**CONTRIBUTIONS TO GENERAL ALGEBRA 2**
**Proceedings of the Klagenfurt Conference, June 10–13, 1982**
**Verlag Hölder–Pichler–Tempsky, Wien 1983 – Verlag B. G. Teubner, Stuttgart**

**M. V. Volkov**

# ON THE JOIN OF SEMIGROUP VARIETIES

In [1] I have proved a general result concerning the join of varieties of universal algebras which implies, in particular, that, if $M$ and $N$ are semigroup varieties, and $N$ is a nilpotent variety, then $M \cup N$ is finitely based if and only if $M$ is finitely based. The "applied" importance of this result may be explained by the following. Take as $N$ the variety $N_k$ consisting of all semigroups which are nilpotent of degree k. Since $N_k$ satisfies exactly identities u=v having the property that the lengths of the words u and v are no less than k, identities of $M \cup N_k$ are identities of $M$ with the same property. Thus, by investigation the finite basis condition on identities of the variety $M$, we may consider only "sufficiently long" identities. This idea has been applied, for example, by G.Pollák and myself in [2].

In [3] I have introduced a new numerical characteristic of a semigroup word w as follows. Let the length of w be n, and let exactly m different letters occur in w. The number n-m is called the <u>level</u> of the word w and will be denoted by lev(w). Denote by $L_k$ the variety defined by all identities u=v such that lev(u)≥k and lev(v)≥k. Clearly, $L_k$ plays the same part as regards the level that $N_k$ plays as regards the length, and $N_k \subseteq L_k$. In this paper I shall prove

<u>THEOREM 1</u>. Let $M$ and $L$ be semigroup varieties, and let $L \subseteq L_k$ for some k. If the variety $M \cup L$ is finitely based, then $M$ is also finitely based.

Theorem 1 generalizes the "only if" part of the above-mentioned result. I do not know whether analogous generalization of the "if" part holds and shall prove here only a partial result. To formulate it, denote by $N^1$ the three-element semigroup on the set 0,1,2 with multiplication table

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 0 |

THEOREM 2. Let $M$ and $L$ be semigroup varieties, and let $L \subseteq L_k$ for some k. If $M$ is finitely based and $N^1 \in M$, then $M \vee L$ is also finitely based.

Theorems 1 and 2 will be proved in sections 1 and 2 respectively, in section 3 I shall prove a corollary concerning the notion of the distance in the lattice of semigroup varieties.

## 1. Proof of Theorem 1

To prove Theorem 1, I need some lemmas and constructions from [3].

LEMMA 1 ([3], Main Theorem). Let $\Sigma=\{u_i=v_i \mid i\in I\}$ be a system of semigroup identities such that there is a number k such that, for any $i\in I$, $lev(u_i)<k$ and $lev(v_i)<k$. Then $\Sigma$ is finitely based.

If a letter x appears in a word w only once, it will be called non-repeated in w; if x occurs in w at least twice, it will be called repeated in w. Maximum subwords of a word w which contain only non-repeated letters (in w) are called blocks of w. Associate with w a distribution word dw(w) by replacing every block of w by a new letter, and a block vector bl(w) whose components are the lengths of the blocks of w from left to right. The graphical identity of words will be denoted by $\equiv$, the endomorphism semigroup of the free semigroup of countable rank F is denoted by End(F).

LEMMA 2. Let u and v be words such that $dw(u)\equiv dw(v)$ and $bl(u)\leq bl(v)$ component-wise. Then an endomorphism $\phi\in End(F)$ exists such that $u^\phi\equiv v$.

This lemma was proved in [3] as the second step in the proof of lemma 3.

Theorem 1 follows from the following

LEMMA 3. Let $M$ and $L$ be semigroup varieties, and $L \subseteq L_k$ for some k. Then $M$ is finitely based in the variety $M \cup L$.

