The Finite Basis Problem for Kauffman Monoids

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NSAC, August 18th, 2009

Outline

- The Finite Basis Problem
- Wire monoids and Kauffman monoids
- Identities in Kauffman monoids
- Open problems

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We consider monoids (semigroups with 1) but we treat them as semigroups (algebras of type $\langle 2 \rangle$).

An identity is a pair of semigroup words (u, v) usually written as a formal equality $u \doteq v$.

A monoid M satisfies an identity $u \doteq v$ (or: $u \doteq v$ holds in M) if every evaluation of letters involved in the words u and v at some elements of M produces equal values in M.

Example: the identities $xy \doteq yx$ and $x \doteq x^2$ hold in the monoid $\langle \{0, 1\}; \cdot \rangle$ while the identity $x \doteq y$ does not.

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Indeed, if an identity $u \doteq v$ holds in $\langle\{0,1\};\cdot\rangle$, then u and v have the same letters. But the laws $xy \doteq yx$ and $x \doteq x^2$ allow one to reduce any word to the product of its letter, each taken once, in some fixed order.

$$xyxyzytzx \xrightarrow{xy\doteq yx} x^3y^3z^2t \xrightarrow{x\doteq x^2} xyzt$$

Hence, whenever two words involve the same letters, they can be reduced to the same product, and thus, to each other.

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Perkins's Example

If a monoid is not finitely based, it is said to be nonfinitely based.
A finite monoid can be nonfinitely based (Peter Perkins, 1969).
Perkins's example is the 6-element monoid B_2^1 (the Brandt monoid) formed by the following 2×2-matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The example published exactly 40 years ago is extremely transparent and natural. It also turns out to be minimal with respect to size of monoid. There are only two 6-element examples (Edmond Lee and Jian Rong Li).

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Main Problem

Semigroups and monoids are the only "classical" algebras for which finite nonfinitely based objects exist: finite groups, associative or Lie rings, lattices are finitely based.

The Finite Basis Problem (FBP) for a class K of monoids asks which monoids in K are finitely based and which are not. Its algorithmic version for the class of finite monoids is known as Tarski's problem. For general algebras (and even for groupoids) the corresponding problem is shown by Ralph McKenzie to be undecidable. For monoids (and semigroups) it is open.

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The area has been progressed a lot over the last decade.
A (partially) updated version of the above survey is available through my webpage:

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The Brandt monoid B_2^1 is inherently nonfinitely based (Mark Sapir).

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- Critical monoids: a series of monoids M_n, $n = 1, 2, \ldots$, such that each M_n does not belong to the variety Var M while all n-generated submonoids of M_n belong to Var M.

A new method (that can be thought of as a variation of the interpretation method) has been recently found by Marcel Jackson and Ralph McKenzie (Interpreting graph colorability in finite semigroups, *Int. J. Algebra Comput.* Vol.16, 119–140, 2006).

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- Complexity analysis: if the finite membership problem for the variety Var M is computationally hard, then M must be nonfinitely based.

FBP vs FMP

The Finite Membership Problem (FMP) for a variety $\mathbf{V}$ is a combinatorial decision problem whose input is a finite monoid N and whose question is whether N belongs to the variety $\mathbf{V}$.

FMP for varieties and pseudovarieties play a central role in the modern theory of monoids.

FMP is always decidable for finitely generated varieties: by the HSP-theorem $N \in \operatorname{Var} M$ iff N is a homomorphic image of the free $|N|$-generated semigroup of $\operatorname{Var} M$ and the free semigroup has at most $|M|^{(|M|^{|N|})}$ elements.

The complexity of FMP is not known (for general algebras it can be extremely high as shown by Marcin Kozik).

But, if $\mathbf{V}$ has a finite identity basis Σ, say, then in order to check whether or not N belongs to $\mathbf{V}$ it suffices to verify whether or not all identities in Σ hold in N, and this is a polynomial (in $|N|$) procedure.

