IDENTITIES OF SEMIGROUPS

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§ 0. INTRODUCTION

a) General remarks. The theory of varieties of algebraic systems, beginning with the first classical works of Birkhoff [116], has in the last decade become one of the central directions of contemporary algebra. It has a wealth of problems and has developed dynamically and fruitfully. Research on varieties of semigroups holds a prominent place in this direction. The first papers in which the statement of problems and facts related to varities of semigroups appeared in the 1950's, in particular, the papers of Green and Rees [128], Mal'tsev [54], McLean [150], Kalicki and Scott [136], Kimura [141,142], and Yamada and Kimura [189]. In the 1960's the papers of Adyan [2], ~~Birkhoff~~ Biryukov [12-14], Naik [156], and Neumann and Taylor [157] appeared, in which the principal theme was semigroup identities. By the end of this decade approximately thirty papers were published which were particularly devoted to questions in the theory of semigroup varieties. We can say that the second half of the 1960's marked the beginning of a systematic investigation of varieties of semigroups which up to the present has not decreased in intensity.

The interest in the theory of varieties of semigroups is two sided: on one side, it can be considered as a natural and important part of the "universal" theory of varieties of algebraic systems and as one of the areas where various general statements of problems are put to the test or where the bounds of applicability of various statements are established. On the other side, the language of varieties is a powerful means for the study and classification of semigroups, which defines the important role of the theory of varieties of semigroups as an interesting and useful branch of the algebraic theory of semigroups on the whole.* As a rule, the actual achievements in the theory of varieties of semigroups show one (and sometimes both) of these facets. Additional interest arises from the comparison of those facts or separate questions on semigroup varieties which are related to general problems, from situations for related types of algebraic systems, above all for groups and (associative) rings. As experience shows, the spectrum of the possible situations here is maximally diverse: identically sounding answers in some areas alternate with opposite answers in others; problems easily solved for some types of systems turn out to be very complicated for others; problems solved long ago for some types of systems continue to remain open for others. We mention that each variety of periodic groups is also a variety in the semigroup signature; hence it follows that some problems on varieties of semigroups can be included in the case of (periodic) groups. This circumstance is one of the connections between the theory of varieties of semigroups and the theory of varieties of groups; other connections will be mentioned later. But naturally, the principal features of the subject considered by us are determined by semigroup-theoretical characteristics, which appear both in the syntactic singularities of semigroup language and in the presence of an entire series of types of semigroups which were distinguished in the course of the preceding development and which play an essential role in the general theory. Among such types are those distinguished by abstract conditions (inverse, Clifford, completely simple, idempotent, finitely-defined semigroups, etc.) as well as those which are of interest by virtue of their concrete nature (semigroups of transformations, semigroups of matrices).

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*It is clear that both sides can be also related to the theory of varieties of algebraic systems of any other concrete type (groups, rings, lattices, and so on). Regarding the second of these, see the remark of Neumann in the preface to the book [75] on the uses of the theory of groups and the induced classification of groups with respect to properties of varieties. See also the statement of Mal'tsev in the first chapter "Varieties" of the book [56] on the wealth of the language of identities which allows many subtle properties of systems and their classes to be expressed.

We particularly note that for two of the above-named types of semigroups there is a particularly natural and characteristic approach from the point of view of the theory of varieties. We are referring to inverse and Clifford semigroups. In each such semigroup it is possible to consider the unary operation $x \to x^{-1}$, which for inverse semigroups takes an element to its inverse, and for Clifford semigroups takes an element to the reciprocal element in the corresponding maximal subgroup. The class J of all inverse semigroups and the class K of all Clifford semigroups will be varieties in the signature consisting of the binary operation of multiplication and the indicated unary operation. Each of the varieties J and K contain the variety $\mathfrak{G}$ of all groups as a subvariety. An analysis of the varieties of inverse and Clifford semigroups really began in the 1970's. This was, naturally, very closely related with the analyses of ordinary varieties of semigroups, not only from the point of view of the formulation of problems, but also partly from the point of view of techniques. However, the specifics of each of these two cases stand out. We note, though, that since each variety of periodic Clifford semigroups is also an ordinary semigroup variety (the identity $x = x^{n+1}$ is satisfied in it for some n, whence $x^{-1} = x^{n-1}$), in the periodic case the theory of varieties of Clifford semigroups is simply a section of the theory of ordinary semigroup varieties. But for inverse semigroups the imposition of a periodicity condition does not suppress the characteristics of the extended signature. If $\mathfrak{X}$ is an arbitrary variety of inverse or Clifford semigroups, then its properties to a specific degree can depend on properties of the group variety $\mathfrak{X} \cap \mathfrak{G}$ (in the case of Clifford semigroups each semigroup in $\mathfrak{X}$ is, as a semilattice of rectangular bundles, made up of groups lying in $\mathfrak{X} \cap \mathfrak{G}$). This determines another natural relation with varieties of groups.

We note that varieties of inverse semigroups can be considered also in the somewhat more general context of involuted semigroups, i.e., semigroups with an additional unary operation $x \to x^*$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. In recent years varieties of involuted semigroups have been the subject of much research.

At present there have been more than 400 papers and books published which were devoted to various aspects of the theory of varieties of semigroups. A wealth of very diverse material has been accumulated here. Naturally, the need to review and systematize this material arises. In particular, such is the plan of the first author, who began to compile a survey several years ago together with collaborators. It was planned that in the survey four large parts be distinguished: 1) identities of semigroups, 2) semigroup varieties and the structure of semigroups, 3) free semigroup varieties, 4) lattices of semigroup varieties (in the original outline, presented in the survey [102], part 2 had a different name). In the course of the work, however, it became clear that the material reviewed was impossible to place in the reasonable limits of one paper, and it was decided that a separate paper would be dedicated to each part of the survey. The reader's attention is directed to the first paper in the planned cycle.

In two surveys on varieties of semigroups previously published (Evans [123], Aizenshtat and Boguta [7]), the principal theme, which is evident in the titles, was lattices of varieties. Separate questions touching on semigroups of identities per se, were considered only in passing. Some information on identities of semigroups is contained, naturally, in the survey of Taylor [181] (see also his abridged version, Proposition 4 in [127]), but this survey has a clearly expressed universal algebraic direction, and semigroups appear only as one of the numerous types of algebraic systems in which some or the other general problems in equational logic are illustrated. If we take into account that the majority of all essential results on semigroup identities was obtained in recent years,* then it becomes especially urgent that this material be systematized. Naturally, we note that among the above indicated four directions there are diverse relations. Therefore, the thematic bounds between them are relative to some degree. (For example, it is clear that identities are present, in any case, in all considerations of varieties.) Nevertheless, for each of these directions we draw a self-evident circle of questions forming a basis of the corresponding problems. Problems relating to the direction "Identities" are those which we can say have a syntactic character, first of all grouped around the finite basis property (test for its presence or absence for various varieties or systems of identities, the hereditary finite basis property, etc.), as well as questions of the description of bases of identities or (what is usually easier) of systems of all identities of semigroups, are of principal interest in some or the other relation, or, for example, simply questions on the satisfiability of nontrivial identities for semigroups under consideration. At the same time, statements of problems which are related to the influence of identities on the structure and properties of semigroups are left to the second direction.

Working over the papers, we strived for reasonable completeness, desiring both to demonstrate the basic achievements in the analysis of identities of semigroups as well as simultaneously to put the problems in order, paying attention also to the classification of problem statements, and to a consideration of the possible perspectives for further development (including concise formulations for several unsolved problems and open questions). In addition, we found it appropriate to interlace information related to the corresponding questions for varieties of inverse and Clifford semigroups with the text devoted to ordinary semigroup varieties. These places are marked by small type**. Concerning information on varieties of involuted semigroups, restrictions on the size of this survey prevent this here. At least one of the primary projects, to place the cases of the proof in order (either for assertions especially important ideologically or demonstrating some important method) has also not yet been done. In the formative stage of this paper we tried to include material related to the algorithmic aspects of this theme; with the exception of several open questions which seemed appropriate in the corresponding sections. We note, however, that this topic perhaps deserves a separate survey. Concerning the references, they contain practically all the works on material surveyed in this paper, plus other papers related to those referred to in the text of this paper. In addition, as a rule announcements are not included in those cases for which results are presented in later publications. We note that this paper also contains a series of entirely fresh results which have not been published previously.

The table of contents gives a clear overview of the material covered in this paper. We note that several sections are thematically divided into subsections which for brevity have not been indicated in the table of contents. Theorems, propositions, as well as examples, problems and questions are each numbered sequentially in each section of the text by pairs of indices.

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*This is, however, not as true for other directions in the theory of varieties of semigroups.

**We note that in the recent book of Petrich, *Inverse Semigroups* [164] in fact more than two chapters are devoted to varieties and questions related to the above directions 2-4 are treated quite completely. The theme "Identities," however, is not adequately elucidated. Other information on varieties of inverse semigroups (reflecting the early period) was presented in the mini-survey of Reilly [178]. In Petrich [163] basic facts on lattices of varieties of Clifford semigroups, obtained up to the beginning of the 1980's, are collected.

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b) **Preliminary information, definitions and fundamental notation.** The more or less standard information from general algebra is sufficient for understanding of this paper in many cases. As a rule, the needed definitions are presented. The exceptions are certain information from the theory of varieties of algebraic systems and from the theory of semigroups, which are assumed familiar to the reader. (Examples of the first exception are the notions of a free system, completely invariant congruence, axiomatic rank, and the meaning of lattice operations over varieties; examples of the second exception are the notions of inverse, Clifford and completely simple semigroups, bundles of semigroups, Rees semigroups of matrix type and so on.) References for information are traditional, in particular [56,42,37,50].* We also recommend the corresponding paper from the *Mathematical Encyclopedia* [61].

We shall chiefly use standard terminology. We agree on one term for the identity u = v in which the sets of letters occurring in the words u and v coincide. In the literature such an identity is called **homotypic**, and also regular or normal. We will use only the first term, because 1) historically it probably was the first (see the pioneering papers [141,142]; 2) it is suitably mnemonic; 3) in contrast to the other two terms, which have many different meanings in algebraic terminology, it does not have homonyms. An identity which is not homotypic is called **heterotypic**. An identity u = v is called **balanced** if each letter in the words u and v occurs the same number of times. It is easy to see that a variety defined by an arbitrary system of balanced equations will be **supercommutative**, i.e., will contain the variety $\mathfrak{A}$ of all commutative semigroups, and conversely, each identity of an arbitrary supercommutative variety will be balanced. Suppose w is a word, x is a letter not in the list of w. Obviously, the pair of identities wx = xw = w is satisfied in some semigroup S if and only if S contains zero 0, and each value of the word w in S is equal to 0. We will therefore write the indicated pair of identities as one symbolic identity w = 0. Identities of the form w = 0, as well as varieties defined by identities of this form, will be called **0-reduced**. If θ is some abstract property of semigroups, we call any variety consisting of θ-semigroups a θ-variety. The expressions "periodic variety," "nilpotent variety," "locally finite varity," and so on will be understood in this sense. In this connection we note that in contrast the term "Clifford variety" means a semigroup (i.e., of signature $(\cdot)$) variety consisting of Clifford semigroups, but the expression "variety of Clifford semigroups" means a variety in the extended signature $(\cdot, {}^{-1})$. An analogous remark could be made for inverse semigroups, however in fact, we do not have the term "inverse variety," since it is easy to understand that if a semigroup variety consists of inverse semigroups then it is a Clifford variety of a special form: it consists of semilattices of groups.

If S is a semigroup, C is the class of semigroups, Σ is a system of identities, then var S, var C and var Σ will respectively denote the variety generated by S, C or defined by the identities in Σ. If $\mathfrak{B}$ is a variety then $\mathrm{eq}\,\mathfrak{B}$ will denote the set of all identities satisfied in semigroups in $\mathfrak{B}$. If $\mathfrak{B} = \mathrm{var}\,S$, $\mathfrak{B} = \mathrm{var}\,C$, $\mathfrak{B} = \mathrm{var}\,\Sigma$, instead of $\mathrm{eq}\,\mathfrak{B}$ we simply write $\mathrm{eq}\,S$, $\mathrm{eq}\,C$, $\mathrm{eq}\,\Sigma$. If the system of identities Σ_0 is such that $\mathrm{eq}\,\Sigma_0 = \mathrm{eq}\,S$, then it is called a basis of identities

*We note that in the modified American edition of [145] the latter has a special chapter devoted to identities of semigroups.

of the semigroup S. Analogously, a basis of identities of the class of semigroups and systems of identities are defined. The mechanism for obtaining identities consisting of eq Σ from the identities Σ is well-known (see for example [45], Ch. 3, §7.2). In this connection we note that in a series of papers on varieties of semigroups (including the survey [7]) the paper [13] was a primary reference. Meanwhile this mechanism was described in the fundamental work of Birkhoff [166] (see Definition 5 and Theorem 9).

A semigroup S, the class of semigroups C (in particular, a variety), and the system of identities Σ are called finitely based if they have a finite basis of identities; not finitely based semigroups, and so on, will sometimes be called infinitely based.

We adopt the following abbreviations for frequently encountered phrases: f.b. - finitely based (variety), finitely based (semigroup), f.d. - finitely defined (semigroup), l.f. - locally finite (variety), s.i. - semigroup identity, r.s.m.t. - Rees semigroup of matrix type. For r.s.m.t. over a semigroup S with sandwich matrix P we will use the canonical notation M(S; I, Λ, P). For r.s.m.t. with a zero over a group with adjointed zero G^0 and sandwich matrix P we will use the notation $M^0(G; I, \Lambda, P)$. The graphic congruence of words will be denoted by ≡, the join of the varieties $\mathfrak{B}$ and $\mathfrak{W}$ in the lattice of varieties, by the symbol $\mathfrak{B} \vee \mathfrak{W}$.