Proof. Any identity $u=v$ holding in $M$ and such that $lev(u) \geq k$ and $lev(v) \geq k$ holds in $M \cup L$. Therefore, $M$ may be given in $M \cup L$ by identities $u=v$ where $lev(u) < k$. Denote by $\Sigma_o$ the set of all identities $u=v$ holding in $M$ such that $lev(u) < k$ and $lev(v) < k$, and by $\Sigma_1$ the set of all identities holding in $M$ such that $lev(u) < k$ and $lev(v) \geq k$. The system $\Sigma_o$ is finitely based by Lemma 1. Hence, to prove our lemma it remains to verify that the system $\Sigma_1$ is finitely based in $M \cup L$. Clearly, any word u with $lev(u) < k$ has less than k repeated letters. Chaingeing the numeration of letters we may assume that the left part of any identity of $\Sigma_1$ has as repeated letters only letters of the set $x_1, \ldots, x_{k-1}$. Furthermore, it is easy to calculate that the length of the word $dw(u)$ where $lev(u) < k$ is less than 3k-1. Since we can take the same new letter z for all such words to construct distribution words, we may assume that distribution words of left parts of identities of $\Sigma_1$ depend on the letters $x_1, \ldots, x_{k-1}, z$, only. However, the set of words of length 3k-1 depending on k given letters is finite. Furthermore, the dimension of the block vector $\underline{bl}(u)$ where $lev(u) < k$ is less than 2k. Consider now a relation on the set $\Sigma_1$: $u=v \quad u'=v'$ iff $dw(u) \equiv dw(u')$ and $\underline{bl}(u) \leq \underline{bl}(u')$ componentwise. The fact that there are only finitely many different distribution words and the dimensions of block vectors are restricted implies immediately that in every infinite sequence $\tau_1, \ldots, \tau_m, \ldots$ of identities of $\Sigma_1$ there exist elements $\tau_m$ and $\tau_n$ such that $m<n$ and $\tau_m \quad \tau_n$. It is known ([4], Lemma III.2.8) that this is equivalent to the fact that $<\Sigma_1, \quad >$ is a well-quasi-ordered set in G.Higman's sense [5]. It follows from properties of well-quasi-ordered sets that there are $\tau_1, \ldots, \tau_n \in \Sigma_1$ such that, for any $\tau \in \Sigma_1$, there exists $i \in \{1, \ldots, n\}$ such that $\tau_i \quad \tau$. These $\tau_1, \ldots, \tau_n$ form a basis for $\Sigma_1$. In fact, it is sufficient to verify that, if $u=v \quad u'=v'$, then $u=v$ and identites of the variety $M \cup L$ imply $u'=v'$. By Lemma 2 an endomorphism $\phi$ exists such that $u^{\phi} \equiv u'$. The identity $u'=v^{\phi}$ follows from $u=v$, therefore, the identity $v'=v^{\phi}$ holds in $M$. Since $lev(v') \geq k$ and $lev(v^{\phi}) \geq lev(v) \geq k$, it holds in $M \cup L$, too. Hence, the identities $u'=v'$ and $u'=v^{\phi}$ are mutually equivalent in $M \cup L$.

The lemma is proved.

## 2. Proof of Theorem 2

The following lemma will clarify the role of the semigroup $N^1$.

LEMMA 4. An identity u=v holds in $N^1$ iff the words u and v have the same repeated and the same non-repeated letters.

Proof. Necessity. $N^1$ contains, obviously, the two-element semilattice as subsemigroup. This implies that the same letters occur in the words u and v. Suppose now that a letter x is repeated in u but non-repeated in v. Replacing x by 2 and the other letters occuring in u and v by 1 we obtain 0=2. Contradiction.

Sufficiency follows easily from the observation that the identities $xy=yx$, $x^2=x^3$ hold in $N^1$.

An endomorphism $\phi\in End(F)$ is said to act flatly on a word $w\in F$ if $lev(w^\phi)=lev(w)$. It follows from Lemma 4 that, if an identity u=v holds in $N^1$, then tne same endomorphisms act flatly on u and on v.

For brevity, we shall use the symbols $y_1, y_2, y_3, y_4$ to denote letters or the empty word. For example, the expression $y_1xy_2xy_3$ where x is a "normal" letter can be understood if necessary as $x^2$ or xyx etc.