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Temperley–Lieb Algebras

Neville Temperley and Elliott Lieb (Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the percolation problem, *Proc. Roy. Soc. London* Ser. A 322, 251–280, 1971) motivated by some problems in statistical mechanics have introduced what is now called Temperley–Lieb algebras. These are associative linear algebras with 1 over $\mathbb{C}$. Given n and $c \in \mathbb{C}$, the algebra $TL_n(c)$ is generated by $n-1$ generators $h_1, \dots, h_{n-1}$ subject to the relations

$$h_i h_j = h_j h_i \quad \text{if } |i-j| \geq 2,$$

$$h_i h_j h_i = h_i \quad \text{if } |i-j| = 1,$$

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Temperley–Lieb Algebras

Neville Temperley and Elliott Lieb (Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the percolation problem, *Proc. Roy. Soc. London* Ser. A 322, 251–280, 1971) motivated by some problems in statistical mechanics have introduced what is now called Temperley–Lieb algebras. These are associative linear algebras with 1 over $\mathbb{C}$. Given n and $c \in \mathbb{C}$, the algebra $TL_n(c)$ is generated by $n-1$ generators $h_1, \ldots, h_{n-1}$ subject to the relations

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Temperley–Lieb Algebras contd

It is easy to see (and has been soon observed) that algebra $TL_n(c)$ is spanned by the monoid K_n with n generators $c, h_1, \dots, h_{n-1}$ subject to the relations

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The monoids K_n are called the Kauffman monoids. Lois Kauffman (An invariant of regular isotopy, *Trans. Amer. Math. Soc.* 318, 417–471, 1990) has independently invented these monoids as geometrical objects. We present Kauffman's approach in a slightly more general setting.

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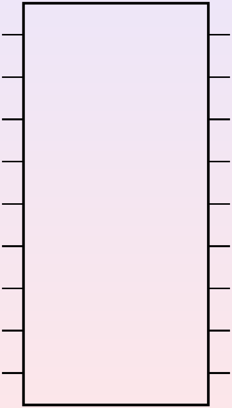
Wire Monoids

Fix n and consider "chips" with $2n$ pins, n on each side. Pins are connected by n wires. To multiply two chips, we connect the right hand side pins of the first with the left hand side pins of the second.



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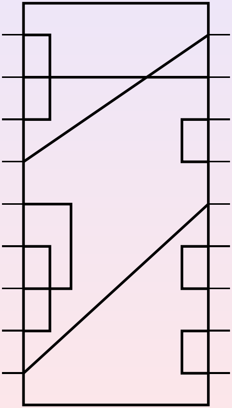

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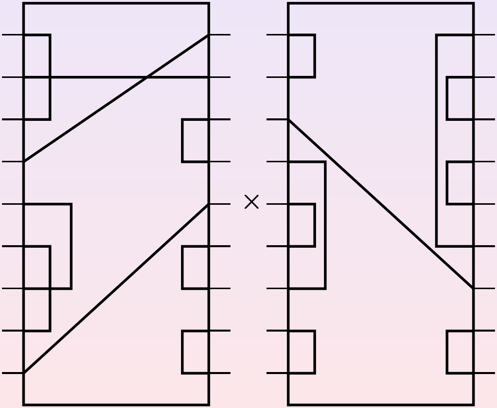

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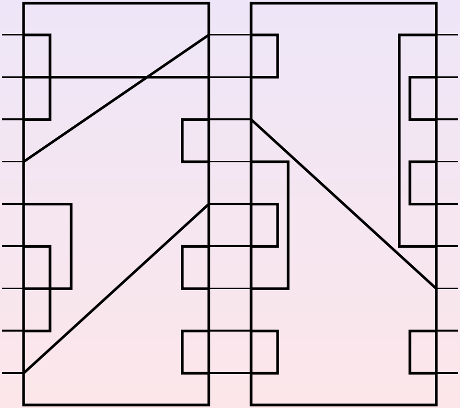