CHAPTER I. THE FINITE BASIS PROPERTY FOR SYSTEMS OF IDENTITIES

1. SURVEY OF EXAMPLES OF INFINITELY BASED SYSTEMS OF IDENTITIES

The first examples of systems of s.i. which are not f.b. were presented by Biryukov in [15] and Austin in [114]. These examples in Table 1.1 below are denoted by A8 and A1 respectively. Up to the present time there have been more than 40 such examples published, and their number continues to increase. It is clear that the construction of new not f.b. systems has long ceased to be an end in itself, and these systems are now used as aids in the solution of various problems in the theory of varieties of semigroups. The most intensive "consumers" of not f.b. systems are papers in which the form of identities giving hereditarily f.b. varieties is defined (to this end systems A4, A5, B3-B7, B9, B10 and B16 have been used - for details see Chapter III), and papers in which continuality of some or other fragments of the lattice of varieties of semigroups is proved (here the systems A1, A10-A13, B1, B13, C1, E2-E4, F2 are applied). In spite of the large number of known not f.b. systems, it is not difficult to note that a very restricted number of methods is used for their construction. It is possible to distinguish five methods:

a) the use of "twisted" subwords of the form $x_1x_2\ldots x_nyx_n\ldots x_2x_1$;

b) the repetition of words of one type (usually of type x^2 or xyx) over different variables;

c) the use of subwords of the form yx^ny with increasing exponent n;

d) the use of subwords of the form x^py^p where p runs through the set of prime numbers;

e) the use of aperiodic sequences.

In correspondence with the method of construction we will classify not f.b. systems of identities as of the form A, B, C, D, E. They are enumerated in Tables 1.1-1.5. Inside the tables the systems are subdivided into those con-

Table 1.1

Systems of Type A

Type	Symbol	Form	Reference
Balanced	A1	$(a_n y)^2 = y a_n y a_n^*,\ n = 2, 3, \dots$	[114]
	A2	$y a_n y a_n^* = y a_n^* y a_n,\ n = 1, 2, \dots$	[159]
	A3	$y^2 a_n a_n^* = a_n a_n^* y^2,\ n = 1, 2 \dots$	[131]
	A4	$yza_n za_n^* zyz = zyza_n za_n^* zy,\ n = 2, 3, \dots$	[52]
	A5	$a_n y a_n^* y = y a_n y a_n^*,\ n = 1, 2, \dots$	[170]
	A6	$a_n y_1 \dots y_n a_n^* y_n \dots y_1 = a_n^* y_1 \dots y_n a_n y_n \dots y_1,\ n = 2, 3, \dots$	[34]
	A7	$y^2 a_n a_n^* a_n y = a_n a_n^* y^2 a_n y,\ n = 1, 2, \dots$	[112]
Homotypic not balanced	A8	$y a_n z y a_n^* y z = y a_n z a_n^* y z,\ n = 1, 2, \dots$	[15]
Heterotypic	A9	$x^3 = xyx^2 = x^2yx = xyxzx = 0,\ a_n a_n^* = a_n^* a_n,\ n = 4, 5, \dots$	[134]
	A10	$a_n y z^2 a_n y = y a_n y a_n^*,\ n = 2, 3, \dots$	[6]
	A11 (m), $m > 1$	$y^{m-1} z a_n^* y t a_n z a_n^* t y = v_n,\ n = 1, 2, \dots,$ where v_n is an arbitrary word containing the subword w^m	[46]
	A12	$yza_n za_n^* t = 0,\ n = 1, 2, \dots$	[59]
	A13	$y a_n y a_n^* = 0,\ n = 2, 3, \dots$	[60]

Here a_n denotes the word $x_1 x_2 \dots x_n$, a_n^* the word $x_n \dots x_2 x_1$.

Table 1.2

Systems of Type B

Type	Symbol	Form	Reference
Balanced	B1	$z^2 y^4 b_n y^4 = y^4 b_n y^4 z^2,\ n = 1, 2, \dots$	[91[
	B2	$z^3 y^3 b_{2n-1} t^3 = y^3 z^3 b_{2n-1} t^3,\ n = 1, 2, \dots$	[181]
	B3	$y b_n y = y b_n^* y,\ n = 2, 3, \dots$	[170]
	B4	$y c_n y = y c_n^* y,\ n = 2, 3, \dots$	[170]
	B5	$yzy b_{2n} = yzy b_{2n-2} x_{2n}^2 x_{2n-1}^2,\ n = 1, 2, \dots$	[170]
	B6	$y^2 c_{2n} = y^2 c_{2n-2} x_{4n-1} x_{4n} x_{4n-1} x_{4n-3} x_{4n-2} x_{4n-3},\ n = 1, 2, \dots$	[170]
	B7	$y_1 z_1 z_2 y_1 c_n y_1 z_3 z_4 y_2 = y_2 z_1 z_2 y_1 c_n^* y_2 z_3 z_4 y_2,\ n = 2, 3, \dots$	[170]
	B8	$y^2 b_n y = y b_n y^2,\ n = 1, 2, \dots$	[171]
	B9	$y^3 b_{2n} = y^3 b_{2n-2} x_{2n}^2 x_{2n-1}^2,\ n = 1, 2, \dots$	[172]
	B10	$tyzt b_{2n} = tyzt b_{2n-2} x_{2n}^2 x_{2n-1}^2,\ n = 1, 2, \dots$	[173]
	B11	$x_1 x_{n+1} \cdot x_2 x_1 x_3 x_2 \cdot x_4 x_3 x_5 x_4 \cdot \dots \cdot x_n x_{n-1} x_{n+1} x_n =$ $= x_{n+1} x_1 \cdot x_2 x_1 x_3 x_2 \cdot x_4 x_3 x_5 x_4 \cdot \dots \cdot x_n x_{n-1} x_{n+1} x_n,\ n = 1, 2, \dots$	[112]
Homotypic unbalanced	B12	$yx^4y = yx^5y,\ yzy b_n z^4 y = y^2 z b_n z^4 y,\ n = 1, 2, \dots$	[92]
	B13	$ztzy_1 \dots y_n z b_n z = zt^2 z y_1 \dots y_n z b_n z,\ n = 1, 2, \dots$	[93]
	B14	$x^m = x^{m+d},\ yx_1^l x_2^l \dots x_n^l y = yx_n^l \dots x_2^l x_1^l y,$ $m < l < m + d,\ m > 1,\ d$ divides $l,\ n = 2, 3, \dots$	[171]
	B15 (p,) p — odd prime	$sxt = sx^{p+1}t,\ x^p y^p = y^p x^p,\ x^2 = x^{p+2},$ $sx^{p-1}y^{p-1}xyzt = szx^{p-1}y^{p-1}xyt,$ $sx_1^{p-1}y_1^{p-1}x_1 y_1 \dots x_n^{p-1} y_n^{p-1} x_n y_n s^{p+1} =$ $= sx_1^{p-1}y_1^{p-1}x_1 y_1 \dots x_n^{p-1} y_n^{p-1} x_n y_n s,\ n = 1, 2, \dots$	[79]
	B16	$y^2 z^2 b_n y = y^2 z^2 b_n z,\ n = 1, 2, \dots$	[172]
	B17	$x^3 = x^4,\ (y^2 b_n)^2 y = y^2 b_n y,\ n = 1, 2, \dots$	[66]

Here b_n denotes the word $x_1^2 x_2^2 \dots x_n^2$, b_n^* the word $x_n^2 \dots x_2^2 x_1^2$, c_n the word $x_1 x_2 x_1 x_3 x_4 x_3 \dots x_{2n-1} x_{2n} x_{2n-1}$, c_n^* the word $x_{2n-1} x_{2n} x_{2n-1} \dots x_3 x_4 x_3 x_1 x_2 x_1$.

Table 1.3

Systems of Type C

Type	Symbol	Form	Reference
Balanced	B1	$xyzt = xzyt,\ yx^ny = xyx^{n-2}yx,\ n = 2, 3, \dots$	[159]
	B2	$xyx^ny = yx^nyx,\ n = 2, 3, \dots$	[51]
	B3	$xyzt = xzyt,\ x^2y^2 = y^2x^2,\ x^2zy^2 = y^2zx^2,\ yx^py = y^2x^p$, p runs through the set of all primes	[185]
	B4	$x_1x_2^2x_1x_2x_1^2x_2^nx_1 = x_1x_2^2x_1^2x_2x_1x_2^nx_1,\ n = 2, 3, \dots$	[112]

Table 1.4

Systems of Type D

Type	Symbol	Form	Reference
Balanced	D1	$(x^py^p) = (y^px^p)^2$, p runs through the set of all primes	[131]

Table 1.5

Systems of Type E

Type	Symbol	Form	Reference
Homotypic unbalanced	E1	$xy^5x^2u_nxy^5x^2 = xy^5x^2u_{n+1}xy^5x^2,\ n = 1, 2, \dots$ Here the word u_n is obtained from the initial interval of length n of the Arshon sequence [9] constructed from the digits 1, 2, 3 replacing i by $xy^{i+1}x^2$.	[71]
	E2 (σ), σ arb. eq. on N	$x^2 = x^3,\ yz^2p_nx^2y = yz^2p_mx^2y,\ (m, n) \in \sigma$ $(p_n)_{n \in N}$ is an arbitrary sequence of words of x, y, z such that $\lvert p_n \rvert \neq \lvert p_m \rvert$ for $n \neq m$ and zp_nx does not contain subwords of form u^2.	[119]
	E3	$y^4v_{2n-1}x^4 = y^4v_{2n}x^3 = y^4v_{2n+1}x^4,\ n = 1, 2, \dots$ v_n is n-th word of Morse-Hedlund seq. [155], constructed of letters x, y.	[90]
Heterotypical	E4 (σ), σ arb. eq. on N	$x^2 = 0,\ xzyxzxyzxzyxyzyxzxyxzyw_nyxzx =$ $= xzyxzxyzxzyxyzyxzxyxzyw_myxzx,\ (m, n) \in \sigma$. If v_n is as in E3 and $v_n \equiv a_1a_2 \dots a_k$, where $a_1, a_2, \dots, a_k \in \{x, y\}$, define $b_1 \equiv a_1$, $b_{i+1} \equiv a_{i+1}$, if $a_{i+1} \not\equiv a_i$ and $b_{i+1} \equiv z$, if $a_{i+1} \equiv a_i$. Then by definition $w_n \equiv b_1b_2 \dots b_k$.	[132]

Table 1.6

Not f.b. Identities of Inverse Semigroups

Symbol	Form	Reference
E1 (M), M — infinite subset in N	$(x_1x_2 \dots x_nx_1^{-1}x_2^{-1} \dots x_n^{-1})^2 = x_1x_2 \dots x_nx_1^{-1}x_2^{-1} \dots x_n^{-1}$, n runs through M	[31]
	$x^2 = x^3,\ r_n^2 = r_n,\ n = 1, 2, \dots$ If $p = p_{n+2}$ is the (n+2)-th prime, $\overline{m}$ is the remainder of m mod p, $\rho_n = (\overline{0}, \overline{1}, \dots, \overline{p-1}, \overline{0}, \overline{2}, \dots, \overline{2(p-1)}, \dots, \overline{0}, \overline{k}, \dots, \overline{k(p-1)}, \dots, \overline{0}, \overline{p-1}, \dots, \overline{(p-1)(p-1)})$, then by definition r_n is obtained from ρ_n by replacing $\overline{i}$ by x_{i+1}, $i = 0, 1, \dots, p-1$.	[33]

sisting of balanced identities, homotypic identities not containing balanced identities, homotypic identities not containingbalanced identities, and finally heterotypic identities.

Not f.b. systems of identities of signature $\langle\cdot, ^{-1}\rangle$, defining nongroup varieties of inverse semigroups are collected in Table 1.6.

§2. TEST FOR THE FINITE BASIS PROPERTY

Suppose Σ is an infinite system of s.i.. How can it be determined whether it is f.b.? There are many tests which answer this question in a series of special cases. The idea for a great number of these tests consists in the following. In Σ there is a finite subsystem Σ_0 with the property that any system Σ, containing Σ_0 is finitely based. The system Σ_0 (and the variety defined by it) is called hereditarily f.b.. Results dealing with hereditarily f.b. systems and varieties are enumerated and analyzed in Chapter III. We now assume that the system Σ is such that each of its finite subsystems can be included in some non f.b. system. If in the above-mentioned situation the finite basis property is induced by reasons of a "local" nature, then it must be ensured by some "global" properties of all identities of the system Σ on the whole. We present examples of such properties.

We call the level of the word w the difference between its length and the number of different letters appearing in w. The level of the identity u = v is the maximum of the levels of the words u and v.

Theorem 2.1 (Volkov [187]). Suppose the system of s.i. Σ is such that the levels of all identities appearing in it are uniformly bounded. Then Σ is finitely based.

We note a very important special case of Theorem 2.1. An identity of the form $x_1x_2\ldots x_n = x_{1\tau}x_{2\tau}\ldots x_{n\tau}$, where τ is a permutation of the set {1, 2, ..., n}, is called a permutation identity. Clearly, the level of a permutation identity is equal to zero. We therefore obtain

Corollary (Pollak [167]). Each system of permutation identities is finitely based.