Fix a word w and construct three sets $\Psi_1(w)$, $\Psi_2(w)$, $\Psi_3(w)$ of endomorphisms which act on w non flatly:

$\Psi_1(w)$ is the set of all endomorphisms which transfer a repeated (in w) letter z in the word $z_1z_2$ where the letters $z_1$ and $z_2$ do not appear in w, and which act trivially on the rest of the letters;

$\Psi_2(w)$ is the set of all endomorphisms mapping a non-repeated (in w) letter x on the word $y_1xy_2xy_3$ and acting trivially on the rest of the letters;

$\Psi_3(w)$ is the set of all endomorphisms constructed as follows. We take two different letters x and z occuring in w and transfer x in $y_1xy_2$ and z in $y_3xy_4$ where $y_1, y_2$ (resp. $y_3, y_4$) can be non-empty only if x (resp. z) is a non-repeated letter in w, and we do not change the rest of the letters.

By constructing $\Psi_2(w)$ and $\Psi_3(w)$ we assume that, if $y_i$ is a letter it does not occur in w. Clearly, if $\psi\in\Psi(w)=\Psi_1(w)\cup\Psi_2(w)\cup \cup\Psi_3(w)$, then $\mathrm{lev}(w)<\mathrm{lev}(w^{\psi})\le 2\mathrm{lev}(w)+1$. Note that, if an identity $u=v$ holds in $N^1$, then $\Psi(u)=\Psi(v)$ by Lemma 4.

LEMMA 5. Let w be a word and $\xi$ an endomorphism acting non-flatly on w. Then there are endomorphisms $\psi\in\Psi(w)$ and $\zeta$ such that $w^{\xi}\equiv w^{\psi\zeta}$.

Proof. Consider three cases.

1) There is a repeated letter z in w such that the length of the word $z^{\xi}$ is greater than 1. In this case $z^{\xi}$ may be written as $z^{\xi}\equiv u_1u_2$ where $u_1$, $u_2$ are non-empty words. Choose an endomorphism $\psi\in\Psi_1(w)$ such that $z^{\psi}\equiv z_1z_2$, and define an endomorphism $\zeta$ by the rule: $z_i^{\zeta}\equiv u_i$, $x^{\xi}\equiv x^{\zeta}$ for all $x\neq z_1,z_2$. It is clear that $w^{\xi}\equiv w^{\psi\zeta}$.

2) There is a non-repeated letter x in w that $\mathrm{lev}(x^{\xi})\ge 1$. In this case $x^{\xi}\equiv u_1zu_2zu_3$ where $u_1,u_2,u_3$ are (possibly empty) words. There is an endomorphism $\psi\in\Psi_2(w)$ such that $x^{\psi}\equiv y_1xy_2xy_3$ where $y_i$ is empty iff $u_i$ is empty. Defining an endomorphism $\zeta$ by the rule $y_i^{\zeta}\equiv u_i$, $x^{\zeta}\equiv z$, $t^{\zeta}\equiv t^{\xi}$ for all $t\neq x,y_1,y_2,y_3$ we obtain that $w^{\xi}\equiv w^{\psi\zeta}$.

3) $\xi$ transfers each repeated letter in a letter and each non-repeated letter in a word of the level 0. It is clear that in this case there are two letters x and z in w such that the words $x^{\xi}$ and $z^{\xi}$ have a common letter t, i.e. $x^{\xi}\equiv u_1tu_2$ and $z^{\xi}\equiv u_3tu_4$ where $u_1,u_2,u_3,u_4$ are (possibly empty) words. An endomorphism $\psi\in\Psi_3(w)$ exists such that $x^{\psi}\equiv y_1xy_2$, $z^{\psi}\equiv y_3xy_4$, and, as above, $y_i$ is empty iff $u_i$ is empty. If an endomorphism $\zeta$ is given by letting $x^{\zeta}\equiv t$, $y_i^{\zeta}\equiv u_i$, $y^{\zeta}\equiv y^{\xi}$ for any letter $y\neq y_i$, x, then $w^{\xi}\equiv w^{\psi\zeta}$.