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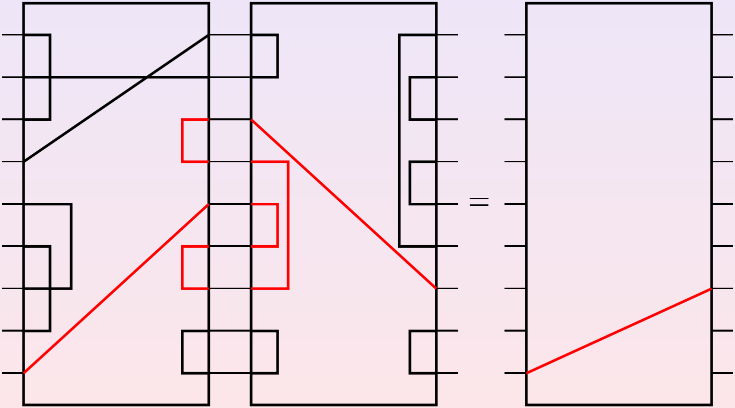

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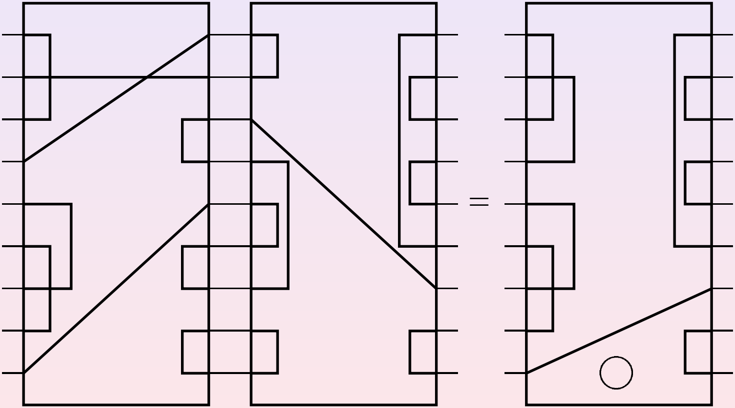

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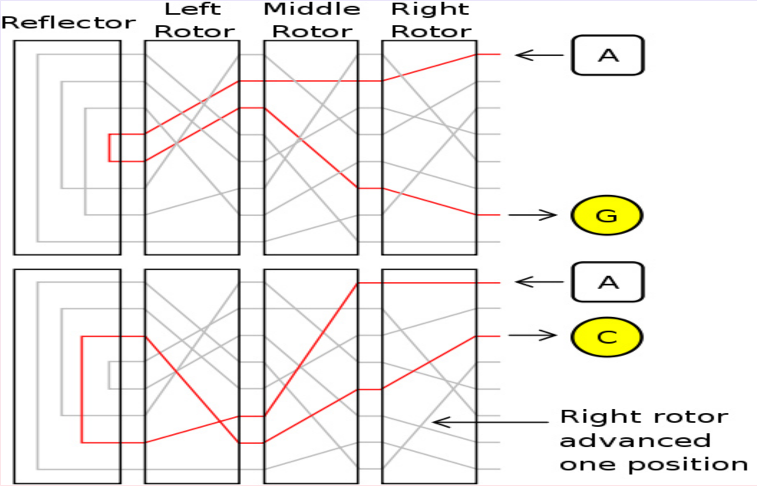

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Richard Brauer's monoids arose in his paper 'On algebras which are connected with the semisimple continuous groups', *Ann. Math.* 38, 857–872, 1937 as vector space bases of certain associative algebras relevant in representation theory.

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Jones monoids are named after Vaughan Jones, the famous knot theorist. We denote by J_n the Jones monoid of chips with n pins.

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Kauffman Monoids as Wire Monoids

Thus the Kauffman monoid K_n consists of n-pin chips with non-crossing wires that may contain circles. Only the number of circles matters, not their location. The monoid K_n is generated by the hooks $h_1, \dots, h_{n-1}$ and the circle c.