The following test for the finite basis property appears here for the first time. We shall say that a letter x is repeated in the word w if it appears no less than two times in the list of w.

Theorem 2.2 (Volkov). Suppose the system of s.i. $\Sigma = \{u_i = v_i\}_{i\in I}$ is such that in each of the words u_i, v_i no more than one letter is repeated. Then Σ is finitely based.

An examination of systems of Type C shows that Theorem 2.2 cannot be extended to systems consisting of identities in which not more than two letters are repeated.

§3. TESTS FOR THE INFINITE BASIS PROPERTY

The proof of the fact that some or the other explicitly described infinite system of s.i. Σ does not have a finite basis is as a rule not very difficult. In the majority of cases it is carried out syntactically by means of the analysis of the hypothetical derivation of an arbitrary identity in Σ from a fixed finite subsystem, more succinctly, semantically by constructing examples of semigroups satisfying any previously defined finite set of identities in Σ, but not lying in var Σ. The situation becomes immeasurably more complicated if the system Σ is not explicitly given. A typical and very important example of such non-explicitly given systems is the system eq S of all identities of some semigroup S. The first test for the infinite basis property applicable to this case was found in the key work of Perkins [159], in which the most important applications of the test were indicated (see §7 below). In order to formulate this test, we give two definitions. A system of s.i. Σ is called closed with respect to cancellation if any identity in Σ is homotypic and the cancellation of all occurrences of some letter in an arbitrary identity in eq Σ leads to either the "empty" identity or an identity in eq Σ. The word w is called an isotherm with respect to Σ if eq Σ does not contain nontrivial identitites, one of the parts of which is w.

Theorem 3.1 (Perkins [159]). Suppose the system of s.i. Σ is such that:

1) Σ is closed with respect to cancellations;

2) the words xyzyx and xzyxy are isotherms with respect to Σ;

3) the system of identities A2 is a consequence of the system Σ;

4) the identities $x^2y=yx^2$, $(xy)^2=xy^2x$ are not consequences of Σ.

Then Σ is not finitely based.

In the course of 15 years Theorem 3.1 remained unique in its kind. Only recently were two new effective tests for the infinite basis property found. The first of these appears here for the first time. We denote the semigroup $\langle a, b \mid aba=a,\ bab=b,\ a^2=a,\ b^2=0\rangle$ by A_2. The semigroup A_2 consists of 5 elements $0, a, b, ab, ba$; it can be thought of as an r.s.m.t. over the one element group E = = {e} with the sandwich matrix $\begin{pmatrix} e & e \\ 0 & e \end{pmatrix}$, or as a semigroup of 2 × 2 matrices $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ with respect to the usual matrix multiplication.

Theorem 3.2 (Volkov). Suppose S is a semigroup, T is a subsemigroup in S. Further, suppose there exist a natural number d and a group G with the identity $x^d = e$ such that

1) $x^d \in T$ for all $x \in S$;

2) $G \in \operatorname{var} S \setminus \operatorname{var} T$;

3) the semigroup A_2 lies in the variety var S.

Then the system of identities eq S is not finitely based.

Somewhat later the following test was discovered. In its formulation the sequence of words $\{Z_n\}$ are used which was introduced by Zimin [29]*: $Z_1 = x_1$, $Z_2 = x_1x_2x_1$, ..., $Z_n = Z_{n-1}x_nZ_{n-1}$.

Theorem 3.3 (Sapir [81]). Suppose the s.i. Σ is such that:

1) the variety $\operatorname{var}\Sigma \cap \operatorname{var}\{x^2 = 0\}$ is locally finite;

2) each of the words Z_1, ..., Z_n, ... is an isotherm with respect to Σ.

Then Σ is not finitely based.

Numerous applications of Theorems 3.2 and 3.3 are collected in §5, as well as in Chapters II and IV below.

Formally speaking, Theorems 3.1-3.3 are comparable in force, i.e., for any i = 1, 2, 3 there exist no f.b. system to which Theorem 3.1 but not the two others is applicable. Nevertheless, we distinguish Theorem 3.3, which is also a test for the essential infinite basis property (see §9 below).

A very efficient test for the infinite basis property of systems of inverse identities was found by Kleiman in [31]. We denote the semigroup $\langle a, b \mid aba = a,\ bab = b,\ a^2 = b^2 = 0\rangle$ by B_2. The semigroup B_2 consists of 5 elements $0, a, b, ab, ba$ and is isomorphic to the semigroup of 2 × 2 matrices $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}$ with respect to the usual matrix multiplication.

Theorem 3.4. Suppose the system of identities Σ (of signature $\langle \cdot, ^{-1}\rangle$) is such that:

1) each identity in Σ is satisfied in the semigroup B_2^1;

2) for some infinite set of natural numbers M the system El(M) is a consequence of Σ.

Then Σ is not finitely based.

In [31] many interesting applications of this theorem were presented, of which we speak in Chapters II and IV.

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*We note that in the pioneering paper of Mal'tsev the sequence of words $\{X_n\}$ arose (under an entirely different pretext, see Theorem 19.2 below), from which the sequence $\{Z_n\}$ is obtained by the identification of two letters.

4. THE IRREDUCIBLE BASIS PROPERTY

A system of s.i. Σ is called irreducible if no identity $\sigma \in \Sigma$ is a consequence of the system $\Sigma \setminus \{\sigma\}$. A system equivalent to some irreducible system is called irreducibly based. Clearly, any finite system is irreducibly based, and therefore the irreducible basis property is a natural generalization of the finite basis property. In this connection, the irreducible basis property can be comparatively easily proved under fairly broad assumptions.

To date, the most general test for the irreducible basis property is the following proposition. It is obtained by a simple combination of results of Aizenshtat [4] and Martynova [58].

Proposition 4.1. Each system consisting of balanced and 0-reduced identities is irreducibly based.

There exist systems not having an irreducible basis. The first such example was the system B12. We note that in B12 all identities except one are balanced. Therefore in combination with Proposition 4.1 this example shows that an intersection of f.b. and irreducibly based varieties may not be irreducibly based (while the intersection of two f.b. varieties is always f.b.). In this connection we note the following

Question 4.1 (Sapir). Is each variety of semigroups the intersection of a finite number of irreducibly based varieties?

Other examples of systems without an irreducible basis are B14 and B15(p).

In [17] it was shown that the system B8, considered as a system of identities of monoids, is not irreducibly based. It is interesting to compare this with Proposition 4.1. The first example of a system of inverse identities (of signature $\langle \cdot, ^{-1} \rangle$) without an irreducible basis was presented in [32], where it was established that for any infinite set of natural numbers M El(M) is such a system. We also note that recently in [36] a system of group identities not having an irreducible basis was constructed.

§5. THE FINITE BASIS PROPERTY AND OPERATIONS OVER VARIETIES

Since the finite basis property is important, and we can say a "positive" property of varieties, the question of its stability with respect to operations over varieties is natural. The presence of such stability turns out to be a very infrequent phenomenon, which in turn generates the question of finding sufficient conditions under which stability nevertheless holds. We separate the statements of related results into two groups depending on the type of operation.

a) The join of varieties. As already mentioned, the intersection of two f.b. varieties will obviously be f.b.. For the join this is not so. In [159] it was shown that a suitable example can easily be constructed using Theorem 3.1, but this example was not published there. The following example, constructed in the 1970's by Karnofski, also remained unpublished for a long time (it appeared only in the survey [181]): the not f.b. variety var B2 is the join of the f.b. varieties var $\{x^3y^3 = y^3x^3\}$ and var $\{(xyz)^2 = x^2y^2z^2,\ x^3y^3z^2t^3 = y^3x^3z^2t^3\}$. Theorems 3.2 and 3.3 as well as several other results lead to the discovery of a large number of pairs of f.b. varieties without an f.b. join, so that it became clear that similar examples are not exceptional, but on the contrary show some regularity. The attempt to reveal this regularity lead the second author of this survey to the following result, published here for the first time.

Theorem 5.1. For any f.b. variety $\mathfrak{V}$, different from the class of all semigroups, there exists f.b. varieties $\mathfrak{X}$ and $\mathfrak{Y}$ such that the join $\mathfrak{V}$, of $\mathfrak{X}$ and $\mathfrak{Y}$ is not finitely based.

Theorem 5.1 follows from Theorem 3.2. It its formulation it is impossible, generally speaking, to restrict to one f.b. in place of two "additional" f.b. varieties, since there exist varieties $\mathfrak{V}$, such that for any f.b. variety $\mathfrak{X}$ the join of $\mathfrak{V}$ and $\mathfrak{X}$ is finitely based. We call the variety $\mathfrak{V}$ with such a property stably f.b.. It is known that the variety of all semilattices [68] and any nilpotent variety [188] are stably f.b.. On the other hand, by means of Theorems 3.2 and 3.3 it is possible to show that the following varieties cannot be stably f.b.: any variety, different from the class of all semigroups, containing the semigroup A_2, any supercommutative variety such that the intersection with var $\{x^2 = 0\}$ is locally finite, any variety generated by a nontrivial group of finite exponent, any *l*.f. variety containing a nonabelian group or the variety of all idempotent semigroups. These facts are in part attributed to Sapir and in part to the second author of this survey.

We see that both the stable finite basis property as well as its negation distinguish interesting classes of f.b. manifolds, which gives interest to the following.

Problem 5.1. Describe all stably f.b. varieties of semigroups.

We note that from results of [30] and [143] it follows that varieties of all groups, all semilattices, as well as the variety var L_2, are stably f.b. varieties of inverse semigroups. An example of two f.b. varieties of groups without an f.b. join is known [35]. LB

b) Products of varieties. The product of varieties of semigroups $\mathfrak{X}$ and $\mathfrak{Y}$ in the sense of Mal'tsev is denoted by $\mathfrak{X} \circ \mathfrak{Y}$. As mentioned in [55], the class

$\mathfrak{X}\circ\mathfrak{Y}$ is not always a variety, and therefore along with the multiplication $\circ$ for varieties of semigroups we also consider the multiplication $*$, where $\mathfrak{X}*\mathfrak{Y}$ is the variety generated by the class $\mathfrak{X}\circ\mathfrak{Y}$. The first explicit example showing the instability of the property "f.b." with respect to the multiplication $*$ was presented by Sukhanov in [88], where by means of Theorem 3.1 it is shown that the variety $\mathfrak{A}\mathfrak{S}$ is not finitely based ($\mathfrak{S}$ is the variety of all semilattices). If Theorem 3.3 is used it is possible to obtain an essentially more general result, namely that the following is infinitely based: any product $\mathfrak{V}*\mathfrak{W}$, where $\mathfrak{V}$ contains a nontrivial semigroup with zero multiplication the variety, all periodic semigroups of which are locally finite, but $\mathfrak{W}$ is an l.f. variety containing $\mathfrak{G}$. From this theorem it follows that if $\mathfrak{V}$ contains $\mathfrak{S}$ and the periodic semigroups in $\mathfrak{V}$ are locally finite, but $\mathfrak{W}$ is locally finite and contains a nontrivial group, then the product $\mathfrak{V}*\mathfrak{W}$ is also not f.b.. These results were obtained by Sapir and are communicated here with his kind permission. Finally, Theorem 3.2 implies the following analog of Theorem 5.1.

Theorem 5.2. For any f.b. variety $\mathfrak{V}$, different from the class of all semigroups, there exist f.b. varieties $\mathfrak{X}$ and $\mathfrak{Y}$ such that the product $(\mathfrak{V}*\mathfrak{X})*\mathfrak{Y}$ is not finitely based.

Theorem 5.2 is attributed to the second author of the present paper and is published here for the first time.

The abundance of examples of f.b. varieties without an f.b. product lead us to hope that the following problem is solvable.

Problem 5.2. Describe all pairs $(\mathfrak{X}, \mathfrak{Y})$ of f.b. varieties such that the variety $\mathfrak{X}*\mathfrak{Y}$ is finitely based.

We now turn to the multiplication $\circ$. At present the answer to the following question is unknown.

Question 5.1. Do there exist f.b. varieties of semigroups $\mathfrak{X}$ and $\mathfrak{Y}$, such that the class $\mathfrak{X}\circ\mathfrak{Y}$ is not an f.b. variety?*

We note that in the majority of presently known cases, when the Mal'tsev product is a variety, the finite basis property of factors implies the finite

*In order to prevent misunderstanding, we point out that known examples of f.b. varieties of groups $\mathfrak{x}$ and $\mathfrak{y}$, the Mal'tsev group product of which we denote $\mathfrak{x}\mathfrak{y}$, is not f.b., do not answer Question 5.1, since the class $\mathfrak{x}\circ\mathfrak{y}$ will not be a variety in this case. We note, however, that although $\mathfrak{x}\mathfrak{y}\neq\mathfrak{x}*\mathfrak{y}$, the absence of a finite basis for $\mathfrak{x}\mathfrak{y}$ implies the absence of such a basis for $\mathfrak{x}*\mathfrak{y}$ also. This gives another series of examples of f.b. varieties of semigroups without an f.b. product (in the sense of the multiplication $*$).

basis property of the product. The following result is in this sense typical. It is obtained by a simple modification of a more particular result of Martynova [58].

Proposition 5.1. If $\mathfrak{V}$ is an f.b. and $\mathfrak{W}$ is an f.b. 0-reduced variety, then the class $\mathfrak{V}\circ\mathfrak{W}$ is an f.b. variety.