The lemma is proved.

To prove Theorem 2 note, first of all, that it is sufficient to verify that the variety $\mathcal{M}\cup\mathcal{L}_k$ will be finitely based for any k. In fact, if $\mathcal{L}\subseteq\mathcal{L}_k$, then $\mathcal{M}\cup\mathcal{L}_k=(\mathcal{M}\cup\mathcal{L})\cup\mathcal{L}_k$, and the variety $\mathcal{M}\cup\mathcal{L}$ is finitely based in $\mathcal{M}\cup\mathcal{L}_k$ by Lemma 3.

We shall use induction. Since $\mathcal{L}_0$ is trivial, $\mathcal{M}\cup\mathcal{L}_0=\mathcal{M}$, and the induction basis follows from the condition of the theorem. By the induction assumption the variety $\mathcal{M}\cup\mathcal{L}_{k-1}$ is finitely based. Let $\Sigma$

be its basis:

$$\Sigma=\Sigma_1\cup\Sigma_2=\{u_1=v_1,\dots,u_l=v_l\}\cup\{a_1=b_1,\dots,a_m=b_m;c_1=w_1,\dots,c_n=w_n\},$$

where $a_i,b_i,c_i$ are words of the level k-1, and $u_i,v_i,w_i$ have the level k. Construct two new systems of identities as follows. The system $\Sigma_3$ consists of all identities

$$xy_1u^{\phi}=xy_1v^{\phi},\ u^{\phi}y_1x=v^{\phi}y_1x,\ zy_1zy_2u^{\phi}=zy_1zy_2v^{\phi},\ zy_1u^{\phi}y_2z=zy_1v^{\phi}y_2z,$$
$$u^{\phi}y_1zy_2z=v^{\phi}y_1zy_2z,\ u^{\psi}=v^{\psi},$$

where u=v runs over $\Sigma_2$, $\phi$ runs over the set of all endomorphisms acting flatly on u, z does not appear in $u^{\phi}$, x runs over the set of all letters occuring in $u^{\phi}$, $y_1,y_2$ can be empty, but, if $y_i$ is non-empty, it does not occur in $u^{\phi}$, and, finally, $\psi$ runs over $\Psi(u)$.

The system $\Sigma_4$ consists of all identities of the kind

$$q_1w_e^{\phi_1}r_1=q_2w_f^{\phi_2}r_2$$

which hold in $MUL_{k-1}$, where $e,f\in\{1,\dots,n\}$, $\phi_1$ acts flatly on $w_e$, $\phi_2$ acts flatly on $w_f$, there are no common letters in the words $q_1r_1$ and $w_e^{\phi_1}$, $q_2r_2$ and $w_f^{\phi_2}$, and $lev(q_1r_1)=lev(q_2r_2)=0$.

By construction, the systems $\Sigma_3$ and $\Sigma_4$ are infinite. However, if s is the maximum of levels of the words $w_1,\dots,w_n$, then it is easy to calculate that the levels of both parts of every identity belonging to $\Sigma_3$ are less than 2s+2, and the levels of both parts of every identity belonging to $\Sigma_4$ are less than s+1. By Lemma 1, $\Sigma_3$ and $\Sigma_4$ are finitely based. Theorem 2 follows now from the following

<u>LEMMA 6</u>. The variety $MUL_k$ can be defined by the system of identities $\Sigma_1\cup\Sigma_3\cup\Sigma_4$.

