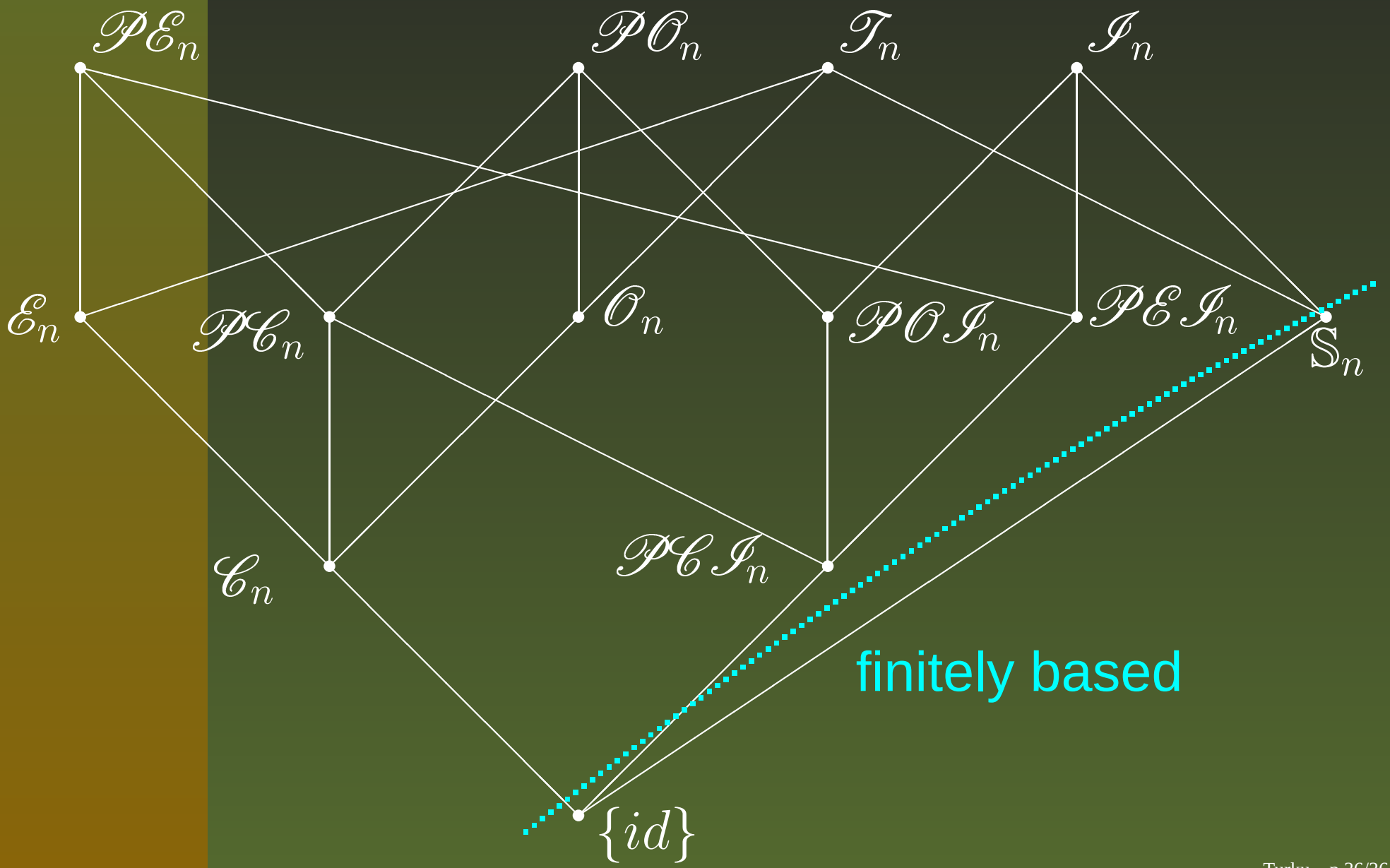
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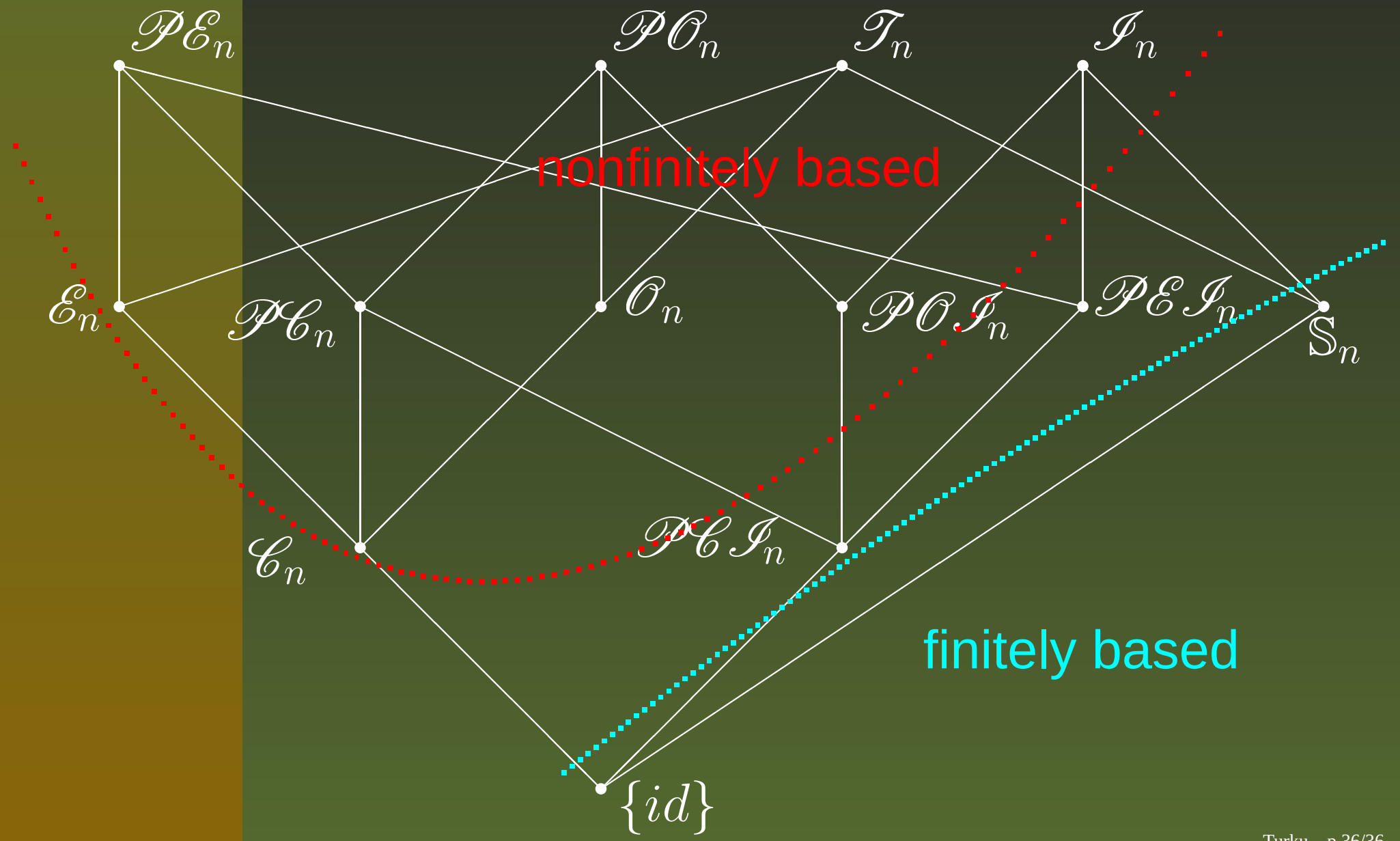
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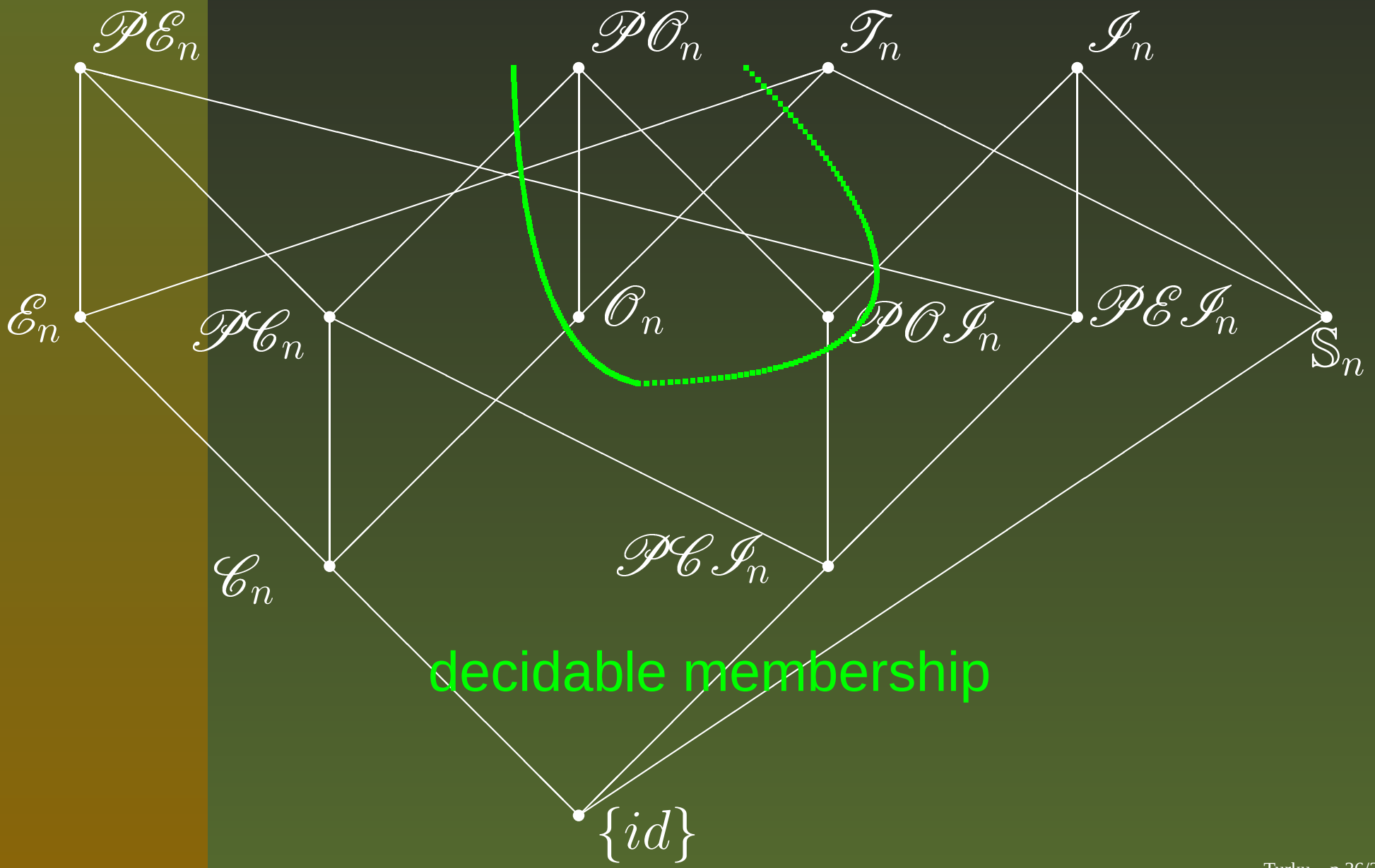
$\mathscr{E}_n$
$\mathscr{PC}_n$
$\mathscr{O}_n$
$\mathscr{POI}_n$
$\mathscr{PEI}_n$
$\mathbb{S}_n$
$\mathscr{C}_n$
$\mathscr{PCI}_n$
finitely based
$\{id\}$

Conclusion


$\mathscr{PE}_n$
$\mathscr{PO}_n$
$\mathscr{T}_n$
$\mathscr{I}_n$
$\mathscr{E}_n$
$\mathscr{PC}_n$
$\mathscr{O}_n$
$\mathscr{POI}_n$
$\mathscr{PEI}_n$
$\mathbb{S}_n$
$\mathscr{C}_n$
$\mathscr{PCI}_n$
decidable membership
$\{id\}$