Another similar result can be extracted from [177], where it is shown that if $\mathfrak{H}$ is an f.b. group variety and $\mathfrak{J}$ is an idempotent variety, then the class $\mathfrak{H}\circ\mathfrak{J}$ is an f.b. variety (for the special case $\mathfrak{J}=\mathfrak{S}$ this was noted earlier in [69]).

For inverse semigroups the analogous questions are touched upon in [115]. However, the assertion made there, that for any f.b. variety of groups $\mathfrak{H}$ and for any f.b. variety of inverse semigroups $\mathfrak{V}$ the Mal'tsev product $\mathfrak{H}\circ\mathfrak{V}$ (in the class of inverse semigroups) is finitely based, is erroneous. Indeed, it is easy to see that the product of two varieties of groups in the class of inverse semigroups coincides with their product in the class of groups, and therefore the assertion in [115] contradicts known examples of f.b. varieties of groups without an f.b. product. We also note a result of [31], that for any nontrivial variety of Abelian groups $\mathfrak{H}$ the product $\mathfrak{S}*\mathfrak{H}$ (in the class of inverse semigroups) is not finitely based.

§6. NUMERICAL CHARACTERISTICS OF BASES OF IDENTITIES

In the literature various numerical parameters of bases of identities of varieties have been considered, and attempts have been made to classify varieties on this basis. As a rule, in relation to such parameters all varieties of groups and rings are too "good," while as "poor" as desired are found without difficulty among varieties of groupoids. It is not ruled out that for the intermediate case of semigroups the analysis of numerical characteristics of systems of identities can turn out to be more fruitful, however at present little has been done in this direction that would allow a judgement on this perspective.

a) The n-basis property. Suppose n is a natural number. A variety is n-based if it has a basis consisting of not more than n identities. It is well-known that each f.b. variety of groups or rings is 1-based. For varieties of semigroups this is not so; moreover for each n it is not difficult to construct an f.b. variety of semigroups which is not n-based. In connection with this the following announcement is very interesting.

Theorem 6.1 (Gerhard and Padmanabhan [126]). An f.b. variety of semigroups satisfying the identity $x^r = x$ for some $r \geq 2$ is 2-based.

Theorem 6.1 simultaneously covers the following known results.

Corollary 1 (Biryukov [16], Gerhard [125] and Fennemore [124]). Any idempotent variety is 2-based.

Corollary 2 (Tarski [179]). Any group f.b. variety is 2-based.

We call a ~~manifold~~ boundedly based if there exists an n such that all its f.b. subvarieties are n-based. Theorem 6.1 shows that each Clifford variety of semigroups is boundedly based. On the other hand, each variety with a finite number of subvarieties will obviously be boundedly based. Varieties of both the first and second type are varieties of finite index, i.e., the indices of nilpotency of their nilpotent semigroups are uniformly bounded. The following is therefore natural.

Question 6.1. Is any variety of finite index boundedly based?

b) The Tarski interval. Following [179], we denote the set of cardinalities of all possible irreducible bases of identities of the variety $\mathfrak{B}$ by $\nabla(\mathfrak{B})$. Clearly, if $\mathfrak{B}$ is not f.b., then either $\nabla(\mathfrak{B})=\varnothing$, or $\nabla(\mathfrak{B})=\{\aleph_0\}$. Therefore it is of interest to examine this set for f.b. varieties. From the Tarski interpolation theorem ([179], see also [180]) it follows that for an f.b. variety $\mathfrak{B}$ the set $\nabla(\mathfrak{B})$ is an interval of the set of natural numbers N (with the usual order). It is not difficult to see that for any f.b. variety $\mathfrak{B}$ of groups or rings $\nabla(\mathfrak{B})=N$. On the other hand, HrNg (see [179]) shows that for any interval $I\subseteq N$ a variety of groupoids $\mathfrak{B}$ can be found such that $\nabla(\mathfrak{B})=I$.

Question 6.2. Is every interval of the set N representable as $\nabla(\mathfrak{B})$ for some suitable variety of semigroups $\mathfrak{B}$?

Results of McNulty [152] and Theorem 6.1 imply that if $\mathfrak{B}$ is an f.b. variety satisfying the identity $x^r = x$ for some $r \geq 2$, then either $\nabla(\mathfrak{B})=N$, or $\nabla(\mathfrak{B})=N/\{1\}$. On the other hand, from [152] it follows that for any supercommutative variety $\mathfrak{B}$ the set $\nabla(\mathfrak{B})$ is finite. We note by results of Bairamov and Makhmudov [10], if $\mathfrak{B}$ is a 1-based supercommutative variety of semigroups, then $\nabla(\mathfrak{B})=\{1\}$. Therefore not every finite interval of the set N is representable as $\nabla(\mathfrak{B})$ for some supercommutative variety $\mathfrak{B}$.

Question 6.3. What finite intervals of the set N are representable as $\nabla(\mathfrak{B})$ for a suitable supercommutative variety of semigroups $\mathfrak{B}$?

c) Axiomatic rank. Finiteness of the axiomatic rank does not in general imply finiteness of a basis of identities. For example, the varieties var C2 and var D1 have axiomatic rank 2 (we note that varieties of axiomatic rank 1 are obviously finitely based). However, for l.f. varieties, in particular for varieties generated by a finite semigroup, these properties are equivalent. The following is discussed in [122].

Question 6.4 (Edmunds). What is the maximum axiomatic rank of f.b. varieties generated by a semigroup of n elements?

It is known that this maximum is greater than or equal to n and that for n = 2, 3, 4 is equal to n [122].

For the variety $\mathfrak{B}$ we denote the variety defined by all identities of not more than n letters in eq$\mathfrak{B}$ by $\mathfrak{B}^{(n)}$. Clearly, $\mathfrak{B}^{(n)}$ is the least variety of axiomatic rank not more than n containing $\mathfrak{B}$.

Question 6.5. Does there exist an f.b. variety of semigroups $\mathfrak{B}$ such that for some n the variety $\mathfrak{B}^{(n)}$ is not finitely based?

The chain $\mathfrak{B}^{(1)} \supseteq \mathfrak{B}^{(2)} \ldots \supseteq \ldots \supseteq \mathfrak{B}^{(n)} \supseteq \ldots$ is called a descending chain of the variety $\mathfrak{B}$. In a series of papers the distribution of strict inclusions in descending chains of varieties of algebras of various signatures was studied. It is interesting that for the case of semigroups it is possible to actually alternate strict inclusions and identities in such chains.

Theorem 6.2 (Jonsson, McNulty and Quackenbush [134]). For any set M of natural numbers greater than 2 there exists a variety of semigroups $\mathfrak{B}$ such that $\mathfrak{B}^{(n)} = \mathfrak{B}^{(n+1)}$ if and only if $n \in M$.

The system of identities A9 plays a key role in the construction of the required variety.

CHAPTER II. FINITELY AND INFINITELY BASED FINITE SEMIGROUPS

§7. FIRST EXAMPLES

The first two examples of not f.b. finite semigroups were published by Perkins in [159]. They are of significant interest both then and now.

Example 7.1. The six-element semigroup B_2^1 is not finitely based.

In [159] this fact is derived from Theorem 3.1. A "semantic" proof (in the terminology of §3) was given in [34]. It also follows easily from Theorem 3.3.

The value of Example 7.1 is above all the large role which the semigroup B_2 plays in the structural theory of semigroups. Its presence or absence among factors of a given semigroup S essentially influences the structure of S (Shevrin [101]). It can be explained a posteriori that Example 7.1 also holds a singular place from the point of view of the theory of varieties, since first it has a minimal number of elements (see §10), and second, the semigroup B_2^1 is essentially infinitely based (see §9).

We describe the second example from [159]. Suppose F is a free semigroup over the alphabet {x, y, z}, I is an ideal in F consisting of all words which are not subwords of the words xyzyx, xzyxy, $(xy)^2$, x^2z, Q is the Rees factor group F^1/I.

Example 7.2. The 25-element semigroup G is not finitely based.

This example also follows immediately from Theorem 3.1. The methods needed for its construction are of independent interest. It is unknown how general this method is. Suppose in general W is a finite set of words in some free semigroup F and I(W) is an ideal in F consisting of all words which are not subwords of W, Q(W) is the Rees factor group $F^1/I(W)$.

Question 7.1 (Sapir). For what finite collections of words W is the subgroup Q(W) not finitely based? Is any set with this property recursive?

The semigroup B_2^1 is inverse, and therefore the question of the finite basis property for its inverse identities naturally arises. By means of Theorem 3.4 in [31] it was shown that B_2^1 is not finitely based as an inverse semigroup either.

§8. FUNDAMENTAL STATEMENTS OF PROBLEMS. FURTHER EXAMPLES

Examples 7.1 and 7.2 were published in 1969 and in the course of 12 years they remained the only examples of not f.b. finite semigroups. In [153] McNulty showed how these examples can be "reproduced" to construct a countable series of not f.b. finite semigroups S_1, S_2, ..., S_n, ... such that $S_i \subset \mathrm{var}\, S_{i+1}$ for all i = 1, 2, ... (a simple proof of this result is given in [20]). But then a series of principally new examples was presented by Sapir in [79], where it was shown that the not f.b. variety var B15(p) is generated by a finite semigroup. In later years a great number of new examples has appeared, the most important of which are analyzed below. In addition, several general approaches (Theorems 3.2 and 3.3) were found for the construction of not f.b. finite semigroups. Because of this it is now possible to pose the following fundamental

Problem 8.1. Describe all f.b. finite semigroups.

The question of determining necessary conditions for the finite basis property of finite semigroups is closely related to the examination of the essential infinite basis property; we leave this discussion to the following section. But here we consider sufficient conditions. The problem of seeking such conditions is equivalent to the problem of finding those classes of semigroups in which each finite semigroup is finitely based. A shining example of such a class, by virtue of the well-known result of Oates and Powell [158], is the class of all groups. It is therefore natural that the attention of investigators be focused first on the classes of semigroups which are in some sense close to groups, in particular, the class of Clifford semigroups. Over a long period, the question of whether an arbitrary finite Clifford semigroup was f.b., posed in [82] by the first author, remained an open question. Very recently Mashevitski [148] gave a negative answer to this question. The corresponding Example 8.1 uses a finite group with an isolated element constructed in [117] without a finite basis of identities in the class of all groups with an isolated element.

Example 8.1. Suppose G is a finite group with an isolated element g of [117]. The r.s.m.t. R over G with sandwich matrix $\begin{pmatrix} e & g \\ e & e \end{pmatrix}$ does not have a finite basis of identities.

Nevertheless, the finite basis property of a finite Clifford semigroup can be guaranteed under sufficiently broad conditions. We recall that a semigroup is called **orthodox** if the set of all its idempotents is a semigroup.

Theorem 8.1 (Rasin [78]). A finite orthodox Clifford semigroup is finitely based.

Another important class of Clifford semigroups in which all finite semigroups are finitely based is the class of central completely simple semigroups (a Clifford semigroup is called **central** if the product of any two of its idempotents lies in the center of the maximal subgroup to which it belongs). This fact, evidently not put in explicit form, follows easily from results of Jones [133]. More details on bases of identities for central completely simple semigroups appear in §20. The central property is a natural generalization of the orthodox property, and therefore it is possible to try to extend the result of Theorem 8.1 to the class of finite central Clifford semigroups. However, the answer to even the following special case is as yet unknown.

Question 8.1 (Rasin). Suppose C is a cyclic group of prime order, M is an r.s.m.t. over C with sandwich matrix $\begin{pmatrix} e & a \\ e & e \end{pmatrix}$, where a is a non-identity element in C. Is the semigroup M^1 f.b.?

Another class of semigroups related to groups is the class of inverse semigroups. In this class Problem 9.1 has been completely solved. We have

Theorem 8.2 (Volkov [21]). A finite inverse semigroup S has a finite basis of identities if and only if the variety var S does not contain the semigroup B_2^1.

We note that the statement (but not the proof!) of Theorem 8.2 remains true in the case when the inverse semigroups are considered as algebras of signature $\langle \cdot, {}^{-1} \rangle$. Sufficiency in this case follows from [143], and necessity, by developing an idea of [31], was recently proved by Sapir [81]. Thus, for finite inverse semigroups the finiteness of a basis of s.i. and finiteness of a basis of inverse identities turn out to be equivalent properties. It is important to note that for arbitrary inverse semigroups these properties are by no means equivalent; the corresponding examples can be already found among groups (see §19 below).

In connection with Theorems 8.1 and 8.2 we note the following.

Question 8.2. Is it true that a finite regular orthodox semigroup S has a finite basis of identities if and only if $B_2^1 \notin \mathrm{var}\, S$?

If the answer is positive, we obtain a simultaneous generalization of Theorems 8.1 and 8.2. We note, as was recently shown by the second author, that a finite orthodox completely 0-simple semigroup is finitely based. At the same time there exist not f.b. finite completely 0-simple semigroups. The first example of this type was found by Mashevitski in [148]: it is an r.s.m.t. with 0 over a two-element group with sandwich matrix $\begin{pmatrix} e & e \\ 0 & e \end{pmatrix}$. See also §20.

We mention several other more special results related with Problem 8.1. In [188] the finite basis property was proved for each finite unipotent semigroup. We note that in the semigroup Q of Example 7.2 there are exactly two idempotents, therefore to strengthen this result by weakening the restriction on the number of idempotents is impossible. We note Theorem 3.3 easily implies that the semigroup A_2^1 is not finitely based. This is the first example of a not f.b. finite idempotent generated semigroup; it is also interesting that in A_2^1 all elements except one are idempotents.

We proceed now to the examination of other basic statements of problems related to f.b. finite semigroups. We first note .