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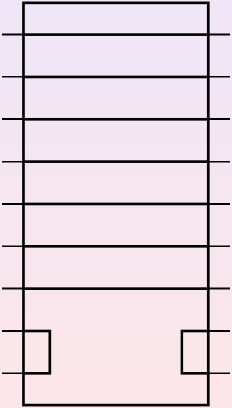

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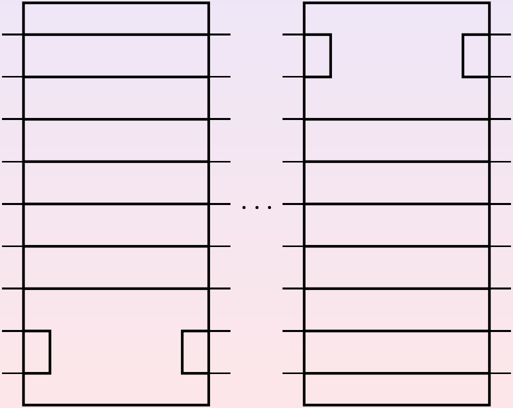

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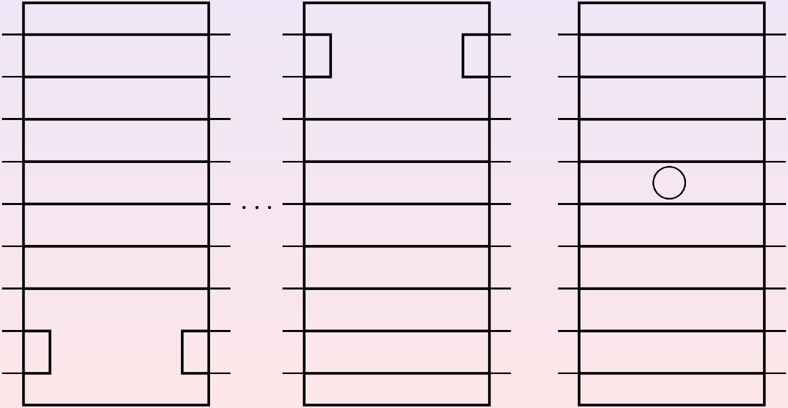

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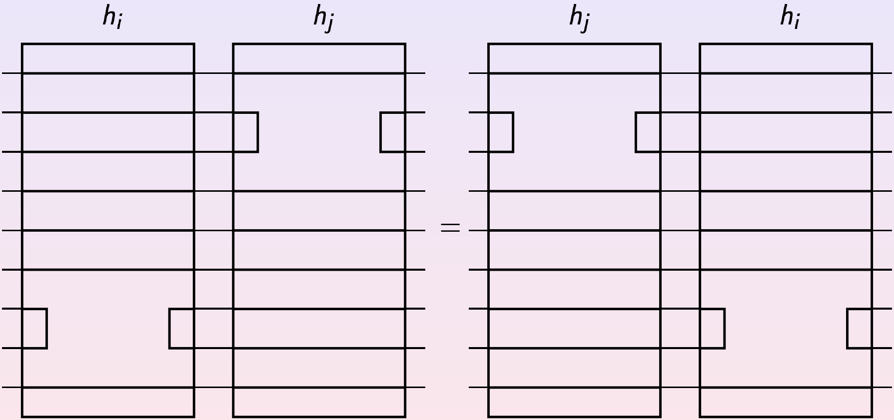

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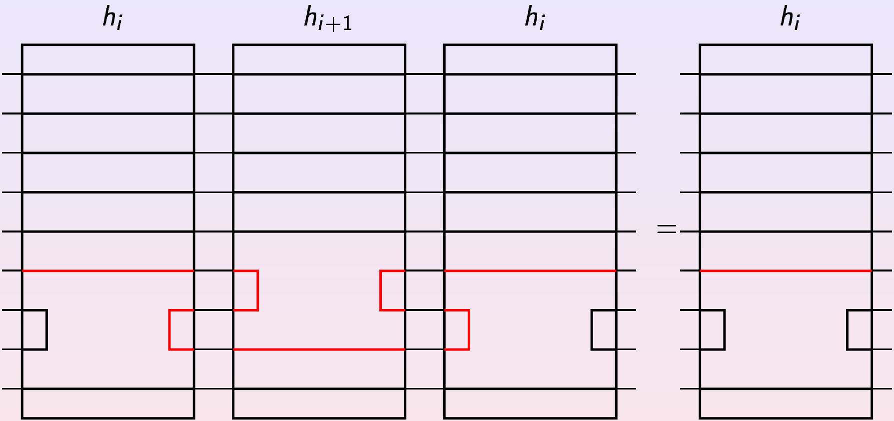