<u>Proof</u>. If an identity belongs to $\Sigma_1\cup\Sigma_3\cup\Sigma_4$, it holds in $MUL_{k-1}$, and, by construction, the levels of both its parts are no less than k. Hence, it holds also in $MUL_k$. It remains to verify that any identity u=v holding in $MUL_k$ is a consequence of the system $\Sigma_1\cup\Sigma_3\cup\Sigma_4$. If u=v holds in $MUL_k$, it holds also in $MUL_{k-1}$, and, hence, it is a consequence of the system $\Sigma$. It is well-known that in this case there exists a "proof" of u=v modulo $\Sigma$, i.e. a sequence of words $d_0,\dots,d_{p+1}$ such that $u\equiv d_0$, $v\equiv d_{p+1}$, and for any $i\in\{0,\dots,p\}$, $d_i\equiv qg^{\phi}r$, $d_{i+1}\equiv qh^{\phi}r$ for some words q,r, some endomorphism $\phi$ and some identity g=h belonging to $\Sigma$. Note that $q,r,\phi$ and g=h depend

on i! First of all, we shall prove that, if $lev(d_i)\geq k$, $lev(d_{i+1})\geq k$, then the identity $d_i=d_{i+1}$ is a consequence of the system $\Sigma_1\cup\Sigma_3$. It is clear in the case that g=h belongs to $\Sigma_1$. In the opposite case there are two possibilities: $\phi$ acts on g flatly or $\phi$ acts on g nonflatly. In the later case Lemma 5 may be applied, and there exists an endomorphism $\psi\in\Psi(g)=\Psi(h)$ such that $g^\phi\equiv g^{\psi\zeta}$ and $h^\phi\equiv h^{\psi\zeta}$ for some endomorphism $\zeta$. The fact that the same endomorphism $\zeta$ may be taken for both words g and h follows from Lemma 4 and from the proof of Lemma 5. We see that the identity $d_i=d_{i+1}$ is a consequence of the identity $g^\psi=h^\psi$ which belongs to $\Sigma_3$. Let now $\phi$ act on g and on h flatly. Since g=h lies in $\Sigma_2$, the level of at least one of words $g^\phi$ and $h^\phi$ is less than k. However, the levels of words $d_i$ and $d_{i+1}$ are no less than k. The level may increase only if either the words qr and $g^\phi$ have a common letter or the level of the word qr is greater than 0. It is easy to see that, in the first case, $d_i=d_{i+1}$ follows from one of the identities

$$xy_1g^\phi=xy_1h^\phi,\ g^\phi y_1x=h^\phi y_1x \qquad (*)$$

where x appears in $g^\phi$, and, in the second case, this identity follows from one of identites

$$zy_1zy_2g^\phi=zy_1zy_2h^\phi,\ zy_1g^\phi y_2z=zy_1h^\phi y_2z,\ g^\phi y_1zy_2z=h^\phi y_1zy_2z \qquad (**)$$

where z does not appear in $g^\phi$. Since the identities (*) and (**) lie in $\Sigma_3$, our statement is proved. Therefore, we may assume that there are indices i and j such that $lev(d_o)=lev(u)\geq k$, $lev(d_1)\geq k,\ldots$ $\ldots$, $lev(d_i)\geq k$, $lev(d_{i+1})=k-1$ and $lev(d_{p+1})=lev(v)\geq k$, $lev(d_p)\geq k,\ldots$ $\ldots$, $lev(d_j)\geq k$, $lev(d_{j-1})=k-1$. Let us consider the identity $d_i=d_{i+1}$. It is clear that this identity may be only of the form $q_1w_e^{\phi_1}r_1=q_1c_e^{\phi_1}r_1$, where $\phi_1$ acts on $c_e$ and on $w_e$ flatly, $lev(q_1r_1)=$ $=0$ and the words $q_1r_1$ and $w_e^{\phi_1}$ have no common letter, since, if at least one of these conditions is not satisfied, the level of $d_{i+1}$ cannot be less than k. Analogously, the identity $d_j=d_{j-1}$ may be only of the form $q_2w_f^{\phi_2}r_2=g_2c_f^{\phi_2}r_2$ with the same restrictions on $q_2r_2$ and $\phi_2$. Since the identity $q_1w_e^{\phi_1}r_1\equiv d_i=d_j\equiv q_2w_f^{\phi_2}r_2$ holds in $MUL_{k-1}$, it belongs to the system $\Sigma_4$. Thus, since the identities $u\equiv d_o=d_1=\ldots=d_i$ and $d_j=\ldots=d_p=d_{p+1}\equiv v$ follow from $\Sigma_1\cup\Sigma_3$, and the identity $d_i=d_j$ belongs to $\Sigma_4$, the identity u=v follows from $\Sigma_1\cup\Sigma_3\cup\Sigma_4$. The lemma is proved.