Question 8.3. Does there exist an algorithm determining for each finite semigroup whether it is finitely based or not?

In other words, Question 8.3 can be stated as: is the set of all f.b. finite semigroups recursive? It is even unknown whether this set is recursively enumerable. Question 8.3 is a specialization of a well-known problem of Tarski for the case of semigroups (see [179,191]): is the set of all f.b. finite algebras of some fixed signature recursive? Recently McKenzie reduced the general problem of Tarski to the case of groupoids, which gives Question 8.3 particular interest. It is not difficult to understand that for any two finite semigroups S and T the condition $T \in \mathrm{var}\, S$ can be algorithmically verified. Therefore an algorithm for recognition of f.b. finite inverse semigroups is obtained from Theorem 8.2.

However, from the "inverse" analog of Theorem 8.2 it follows that this algorithm also recognizes finite inverse semigroups which are finitely based as algebras of singature $\langle \cdot, ^{-1} \rangle$.

In Question 8.3 we are interested in the recognition of f.b. semigroups among finite semigroups. The dual question, of the possibility of an algorithmic recognition of finite semigroups among f.b. semigroups, is no less natural and was also posed by Tarski in [179] (again, for algebras of an arbitrary signature). It can be formulated as follows.

Question 8.4. Does there exist an algorithm determining for each finite system of s.i. Σ whether the variety var Σ is generated by a finite semigroup?

It is known that the question analogous to Question 8.4 for groupoids has a negative answer (Murskii [72]). Very recently Dvoskina (see [192]) answered Question 8.4 positively for the class of commutative semigroups.

The following question has also inspired problems and results in universal algebra. In [73] Murskii established that for any fixed signature "almost all" finite algebras of this signature are finitely based, i.e., the ratio of the number of f.b. algebras of order n to the number of all algebras of order n tends to 1 as n tends to infinity. For the case of groupoids this result was essentially refined in [74], where it was found that the fraction of all n-element not f.b. groupoids among all n-element groupoids is asymptotically equal to n^{-6}. In connection with this the following is appropriate.

Question 8.5. What is the limit of the ratio of the number of f.b. semigroups of order n to the number of all semigroups of order n as n tends to infinity? What is the (asymptotic) fraction of all n-element not f.b. semigroups among all n-element semigroups?

The following question was mentioned in [82], Problem 2.51a.

Question 8.6. Does there exist a finite semigroup with an infinite irreducible basis of identities?

Finite semigroups without an irreducible basis of identities are known, for example the finite semigroup of [79] generating the variety var B15(p). However, for the majority of not f.b. finite semigroups encountered in the literature (in particular, for the semigroup B_2^1) the question of the presence of an irreducible basis remains open.

§9. INHERENTLY INFINITELY BASED SEMIGROUPS

An l.f. variety is called **inherently infinitely based** if each l.f. variety containing it has no finite basis of identities. A finite semigroup is called **inherently infinitely based** if such a variety is generated by it. The notion of an inherently infinite basis property appeared in the papers of Murškii [74] and Perkins [160] (see also [161] and [154]) for the case of groupoids, where the first examples of essentially infinitely based groupoids were presented.

Recently Sapir succeeded in describing inherently infinitely based semigroups. The following theorem which contains this description is originally published here.

Theorem 9.1. A) If a finite semigroup is inherently infinitely based then an inherently infinitely based subsemigroup with identity can be found in it.

B) Suppose S is a finite semigroup with identity, d is the period of S. The semigroup S is inherently infinitely based if and only if for some element $a \in S$ and some idempotent $e \in SaS$ the elements eae and $ea^{d+1}e$ do not lie in one residue class of the maximal subgroup H_e with identity e with respect to its last hypercenter.

From Theorem 9.1 a series of very interesting corollaries follows. First of all, we shall indicate a concrete inherently infinitely based finite semigroup.

Corollary 1. The semigroup B_2^1 is inherently infinitely based.

Thus, each finite semigroup S such that $B_2^1 \in \operatorname{var} S$, does not have a finite basis of identities; in particular, for any finite semigroup S the semigroup $S \times B_2^1$ is not finitely based. We obtain the following result which characterizes the "relative distribution" of the classes f.b. and not f.b. finite semigroups.

Corollary 2. Each finite semigroup is imbedded in a not f.b. finite semigroup*. Not every finite semigroup is imbedded in an f.b. finite semigroup.

By combining Corollary 1 with results of [101], we obtain the following necessary condition for the finite basis property of a finite semigroup.

Corollary 3. Suppose S is an f.b. finite semigroup. Then for any idempotent e of S the semigroup eSe is a semilattice of Archimedean semigroups.

In connection with Question 8.3 we note, finally, that Theorem 9.1 obviously implies

Corollary 4. There exists an algorithm which for each finite semigroup determines if it is inherently infinitely based or not.

Theorem 9.1 reveals the intrinsic structure of inherently infinitely based semigroups. Sapir recently obtained a characterization of such semigroups in the language of identities.

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*This result was proved earlier by the second author; it follows immediately from Proposition 11.2 or from Theorem 22.1.

Theorem 9.2. A finite semigroup S is inherently infinitely based if and only if each of the words

$$Z_1 \equiv x_1,\ Z_2 \equiv x_1x_2x_1,\ \ldots,\ Z_n \equiv Z_{n-1}x_nZ_{n-1},\ \ldots$$

is an isotherm relative to eq S.

Sufficiency of the condition of Theorem 9.2 follows immediately from Theorem 3.3. In this connection, only the fact that the variety var S is locally finite is essential, i.e., the condition of Theorem 9.2 will be a sufficient condition for the inherently infinite basis property for any *l*.f. variety. However, in the proof of necessity the fact that the class $\mathfrak{H}$ of all groups in var S is an f.b. variety is used (this follows from results of [139] and [158]). Thanks to this, from the system of identities eq S it is possible to extract a finite subsystem Σ such that the class of all groups in var Σ coincides with $\mathfrak{H}$. If now $Z_n = w$ is a nontrivial identity satisfied in S, then the variety

$$\operatorname{var}\Sigma \cap \operatorname{var}\{Z_n = w\}$$

is finitely based, contains S and by a result of [81] is *l*.f.

It is not difficult to understand that in the argument presented it is essential only that local finiteness of the variety $\mathfrak{H}$ can be guaranteed by imposing a finite number of identities. Thus, the question of whether each inherently infinitely based variety of semigroups satisfies the condition of Theorem 9.2 is equivalent to question, well-known in the theory of varieties of groups, of whether each *l*.f. variety of groups is contained in an f.b.l.f. variety of groups. This latter question can also be formulated as follows.

Question 9.1. Do there exist inherently infinitely based varieties of groups?

From results of Kostrikin [43] it follows that the class $\mathfrak{K}_p$ of all *l*.f. groups of prime exponent p form a variety. The question of its finite basis property, mentioned in [75], Problem 4, is open up to the present. It is not difficult to verify that if $\mathfrak{K}_p$ does not have a finite basis of identities then it necessarily will be an inherently infinitely based variety.

From Zorn's lemma it easily follows that there exist minimal inherently infinitely based varieties and that each inherently infinitely based variety contains a minimal such variety. We shall indicate one such variety. Suppose $Z \equiv x_1x_2x_1x_3x_1x_2x_1x_4\ldots$ is an infinite sequence with initial interval being the words $Z_1, Z_2, \ldots, Z_n, \ldots$ The set of all elements of a free semigroup of countable rank F_ω which are not subwords of the sequence Z forms an ideal I(Z) in F_ω. The variety $\mathfrak{Z}$, generated by the Rees factor group $F_\omega/I(Z)$ will also be a minimal inherently infinitely based variety. Moreover, as it is not difficult to see, it will be the smallest variety satisfying the condition of Theorem 9.2. Therefore the question of whether there is a variety 3 which is the smallest among all inherently infinitely based varieties of semigroups is equivalent to Question 9.1. The variety 3 is not generated by a finite semigroup. It is natural to try to find minimal inherently infinitely based varieties generated by a finite semigroup. An example of such a variety is the variety var B_2^1. Moreover, in a very broad class of varieties this variety is the smallest, as the following result of Sapir, published originally here, shows.

Theorem 9.3. Suppose S is a finite semigroup such that all its subgroups are nilpotent. Then S is inherently infinitely based if and only if $B_2^1 \in \operatorname{var} S$.

However, there also exist essentially infinitely based semigroups of S such that $B_2^1 \notin \operatorname{var} S$. The corresponding example was found by Sapir; it is quite complicated, and we will not reproduce it here. The following naturally arises.

Problem 9.1. Describe all inherently infinitely based varieties which are minimal among those generated by finite semigroups.

§10. SEMIGROUPS WITH A SMALL NUMBER OF ELEMENTS

The small number of elements in the not f.b. semigroup B_2^1 focuses attention on the finite basis problem for semigroups of order ≤5. This problem was mentioned, in particular, in [44] (Problem 2.14 d) and in [82] (Problem 1.63). Its complete solution occupied around fifteen years and required significant efforts: we note that (to within isomorphism or antiisomorphism) there are 18 three-element, 126 four-element and 1160 five-element semigroups (see for example [135]). The result of these efforts is

Theorem 10.1. Each semigroup of order not exceeding 5 is finitely based.

The history of the proof of Theorem 10.1 is the following. The case of two-element semigroups is trivial and furthermore is covered by the general result of Lyndon [147] on the finite basis property of any two-element algebra. The finite basis property of three-element semigroups was proved by Perkins in [159]. Here the following semigroup characteristic is used: there exist a not f.b. three-element groupoid (Murskii [70]). The finite basis property of four-element semigroups was announced by Bol'bot [17], Karnofsky [137] and Plemmons [166], and the first proof was published in [18].* There is another proof in

*In [137] there is an outline of a proof, but as correctly noted in [122], the argument in [137] does not go through for all four-element semigroups.

[122]; we note that in [122] bases of identities are explicitly described for all semigroups of order ≤4. Bases of identities of all four-element noncommutative nonidempotent semigroups were presented (without proof) in [85]. "Sturm" five-element semigroups were introduced by Edmunds in [121], where the finite basis property was proved for five-element semigroups with zero and identity. Tishchenko [182] analyzed the case of five-element monoids. The five-element semigroups A_2 and B_2, which play an especially important role both in the structural theory of semigroups as well as in the theory of semigroup varieties, had been often encountered earlier. Finiteness of a basis of identities of these semigroups was established by Trakhtman in [94] and [95] respectively (for details see §20 below). These results opened the way to the proof of the finite basis property for an arbitrary five-element semigroup, which was announced in [96] (see also [184]). The complete text of this proof has not yet been published.

The difficulties arising in the proof of Theorem 10.1 are not only technical but also of a fundamental nature. In particular, there do not exist any general conditions for the finite basis property which would allow us to replace "manual" with "automatic" arguments. This circumstance is sharply demonstrated by the following.

Theorem 10.2. For each $n \geq 5$ the variety $\mathfrak{F}_n$, generated by all n-element semigroups does not have a finite basis of identities.

Theorem 10.2 was obtained by the second author and is originally published here. It follows from Theorem 3.2 (for n > 5 Theorem 3.3 can be applied). By comparing Theorem 10.1 and Theorem 10.2 (for n = 5), we see that while identities of each separately taken five-element semigroup have a finite basis, the set of all identities for all these semigroups is not finitely based. It is possible to prove that also for n > 5 the variety generated by all f.b. n-element semigroups is not finitely based. We also note that the varieties $\mathfrak{F}_2$ and $\mathfrak{F}_3$ are finitely based (for the first this is obvious, and for the second this was recently shown by Dvoskina). We formulate

Question 10.1. Does the variety $\mathfrak{F}_4$ have a finite basis of identities?

11. THE CLASS OF FINITE FINITELY BASED SEMIGROUPS AND SEMIGROUP-THEORETICAL CONSTRUCTIONS

We denote the class of all finite f.b. semigroups by FFB. The behavior of the class FFB in relation to various operations over semigroups and semigroup-theoretic constructions is naturally interesting. It turns out that the class FFB in this sense is highly irregular. The corresponding results are summarized as follows.

Theorem 11.1. The class FFB is not closed with respect to subsemigroups and factor semigroups, direct products, ideal extensions, left, right and commutative bundles.

We present examples proving Theorem 11.1, but first we describe a construction used for these examples.

a) The construction. Suppose S is an arbitrary semigroup, a and 0 are symbols not lying in S. We consider the set

$$T(S) = S \cup \{a\} \times S^1 \cup S^1 \times \{a\} \cup \{a\} \times S^1 \times \{a\} \cup \{0\}$$

and on it we define an operation by the following rules (below $s \in S^1$, $t \in S$): in S the operation $\circ$ coincides with the original multiplication,

$$(a, s) \circ t = (a, st), \ t \circ (s, a) = (ts, a), \ (a, s) \circ (t, a) = (a, st, a),$$
$$(a, t) \circ (s, a) = (a, ts, a)$$

and all remaining products are equal to 0. It is easy to verify that the operation $\circ$ is associative. Further, suppose W is an arbitrary set of words, W(S) is the set of all values of words in S under substitution of elements of S for letters. We denote the Rees factor group by T(S)/I, where

$$I = \{a\} \times W(S) \times \{a\} \cup \{0\}.$$

The construction of the semigroup T(S, W) was presented by Sapir. The following states a fundamental property of T(S, W), published originally here.

Proposition 11.1. Suppose F is a free semigroup of countable rank in the variety var S, W is a set of words such that $W(F) \neq V(F)$ for any finite set of words V. Then the semigroup T(S, W) does not have a finite basis of identities.