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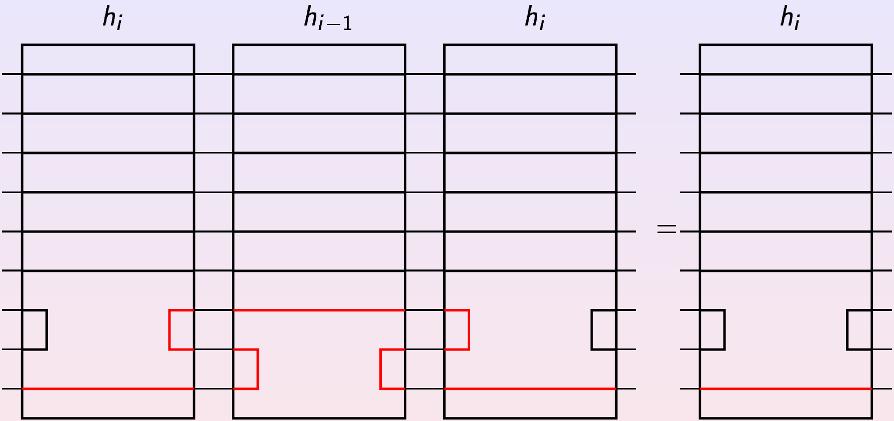
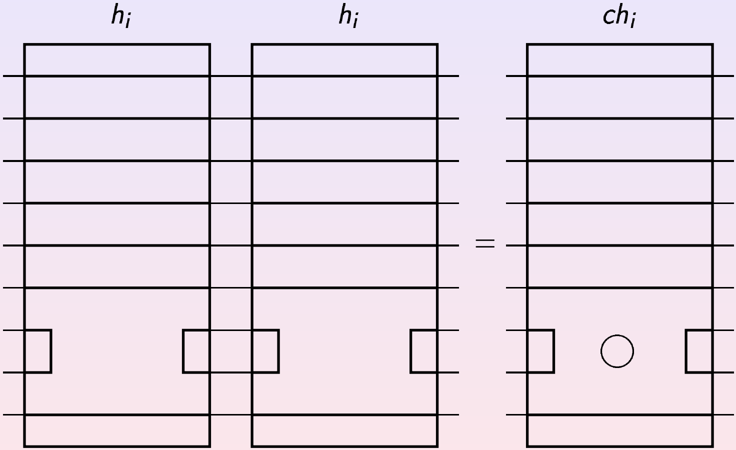


$h_i h_i = ch_i$



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Thus, the "planar" wire monoid generated by the hooks and the circle satisfies the relation of K_n and is therefore a homomorphic image of K_n. In fact, this wire monoid is isomorphic to K_n (requires some work). This connection was realized by Jones who didn't bother himself with a formal proof. Such a proof has first been published by Mirjana Borisavljević, Kosta Došen and Zoran Petrić (Kauffman monoids, *J. Knot Theory Ramifications* 11, 127–143, 2002).