## 3. A corollary

Let us recall that an element x of a partially ordered set P covers an element y∈P if x>y, and x≥z>y implies z=x for any z∈P. We shall say that the distance between two elements x and z of P is finite if either x=z or x>z, and there is a sequence $y_o,\ldots,y_{p+1}$ of elements of P such that $x=y_o$, $z=y_{p+1}$ and $y_i$ covers $y_{i+1}$ for all i=0,...,p. Finally, we shall say that the distance between two elements x and z is ω if x>z, the distance between x and z is not finite, but for any y such that x>y≥z the distance between y and z is finite. A.N.Trahtman [6] has proved that in the lattice of varieties of semigpoups each proper subvariety has a covering variety. I prove here that this lattice satisfies also the following stronger condition:

COROLLARY. Let $M$ be a proper semigroup variety. There exists a variety $K$ such that the distance between $K$ and $M$ is ω.

Proof. The join of all varieties $L_k$, k=0,1,2... coincides with the variety of all semigroups. Therefore, there is k such that $L_k \not\subseteq M$ and, hence, $M \cup L_k \supset M$. We show that the distance between these varieties cannot be finite. In the opposite case a variety $X$ exists such that $M \subseteq X \subset M \cup L_k$ and $M \cup L_k$ covers $X$. All identities of $X$ are identities of $M$, but any identity of $M$ such that the levels of both of its parts are no less than k holds in $M \cup L_k$. Therefore, an identity u=v such that lev(u) ⊲ k holds in $X$. It is obvious that $L_k$ is the join of its nilpotent subvarieties, therefore, a nilpotent subvariety $N \subset L_k$ exists such that $X \not\supseteq N$. We obtain that $M \cup L_k \supseteq N \cup X \supset X$. Since $M \cup L_k$ covers $X$, it follows that $N \cup X = M \cup L_k$. There is a number n having the property that any identity such that the lengths of both of its parts are no less than n holds in $N$. Now consider the identity $x_1 \ldots x_n u = x_1 \ldots x_n v$ where the letters $x_1,\ldots,x_n$ do not occur in u. It holds in $X$ and in $N$, hence, it holds also in $N \cup X = M \cup L_k$, but $\mathrm{lev}(x_1 \ldots x_n u) = \mathrm{lev}(u)$ ⊲ k. Contradiction.

Thus, the set A of all varieties of the interval $[M, M \cup L_k]$ such that the distance between these varieties and $M$ is not finite is non-empty. However, for each variety $A$ in this interval, $A \cup L_k = M \cup L_k$, and $A$ is finitely based in $M \cup L_k$ by Lemma3. It follows immediately that the interval $[M, M \cup L_k]$ satisfies the minimum condition, and, in particular, the set A has a minimum element K which was required.

Corollary is proved.

This work was written when I was a visitor at the Vienna Technical University. I am indebted to Prof. H.Kaiser for the invitation and to all Austrian colleagues for the good organization of my visit.

REFERENCES

[1] M.V.Volkov, On the join of varieties. Simon Stevin (to appear).

[2] G.Pollák and M.V.Volkov, On almost simple semigroup identities. Algebraic theory of semigroups, Coll. Math. Soc. János Bolyai (to appear).

[3] M.V.Volkov, Finite basis theorem for systems of semigroup identities. Semigroup Forum (to appear).

[4] P.M.Cohn, Universal algebra. New York, 1965.

[5] G.Higman, Ordering by divisibility in abstract algebras. Proc. London Math. Soc. (3) 2(1952), 326-336.

[6] A.N.Trahtman, On covering elements in lattices of varieties. Mat.Zametki 15(1974), 307-312 (Russian).

Ural State University
Lenin-avenue 51
620083 Sverdlovsk
USSR