By varying S and W in the described construction, it is possible to obtain examples of not f.b. semigroups (including finite) with interesting properties for the theory of varieties. Other applications of this construction appear in §18.

b) Subsemigroups and factor semigroups. The first example of a finite f.b. semigroup having a not f.b. subsemigroup and a not f.b. factor semigroup was found by Sapir. We present this example. Suppose p and q are distinc odd primes, K and L are nonabelian groups of orders p^3 and q^3 respectively. Further, suppose W_p is the set of all words of the form

$$[x_1^p, y_1^p] \ldots [x_n^p, y_n^p],$$

and W_q is the set of all words of the form

$$[x_1^q, y_1^q] \ldots [x_n^q, y_n^q],$$

where $[x, y] = x^{pq-1}y^{pq-1}xy$. We consider the semigroups $T(K \times L, W_p)$ and $T(K \times L, W_q)$. From Proposition 11.1 it follows that both these semigroups are not finitely based, but it is possible to verify that their 0-direct join has a finite basis of identities. Clearly, however, the 0-direct join of the two semigroups S and T has an ideal isomorphic to S such that the Rees factor semigroup with respect to it is isomorphic to T. Thus, we obtain that the class FFB is not even closed with respect to taking ideals and Rees factor semigroups.

c) Direct products. Ideal extensions. The first example of a not f.b. finite semigroup which is a direct product of f.b. semigroups was constructed by Sapir by means of the construction 11.1a. Another example follows from Theorem 10.2: the direct product of all five-element semigroups is not finitely based. By using Theorem 3.2 it is not complicated to deduce the following more general assertion, obtained by the second author and originally published here.

Proposition 11.2. For any f.b. semigroup S there exist f.b. finite semigroups T and U such that the direct product S × T × U is not finitely based.

As T we can take the semigroup A_2, and as U an arbitrary finite group not lying in var S. Hence it follows that for any nontrivial finite group G the semigroup $A_2 \times G$ is not finitely based. It is easy to see that the 0-direct join of the semigroups A_2 and G^0 is equationally equivalent to $A_2 \times G$, and therefore also does not have a finite basis of identities. Thus, this example shows that the class FFB is also not closed with respect to ideal extensions.

d) Bundles. Example 7.2 shows that there exists a finite f.b. semigroup S such that the semigroup S^1 is not finitely based. Thus the class FFB is not closed with respect to commutative bundles, and moreover with respect to ordinal sums. For comparison we note that for any f.b. semigroup S the semigroup obtained by adjoining zero to S is also f.b. This follows from results of Mel'nik [68]. The following remains open.

Question 11.1. Is there an f.b. finite semigroup which is a commutative bundle of a commutative semigroup?

If the answer turns out to be positive, it would be possible to significantly shorten the proof of Theorem 10.1.

Since there exists a not f.b. finite completely simple semigroup (see Example 8.1) which can be considered either as a left bundle of right groups or as a right bundle of left groups, the class of FFB is also not closed with respect to formation of left and right bundles.

Theorem 11.1 shows that the class FFB is not closed with respect to the operators H of taking homomorphic images, S of taking subsemigroups, P_f of taking finite direct products, B_f of taking finite bundles and so on. The following naturally arises.

Problem 11.1. Describe the closures of the class FFB with respect to the operators H, S, P_f, B_f and their combinations.

In particular we note the following concrete question.

Question 11.2 (Sapir). Does the HSP_f-closure of the class FFB coincide with the class of finite semigroups which are not essentially finitely based?

From results of §9 it follows that the latter class is HSP_f-closed.

CHAPTER III. HEREDITARILY FINITELY BASED VARIETIES

§12. TECHNIQUES

Any nontrivial syntactic proof of the hereditary finite basis property for some or other variety is based in the end on one of two circumstances: first, on the existence of some standard list for elements of a free semigroup in it, and second, on the possibility of suitably completely preordering these standard lists.* It was recently noticed that these arguments can be used in a form suitable for the application of a scheme. We present one of the possible variants of such a scheme.

Suppose $\mathfrak{B}$ is a variety of semigroups, β is a completely invariant congruence on a free semigroup of countable rank F_ω corresponding to $\mathfrak{B}$. We shall say that $\mathfrak{B}$ admits a fine standard form if there exist a subset $M \subseteq F_\omega$, a relation $\le$ on M and a relation $\trianglelefteq$ on the set

$$M^{\#} = \{(u, v) \in M \times M \mid u < v\}$$

such that:

a) the intersection of M with any β-class is nonempty;

b) $\langle M, <\rangle$ is a completely ordered set;

c) $\langle M^{\#}, \trianglelefteq\rangle$ is a completely preordered set;

d) if $(u, v), (z, t) \in M^{\#}$ and $(u, v) \trianglelefteq (z, t)$, then there exists $w \in M$ such that $w < z$ and the pair (z, w) lies in the completely invariant congruence on F_ω generated by the pair (u, v) and β.

*We recall that a preordered set is called completely preordered if in it each strictly decreasing chain and each antichain are finite. This notion goes back to Higman [130], and starting with Cohen [120] was successfully used in the theory of varieties for the proof of the hereditary finite basis property.

Proposition 12.1. If the variety $\mathfrak{B}$ admits a fine standard form, then all its subvarieties are finitely based inside $\mathfrak{B}$; in particular, if $\mathfrak{B}$ is itself finitely based, then $\mathfrak{B}$ is an hereditarily f.b. variety.

Various modifications of Proposition 12.1 appear in [169] and [174]. One type of reform was contained in [159] (we have in mind the so-called (p, q)-block argument applied there). We note, however, that only the selection "in pure form" of assertions similar to Proposition 12.1 gave the possibility of proving a more labor-consuming results on hereditarily f.b. varieties. In order to demonstrate the methods of application of Proposition 12.1, we shall show how it is used to prove the following well-known result.

Proposition 12.2. The variety of all commutative semigroups $\mathfrak{A}$ is hereditarily f.b.

Suppose A is the set of all words of the form $u \equiv x_1^{\alpha_1} \dots x_m^{\alpha_m}$, where $\alpha_1, \dots, \alpha_m$ are nonnegative integers, at least one of which is postiive. Clearly, the intersection of A with each class of the completely invariant congruence α corresponding to A is nonempty, i.e., condition a) is satisfied. With each word u of the indicated form we associated the vector $\exp u = (\alpha_1, \dots, \alpha_m)$, and we set $v \preceq u$ if $\exp v \preceq \exp u$ in the lexicographic sense. The relation $\preceq$ is a total order on A, thus condition b) is satisfied. If $(u, v), (z, t) \in A^*$, where $\exp u = (\alpha_1, \dots, \alpha_m)$, $\exp z = (\beta_1, \dots, \beta_n)$, then we set $(u, v) \trianglelefteq (z, t)$, if there exists a monotone injection ϕ of the set N into itself such that $\alpha_i < \beta_{\varphi(i)}$ for all i = 1, ..., n. By [130] $(A^*, \trianglelefteq)$ is a completely preordered set, consequently condition c) is satisfied. Suppose $(u, v) \trianglelefteq (z, t)$ and ϕ is the corresponding injection. We consider the word

$$y \equiv x_1^{\gamma_1} \dots x_m^{\gamma_m},$$

where $\gamma_j = \beta_j - \alpha_i$, if $j = \varphi(i)$ for some i, and $\gamma_j = \beta_j$ otherwise, and the endomorphism $\hat{\phi}$ of the semigroup F_ω defined by the rule $\hat{\varphi}(x_k) = x_{\varphi(k)}$ for all $k \in N$. Then it is easy to verify that as the word w in condition d) we can take a word in A lying in the same α-class as the word $\hat{\varphi}(v) y$.

Proposition 12.2 was the first result on the hereditary finite basis property for varieties of semigroups. It was proved by Perkins [159], and independently by the first author (published in 1966). This result served as a starting point for all investigations in this direction.

§13. VARIETIES DEFINED BY ONE IDENTITY

All investigation on hereditarily f.b. varieties of semigroups is aimed in the end on the solution of the following general problem.

Problem 13.1. Describe all hereditarily f.b. varieties.

A "stricter" form of Problem 13.1 is the following.

Question 13.1. Does there exist an algorithm which for each finite system of s.i. Σ determines whether the variety var Σ is hereditarily f.b.?

We note that for varieties of groupoids the answer to the analogous question is negative (Murskii [72]).

The greatest progress toward the solution of Problem 13.1 has been achieved for varieties defined by one identity. A fundamental overview of hereditarily f.b. varieties in this class was first given by Lyapin [52]. Namely, in [52] it was proved that over a given finite alphabet there exists only a finite number of balanced identities such that each of them gives an hereditarily f.b. variety. This result was made concrete by Pollak in [168], where 12 series of identities were found which contained all identities defining hereditarily f.b. varieties. Further progress was achieved by Pollak in [170]. In order to compactly state this result, we introduce the relation $\approx$ on the free semigroup F_ω, setting $u \approx v$ if and only if u is obtained from v by some (one-to-one) renumbering of letters.

Theorem 13.1 (Pollak [170]). If the identity u = v defines an hereditarily f.b. variety of semigroups, then one of the following conditions is satisfied:

1) $u \approx xyzx$, $v = x^2$ or vice versa (i.e., $v \approx xyzx$, $u \approx x^2$);

2) $u \approx x_1 \dots x_{k-1} y x_k y x_{k+1} \dots x_m$, $v \approx x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$, $\varepsilon_1, \dots, \varepsilon_n \in \{1, 2\}$, or vice versa;

3) $u \approx xyx$ or $v \approx xyx$;

4) $u \approx x_1 \dots x_n$ or $v \approx x_1 \dots x_n$.

In a certain sense Theorem 13.1 is unimprovable, since each of the classes of identities named in it contains identities actually defining hereditarily f.b. varieties and not lying in the remaining classes. For its proof the not f.b. systems of s.i. A5 and B3-B7 are used. If the identity u = v defines a hereditarily f.b. variety, it must be applicable to some identity of each of the systems, since otherwise one of the varieties var $\{u = v\} \cap$ var A5, ..., var $\{u = v\} \cap$ var would be not f.b. Hence the conclusion is derived from the structure of the words u and v. A further analysis of identities satisfying the conditions of Theorem 13.1 appears in Pollak [169,172,173]. These results are summarized in Table 13.1. From this table it is evident that the identities satisfying conditions 1) or 2) of Theorem 13.1 and defining hereditarily f.b. varieties admit a complete description. For identities satisfying 3) such a description is obtained only under the additional homotypic assumption. The most difficult

Table 13.1

Form of identity	When var $\{u = v\}$ is hereditarily f.b.	When var $\{u = v\}$ is not hereditarily f.b.
$u \approx xyzx,\ v \approx x^2$	Always	Never
$u \approx x_1 \dots x_{k-1}yx_kyx_{k+1}\dots x_m$, $v \approx x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$, $\varepsilon_1, \dots, \varepsilon_n \in \{1, 2\}$	(a) u = v is a balanced identity and either the first or last letters of the words u and v are distinct (b) u = v is an unbalanced identity, and if $u \equiv x_1 \dots x_{k-1}yx_kyx_{k+1}\dots x_m$, then $v \not\equiv x_1 \dots x_{k-1}y^2x_k^2x_{k+1}\dots x_m$, $v \not\equiv x_1 \dots x_{k-1}x_k^2y^2x_{k+1}\dots x$, $m > 1$	(a) u = v is a balanced identity, the words u and v start with the same letter and end with the same letter (b) u = v is an identity of the form $x_1 \dots x_{k-1}yx_kyx_{k+1}\dots x_m = x_1 \dots x_{k-1}y^2x_k^2x_{k+1}\dots x_m$ or is dual, m > 1
$u \approx xyx$	(a) u = v is a homotypic identity and $v \approx x^ky^l$, $k, l < 2$; (b) u = v is a heterotypic identity not satisfied in nontrivial group and $v \not\equiv uw$, $v \not\equiv wu$ for any w	(a) u = v is a homotypic identity and not $v \approx x^ky^l$, $k, l \leqslant 2$; (b) v ≡ uw or v ≡ wu
$u \approx x_1 \dots x_n$	u = v is a heterotypic identity not satisfied in nontrivial groups	Known separate examples

case to describe are identities with condition 4). This case is at present difficult to visualize, since the accumulated examples have not yet revealed a general pattern. The possibility for further progress here depends on the answer to Question 8.1.

Since a complete description of identities defining hereditarily f.b. varieties does not yet exist, it is of interest to obtain such descriptions for certain important classes of identities. In particular, in [174] balanced identities with this property were described.

Theorem 13.2 (Pollak and Volkov [174]). A balanced identity defines a hereditarily f.b. variety if and only if its right and left parts start or end with different letters, and it has either the form $x_1x_2\dots x_m = x_{1\sigma}x_{2\sigma}\dots x_{m\sigma}$, or the form

$$x_1 \dots x_{k-1}yx_kyx_{k+1}\dots x_m = x_{1\sigma}\dots x_{(l-1)\sigma}y^2x_{l\sigma}\dots x_{m\sigma},$$

where σ is a permutation of the set {1, 2, ..., m}.