Similarly, one can show that the Jones monoid J_n is generated by the hooks $h_1, \ldots, h_{n-1}$ subject the relations

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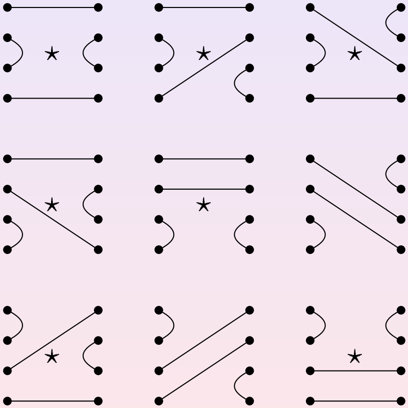

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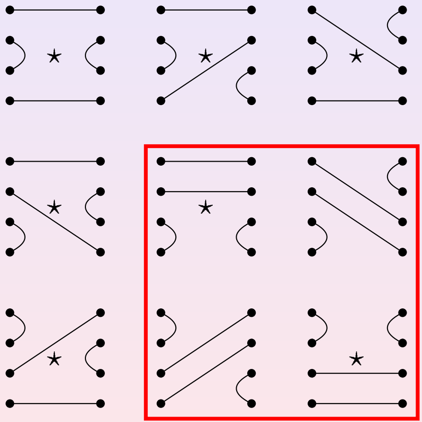

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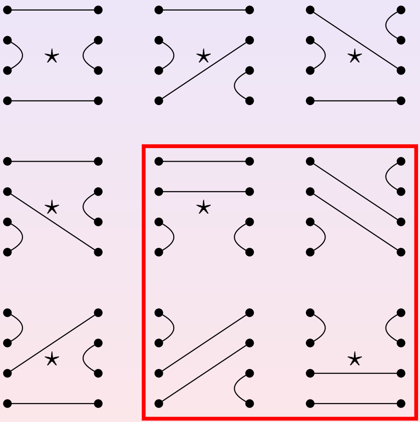

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Here Z_m is the nth Zimin word; recall that this remarkable series of words is are defined as follows: $Z_1 = x_1, \ldots, Z_m = Z_{m-1}x_mZ_{m-1}$. The word $Z_4 = 121312141213121$ (without two last digits) is engraved on the apple in the center of the conference poster.

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However it can be proved that the variety $\mathrm{Var}\, K_n$ possesses a finiteness property that is incompatible with having a finite identity basis. Namely, all semigroups in $\mathrm{Var}\, K_n$ that satisfy the identities $x^2y \doteq yx^2 \doteq x^2$ (that is, $x^2 \doteq 0$) are locally finite.

Theorem (Mark Sapir, 1987)

If all semigroups with $x^2 \doteq 0$ in a variety $\mathbf{V}$ are locally finite and $\mathbf{V}$ has no non-trivial identity of the form $Z_m \doteq W$, then $\mathbf{V}$ is nonfinitely based.

Here Z_m is the nth Zimin word; recall that this remarkable series of words is are defined as follows: $Z_1 = x_1, \ldots, Z_m = Z_{m-1}x_mZ_{m-1}$. The word $Z_4 = 12131214121312 1$ (without two last digits) is engraved on the apple in the center of the conference poster.

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The key ingredient of the proof comes from the recent paper by Kwok Wai Lau and Des FitzGerald (Ideal structure of the Kauffman and related monoids, *Comm. Algebra* 34, 2617–2629, 2006).

They have studied a natural parameterization:

Kauffman chip $\mapsto$

(the Jones chip obtained by erasing circles, the number of circles)

The first component is a homomorphism while the second is not but it can be controlled.

Identities in Kauffman Monoids

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For each $n \geq 4$, the Kauffman monoid K_n is nonfinitely based.

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Open Problem

Both Kauffman and Jones monoids admit a natural unary operation (flipping chips). J_n satisfies the identity $xx^*x \doteq x$. The Brandt monoid B_2^1 also has a natural involution (transposition), and B_2^1 with this operation belongs to the unary semigroup variety generated by K_n or J_n with $n \geq 4$. However Sapir has discovered that B_2^1 is not inherently nonfinitely based as a unary semigroup under transposition. Moreover, there exists no inherently nonfinitely based unary semigroups satisfying $xx^*x \doteq x$, see Karl Auinger, Igor Dolinka, and MV, Equational theories of semigroups with enriched signature, available online under `http://arxiv.org/abs/0902.1155v2`

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Question

Are the Kauffman and Jones monoids finitely based as unary semigroups?