The outline of the proof of Theorem 13.2 is the following. We call a word simple if each letter appears in it not more than one time, and quasisimple if some letter appears two times and the remaining letter not more than one time. In identities satisfying one of the conditions 1)-4) of Theorem 13.1, one of the sides is either a simple word or a quasisimple word. We call an identity simple (quasisimple) if both sides have simple (quasisimple) words. It is comparatively simple to describe simple identities defining hereditarily f.b. varieties; this was done by Aizenshtat in [5]. In [174] an analogous problem was solved for quasisimple identities. If a balanced identity satisfies one of the conditions 1)-4) of Theorem 13.1, then it is simple or quasisimple. Therefore Theorem 13.2 follows from the results mentioned.

Another important class for which it might be hoped that the problem of describing identities defining hereditarily f.b. varieties in it can be more easily solved than the general case is the class of s.i. not satisfied in nontrivial groups. We formulate

Question 13.2. What hereditarily f.b. varieties defined by one identity are not satisfied in nontrivial groups?

It is now unknown whether, for example, the variety $\mathrm{var}\{x^2y=xy\}$ is hereditarily f.b.

§14. PERMUTATION AND QUASIPERMUTATION VARIETIES

An identity of the form $x_1\dots x_{k-1}yx_kyx_{k+1}\dots x_m = x_{1\sigma}\dots x_{(l-1)\sigma}y^2x_{l\sigma}\dots x_{m\sigma}$ is called a quasipermutation. A quasipermutation (permutation) variety is a variety satisfying a quasipermuation (nontrivial permutation) identity. The basic role of permutation and quasipermutation varieties in questions of the hereditary finite basis property is evident from Theorem 13.2. Permutation identities have long been studied (see [5,167,175]). However the notion of quasipermutation identity is recent, but it has proven to be a fruitful generalization of permutaton identity. Many important results on permuation varieties have recently been successfully extended to suitable classes of quasipermutation varieties, and one can expect that work in this direction will continue.

The first result on the hereditary finite basis property of permutation varieties was obtained in [159], where it was proved that a periodic permutation variety is hereditarily f.b. In [19] an analogous fact was proved for permutation varieties satisfying the identity $x^{p+q}y=x^pyx^q$.* In order to formulate the "quasipermutation" generalization of these results, we consider the identities

$$x_1\dots x_ry^2x_{r+1}\dots x_{2r}z_1z_2x_{2r+1}\dots x_{3r} = x_1\dots x_ry^2x_{r+1}\dots x_{2r}z_2z_1x_{2r+1}\dots x_{3r}, \tag{1}$$

$$x_1\dots x_ry^2zx_{r+1}\dots x_{2r} = x_1\dots x_rzy^2x_{r+1}\dots x_{2r}. \tag{2}$$

We note that as is well-known (see for example [159]), each nontrivial permu-

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*In [19] a formally stronger identity $x^ryz = x^rzy$ was imposed, but as mentioned in [185], in permutation varieties the identity $x^{p+q}y = x^pyx^q$ implies an identity of this form.

tation identity implies an identity of the form $x_1 \dots x_r yzx_{r+1} \dots x_{2r} = x_1 \dots x_r zyx_{r+1} \dots x_{2r}$. Therefore in each permutation variety (1) and (2) are satisfied for suitable r.

Theorem 14.1 (Pollak and Volkov [174]). If a quasipermutation variety satisfies at least one of the identities (1) and (2) and at least one of the identities $x^p = x^{p+q}$ and $x^{p+q} = x^p y x^q$, then it is hereditarily finitely based.

Even though the identities (1) and (2) appear specialized, they are used very often in quasipermutation varieties. Thus, if in the identity

$$x_1 \dots x_{k-1} y x_k y x_{k+1} \dots x_m = x_{1\sigma} \dots x_{(l-1)\sigma} y^2 x_{l\sigma} \dots x_{m\sigma}$$

the permutation σ is nontrivial, then it implies an identity of the form (1) or the dual to (1), and if σ is trivial, but $k \neq l$, then it implies an identity of the form (2) from [174]. On the other hand, as examples show, it is impossible to remove the additional conditions in Theorem 14.1. Theorem 14.1 is proved by means of Proposition 12.1.

§15. 0-REDUCED VARIETIES. 0-HEREDITARILY F.B. VARIETIES

A fundamental role of 0-reduced varieties among all varieties of semigroups is explained by the fact that the completely invariant congruences of a free semigroup corresponding to them are Rees congruences. Because of this, 0-reduced varieties have many "pleasant" properties typical of congruence-modular varieties and not in general true for varieties of semigroups. We present one such property related to the hereditary finite basis property and not mentioned earlier in the literature. This example was discovered by the second author.

Proposition 15.1. Suppose $\mathfrak{B}$ and $\mathfrak{N}$ are hereditarily f.b. varieties, and $\mathfrak{N}$ is a 0-reduced variety. Then the join of the varieties $\mathfrak{B}$ and $\mathfrak{N}$ is hereditarily f.b. if and only if it is f.b.

That the analog of Proposition 15.1 for arbitrary hereditarily f.b. varieties is false is well-known: the join of varieties of all commutative semigroups and all rectangular bundles is equal to $\operatorname{var}\{xyzt = xzyt\}$ and is not hereditarily f.b. Proposition 15.1 shows that each advance in the analysis of 0-reduced hereditarily f.b. varieties essentially extends the class of known hereditarily f.b. varieties. In particular, by applying this to the simplest case, when

$$\mathfrak{N} = \mathfrak{N}_k = \operatorname{var}\{x_1 \dots x_k = 0\},$$

we obtain with regard to the results of [188] that for any hereditarily f.b. variety its join with $\mathfrak{N}_k$ is also hereditarily f.b. Since no proper variety contains all varieties $\mathfrak{N}_k$, we have the following fundamentally important result.

Proposition 15.2. There does not exist a maximal variety among hereditarily f.b. varieties.

An extensive class of hereditarily f.b. 0-reduced varieties is indicated in [187]. For any k. $\mathfrak{L}_k$ denotes the variety defined by all possible identities of the form w = 0, where the level (see §2) of the word w is not less than k. It is clear that $\mathfrak{L}_k$ plays the same role in relation to the level as $\mathfrak{N}_k$ plays in relation to the length, and that $\mathfrak{N}_k \subset \mathfrak{L}_k$.

Theorem 15.1 (Volkov [187]). The variety $\mathfrak{L}_k$ is hereditarily f.b.

We note that in [186] it was shown that if $\mathfrak{M}$ is an f.b. variety containing $\mathrm{var}\{x^2 = x^3,\ xy = yx\}$, then the join of $\mathfrak{M}$ and $\mathfrak{L}_k$ will also be f.b. Hence by Proposition 15.1 it follows that if $\mathfrak{M}$ is a hereditarily f.b. variety containing $\mathrm{var}\{x^2 = x^3,\ xy = yx\}$, then the join of $\mathfrak{M}$ and $\mathfrak{L}_k$ will also be hereditarily f.b.

For 0-reduced varieties, along with hereditarily f.b. varieties it is natural to consider 0-hereditarily f.b. varieties, i.e., varieties such that all 0-reduced subvarieties are f.b. This generalization of the hereditary finite basis property was introduced (in different terms) in [168], where all 0-hereditarily f.b. varieties defined by one or two identities were described.

Theorem 15.2 (Pollak [168]). The identities u = v = 0 define a 0-hereditarily f.b. variety of semigroups if and only if one of the following conditions, or their duals, is satisfied:

1) $u \approx x_1 \ldots x_k$ or $v \approx x_1 \ldots x_k$;

2) $u \approx xyx$ or $v \approx xyx$;

3) $u \approx yx_1 \ldots x_k y,\ v \approx x_1 y_1 x_1 z_1 \ldots z_l x_2 y_2 x_2$ or vice versa, i.e., $u \approx x_1 y_1 x_1 z_1 \ldots z_l x_2 y_2 x_2,\ v = {}$
$= yx_1 \ldots x_k y)$;

4) $u \approx xyzx,\ v$ is a subword of $w \approx x^2 y_1 \ldots y_k z^2$, or vice versa;

5) $u \approx xyzx,\ v \approx x^2 y_1 \ldots y_k ztz$ or $v \approx x^2 ztz$, or vice versa;

6) $u \approx x_1 \ldots x_{k-1} y x_k y x_{k+1} \ldots x_m,\ v \approx x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n},\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, 2\}$ or vice versa;

7) $u \approx yx_1 yx_2 \ldots x_k,\ v \approx zx_1^{\varepsilon_1} \ldots x_l^{\varepsilon_l} yz,\ \varepsilon_1, \ldots, \varepsilon_l \in \{1, 2\}$ or vice versa;

8) $u \approx xyxz,\ v \approx yx_l^{\varepsilon_1} \ldots x_l^{\varepsilon_l} y,\ \varepsilon_1, \ldots \varepsilon_l \in \{1, 2\}$ и $\varepsilon_k = 1$ for some k, or vice versa.

From the description of 0-hereditarily f.b. varieties defined by one identity (see conditions 1) and 2) of Theorem 15.2), it follows that these varieties are in fact hereditary f.b. For varieties defined by two identities this is not so: there eixsts a variety with a basis of identities satisfying condition 4) and which is 0-hereditarily f.b. but not hereditarily f.b. (Pollak, 1985 letter to the authors).

§16. CLIFFORD VARIETIES

For a Clifford variety $\mathfrak{V}$ we denote its largest group subvariety, i.e., the subvariety consisting of all groups of the variety $\mathfrak{V}$, by $\mathfrak{G}(\mathfrak{V})$. Properties of $\mathfrak{V}$ in many (but far from all) cases are determined by properties of $\mathfrak{G}(\mathfrak{V})$, and it is natural to study them "modulo groups." Such an approach is one of the developments mentioned in the introduction of connections between the theories of varieties of semigroups and groups. The best variant for this approach is the complete reduction to groups. Such a reduction was successfully carried out in [78] for the problem of describing hereditarily f.b. Clifford varieties in the important special case of orthodox semigroups (see §8).

Theorem 16.1 (Rasin [78]). The variety $\mathfrak{V}$ of orthodox Clifford semigroups is hereditarily f.b. if and only if the variety $\mathfrak{G}(\mathfrak{V})$ is hereditarily f.b.

The limiting special case of this theorem, when the variety $\mathfrak{G}(\mathfrak{V})$ is trivial, is the long known assertion (see Corollary 1 of Theorem 6.1) that the variety of all idempotent semigroups is hereditarily f.b.

The orthodox condition in Theorem 16.1 cannot be removed. Indeed, there exists a finite completely simple semigroup R without a finite basis of identities (see Example 8.1). It is easily verified (see for example [139]) that $\mathfrak{G}$ (var R) = var H, where H is a maximal subgroup in R. By [158] var H is an hereditarily f.b. variety.

A natural generalization of the orthodox property is centrality (see §8). It is unknown whether Theorem 16.1 can be extended to varieties consisting of central Clifford semigroups, i.e., the answer to the following question is unknown.

Question 16.1. Are there hereditarily f.b. varieties of central Clifford semigroups such that all its group varieties are f.b.?

We shall indicate the relation of this question to Question 8.1, and also that the results of Jones [133] (or Petrich and Reilly [165]) imply that for varieties of completely simple semigroups the answer to this question is positive.

Theorem 16.2. The variety $\mathfrak{V}$ of central completely simple semigroups is hereditarily f.b. if and only if $\mathfrak{G}(\mathfrak{V})$ is a hereditarily f.b. variety.

Corollary ([62]). Each variety of completely simple semigroups with Abelian maximal subsemigroups is hereditarily f.b.

The proofs of Theorems 16.1 and 16.2 are based on the analysis of the structure of the lattice of varieties of orthodox Clifford and central completely simple semigroups respectively.

If we consider Clifford semigroups as algebras of signature $\langle\cdot,{}^{-1}\rangle$, then Theorem 16.2 remains true, while the truth of Theorem 16.1 is open to question in this situation.

Question 16.2. Are there hereditarily f.b. varieties of orthodox Clifford semigroups (of signature $\langle\cdot,{}^{-1}\rangle$) such that all group subvarieties are f.b.?

We also note that hereditarily f.b. varieties of inverse semigroups are described "modulo groups" in [31].

Theorem 16.3. The hereditarily f.b. varieties of inverse semigroups are exhausted by varieties of the form $\mathfrak{H}$, $\mathfrak{S}\vee\mathfrak{H}$ and $\operatorname{var}B_2\vee\mathfrak{H}$, where $\mathfrak{H}$ is a hereditarily f.b. variety of groups.

§17. SINGULARITIES OF THE STRUCTURE OF SEMIGROUPS IN HEREDITARILY FINITELY BASED VARIETIES

The hereditary finite basis property has a syntactic character, it is clear a priori however that it must have some influence on the structure of semigroups in the varieties satisfying it. Until very recently this aspect was completely unanalyzed. Actually, the first example containing a property following from the hereditary finite basis property from the point of view of the structural theory of semigroups was published recently in [23]. For an arbitrary semigroup S E(S) denotes the set of all its idempotents, Reg(S) the set of all its regular elements, Gr(S) the union of all its subgroups, M(S) the union of all its submonoids. We note that

$$E(S)\subseteq \mathrm{Gr}(S)\subseteq \mathrm{Reg}(S),\ \mathrm{Gr}(S)\subseteq M(S),$$

where in general all these inclusions can be strict and none of these sets is necessarily a subsemigroup.

Theorem 17.1 (Volkov and Sapir [23]). Suppose $\mathfrak{B}$ is a supercommutative hereditarily f.b. variety, S is an arbitrary semigroup in $\mathfrak{B}$. Then:

1) E(S), Reg(S), Gr(S), M(S) are subsemigroups in S;

2) Reg(S) = Gr(S);

3) M(S) satisfies either the identity $x^2y = xyx$ or the identity $xyx = yx^2$.

By combining Theorem 17.1 with various results from the structural theory of semigroups, it is possible to extract diverse consequences from it which characterize the structure of semigroups of some or other important types in hereditarily f.b. supercommutative varieties. Thus, for example, by using results of Shevrin [100] we obtain immediately that each quasiperiodic semigroup in such a variety is a semilattice of Archimedean semigroups, and from a result of Rasin [177] we deduce that regular semigroups in supercommutative hereditarily f.b. varieties are either left regular or right regular bundles of Abelian groups, etc. We note also that a rather unexpected fact follows from the theorem: in a supercommutative hereditarily f.b. variety all groups must be Abelian. We recall that a variety defined by one of the identities $x^2y = xyx$, $xyx = yx^2$, is hereditarily f.b. (see §13). Therefore from Theorem 17.1 it is also possible to extract a necessary and sufficient condition for a nonperiodic monoid (nonperiodic regular semigroup, etc.) to generate a hereditarily f.b. variety.

Question 17.1. What are necessary and sufficient conditions for a nonperiodic semigroup $S = S^2$ to generate a hereditarily f.b. variety?

It is possible to show that the presence of identities of the form $x^2y = xyx$ or $xyx = yx^2$ is not a necessary condition in this case.

Another semantic property of the hereditary finite basis property is originally published here.

Theorem 17.2 (Sapir). Suppose $\mathfrak{B}$ is a hereditarily f.b. variety of semigroups. Then all null semigroups in $\mathfrak{B}$ are locally finite. If furthermore either $\mathfrak{B}$ is a supercommutative variety or each group in $\mathfrak{B}$ is locally finite, then all periodic semigroups in $\mathfrak{B}$ are locally finite.

§18. LIMIT VARIETIES

A not f.b. variety is called a limit variety if each of its proper subvarieties is finitely based. By Zorn's lemma each not f.b. variety contains some limit subvariety. Hence it follows that a variety of semigroups is hereditarily f.b. if and only if it does not contain limit subvarieties. By describing the limit varieties we thus obtain a description of hereditarily f.b. varieties. This makes clear the importance (and difficulty) of the following problem.

Problem 18.1. Describe all limit varieties of semigroups.

The first explicit example of a limit variety was presented in [185].

Example 18.1. The variety var C3 is a limit variety.

Up to the present this example remains unique among the permutation and even supercommutative varieties. In connection with this we pose the following question.

Question 18.1. a) Do there exist permutation limit varieties different from var C3? b) Do there exist supercommutative limit varieties which are not permutation?

A series of limit varieties was constructed in [79].

Example 18.2. For each simple odd p the variety var B15(p) is a limit variety.

At first glance, between the varieties of Examples 18.1 and 18.2 there is little in common and the constructions follow from different arguments. The remark of Sapir that they can all be obtained by means of the construction 11.1a is therefore unexpected. More precisely, each of these varieties is generated by a semigroup T(S, W) from this construction for a suitable subgroup S and set of words W. Thus, in order to obtain the variety var C3, as S we take an infinite cyclic semigroup and as W we take the set of words $\{x^p | p$ is a prime number$\}$. The variety B15(p) is obtained from S a nonabelian group of order p^3 and W the set of all words of the form $[x_1, y_1][x_2, y_2]\dots[x_n, y_n]$, where $[x, y] = x^{p-1}y^{p-1}xy$. The construction 11.1a also leads to new examples of limit varieties.

Example 18.3. Suppose p is a prime number, G is a group of order p, S is an r.s.m.t. over G with sandwich matrix $\begin{pmatrix} e & g \\ e & e \end{pmatrix}$, where g is a nonidentity element in G, W is the set of all words of the form $x_1^p x_2^p \dots x_n^p$. Then the variety varT(S, W) is a limit variety.

Examples 18.2 and 18.3 show in particular that the set of nongroup limit varieties of semigroups is infinite. Its cardinality is unknown.

Question 18.2. Is the set of limit varieties of semigroups countable?

For completeness we mention that no explicit examples of limit varieties of groups nor the cardinality of the set of such varieties is now known. The corresponding questions were noted in [44], Problem 4.46. But nongroup limit varieties of inverse semigroups was completely described in [31].

Theorem 18.1. The only nongroup limit variety of inverse semigroups is var B_2^1.

CHAPTER IV. IDENTITIES OF SEMIGROUPS OF SEVERAL TYPES

As follows from what was mentioned in the introduction, an important direction is the analysis of identities of semigroups of some or other concrete types which play an essential role in the theory of semigroups. For a fixed type Θ fundamental problems are formulated here:

1) explain under what conditions the type Θ has nontrivial s.i.;

2) find a basis of identities for a given (arbitrary or distinguished by

some reasonable test) semigroup of type Θ or characterize the system of all its identities;

3) describe those semigroups of type Θ which have a finite basis of identities.

It is understood that in specific situations other additional questions may arise; on the other hand, not all three indicated problems will be of interest for each type Θ. This chapter surveys results related to these problems in connection with the following types of semigroups: groups, r.s.m.t., f.d. semigroups, semigroups of transformations, and several others. It is natural that open questions noted in this chapter are as a rule specializations of the problems 1)-3) for the corresponding cases.

§19. GROUPS

In this section we are concerned with s.i. of groups. We note that the consideration of such identities, which is a natural part of our program, is also of essential interest from the point of view of the theory of varieties of groups. Thus, our attention will focus on identities of signature $\langle \cdot \rangle$, while at the same time we will constantly encounter varieties of groups (in the signature $\langle \cdot, {}^{-1} \rangle$) which are not necessarily varieties of signature $\langle \cdot \rangle$, i.e., varieties of semigroups. In order to set off this circumstance we will denote varieties of groups in boldface type.

a) The existence of nontrivial s.i. In Mal'tsev [54], the first paper devoted to s.i. of groups, it was discovered that while each nilpotent group has a (nontrivial) s.i., for a free metabelian group of rank 2 this did not occur. The following general question naturally arises.

Question 19.1. Whic groups (or, what is the same, which varieties of groups) satisfy a nontrivial s.i.?

An obvious example of a variety of groups with an s.i. is the variety $\mathbf{B}_n$ of all groups of exponent $n \neq 0$. Moreover, it is clear that if the variety of groups $\mathbf{V}$ satisfies an s.i. then this is true also for the variety $\mathbf{VB}_n$. As already mentioned, in [54] it was shown that each nilpotent variety $\mathbf{N}$ has an s.i., consequently such an identity is also satisfied in any variety of the form $\mathbf{NB}_n$. It turns out that in a broad class of varieties of groups there are no other varieties with an s.i. The variety of groups is called an SC-variety if it is contained in a product of some number of varieties of solvable groups and varieties generated by a finite group. We denote the variety of all Abelian groups by $\mathbf{A}$ and the variety of all Abelian groups of exponent $n \neq 0$ by $\mathbf{A}_n$. The anwser to Question 19.1 in the class of SC-varieties is given by the following.

Theorem 19.1. For an SC-variety of groups V the following are equivalent:

(1) V satisfies a nontrivial s.i.;

(2) $V \subseteq NB_n$ for some nilpotent variety N;

(3) for no prime p is the variety A_pA contained in V.

Theorem 19.1 has not been published earlier but is obtained by a simple combination of known results. Thus, the implication (2) → (1) was argued above, the implication (3) → (2) was proved in [129]. In order to establish (1) → (3) we note that the variety A_pA contains an interlacing of a cyclic group of order p with an infinite cyclic group. Belyaev and Sesekin [11] showed that this interlacing contains a free subsemigroup of rank 2, and consequently there are no nontrivial s.i. in A_pA.

From Zorn's lemma it is easy to deduce that each variety of groups without an s.i. contains a minimal subvariety with the same property. Theorem 19.1 shows that in particular the varieties A_pA are such minimal varieties without s.i., or as we will say, varieties almost satisfying s.i. Other examples of varieties almost satisfying s.i. are unknown. In this connection the follwoing is fruitful.

Question 19.2. Are the varieties of groups almost satisfying an s.i. exhausted by the varieties A_pA for all possible prime p?

If the answer to question 19.2 were positive, we would obtain, in particular, that each n-Engle group satisfies a nontrivial s.i. This conjecture was discussed by Shirshov [103], where it was given the following more concrete form.

Question 19.3 (Shirshov). Suppose $u_1 \equiv xy,\ v_1 \equiv yx;\ \ldots;\ u_{m+1} \equiv u_m v_m,\ v_{m+1} \equiv v_m u_m$. It is true that for each natural number n an m can be found such that each n-Engel group satisfies the identity $u_m = v_m$?

In [103] it was shown that 2-Engle groups satisfy the identity $u_2 = v_2$, and 3-Engel groups satisfy the identity $u_3 = v_3$.

b) Semigroup bases of group identities.

Proposition 19.1. If a variety of groups satisfies an s.i. then it can be defined by a basis of s.i.

Proposition 19.1 was first published in [144] as a corollary of Theorem 2 in that paper. However, as mentioned in [131], this theorem is false. The truth of Proposition 19.1 was by the same token placed in doubt. Nevertheless, a modification of the argument in [144] proves this proposition. In essence Proposition 19.1 means that if V is a variety of groups such that the variety of semigroups $\mathfrak{V}$ generated by it is different from the class of all semigroups, then the class of all groups in $\mathfrak{V}$ coincides with V.

We shall now indicate a concrete basis for some important varieties having a basis of s.i. Suppose

$$X_0 \equiv x,\ Y_0 \equiv y;\ \ldots;\ X_{n+1} \equiv X_n z_{n+1} Y_n,\ Y_{n+1} \equiv Y_n z_{n+1} X_n;$$

Theorem 19.2 (Mal'tsev [54]). The identity $X_n = Y_n$ defines the variety N_n of all n-degree nilpotent groups in the class of all groups.

An independent (but later) similar semigroup basis for the variety N_n was found in [157]. It consists of the identity $X'_n = Y'_n$, where

$$X'_1 \equiv xy,\ Y'_1 \equiv yx;\ \ldots;\ X'_{n+1} \equiv X'_n z_n Y'_n,\ Y'_{n+1} \equiv Y'_n z_n X'_n;\ \ldots$$

This basis was rediscovered in [48], where it was also asserted that it has a minimal number of letter among all semigroup bases of the variety N_n. The latter assertion is true, however, only for $n \leq 2$. Indeed, it is known (see [75], 34.33) that for $n > 2$ the variety N_n can define a group identity over n letters, but from the algorithm of the proof the Proposition 19.1 it is possible to see that for the transition to a semigroup basis of identities we do not need to use additional letters.

In [103] it was shown that the variety of all 2-Engel groups is defined by the identity $xy^2x = yx^2y$, and the variety of all 3-Engel groups is defined by the identities

$$xy^2xyx^2y = yx^2yxy^2x,\quad xy^2xyxyx^2y = yx^2y^2x^2y^2x.$$

In [104] Shirshov found a semigroup basis for the variety of all groups, which he called nilpotent with respect to a fixed procession of real numbers. In [47] semigroup bases of identities were described for each variety of 2-degree nilpotent groups.

c) **The finite basis property of s.i. of groups.** Suppose V is a variety of groups, $\mathfrak{V}$ is the variety of semigroups generated by it. We shall say that s.i. of V are finitely based if $\mathfrak{V}$ is an f.b. variety. The following naturally arises.

Question 19.4. For which varieties of groups are their s.i. finitely based?

Clearly, Question 19.4 is meaningful only for varieties of groups satisfying a nontrivial s.i. but not for those which are varieties of semigroups. The first example of an f.b. variety of groups such that the s.i. do not have a finite base was presented by Isbell in [131]: the variety defined by the identity $x^2y^2 = y^2x^2$. However this example was rediscovered independently in [109] (see also [112]), where the same variety arose from entirely different arguments, namely as the variety generated by the group $(a, b \mid ab^2a = 1)$. A series of later analogous examples was found in [49] and [80]. However they all are very particular cases of the following general result of Sapir, originally published here.

Theorem 19.3. Suppose the variety of groups V is contained either in the variety N_n for some n or in the variety AL for some l.f. variety L, and suppose the s.i. of the variety V are finitely based. Then V is either Abelian or has a finite exponent.

Obviously each Abelian variety of groups generates an f.b. variety of semigroups. Furthermore, each variety of groups of finite exponent is a variety of semigroups, and therefore the question of the finite basis property of its s.i. is equivalent to the question of the finite basis property for its group identities. Thus if we consider that each subvariety of the variety N_n is finitely based (see [75], 34.14), we see that Theorem 19.3 gives a complete solution to Question 19.4 for nilpotent varieties of groups and answers this question "modulo groups" for subvarieties of varieties of the form AL. It would be very tempting to try to extend this theorem to subvarieties of varieties of the form N_nB_k, since by Theorem 19.1 this would give a solution to Question 19.4 in the class of SC-varieties "modulo groups." We formulate this as the following

Question 19.5 (Sapir). Is it true that each subvariety of the variety N_nB_k such that its s.i. are finitely based is either Abelian or has finite exponent?

§20. REES SEMIGROUPS OF MATRIX TYPE

a) R.s.m.t. over an arbitrary semigroup. By construction r.s.m.t. are described by completely simple and completely 0-simple semigroups. Hence the problem of defining with respect to identities of the semigroup S and matrix P identities of the semigroup M(S;I, Λ, P) acquires a special urgency. Here we show a method found by Sukhanov in [88] for constructing the set of all identities satisfyed in all possible r.s.m.t. over S with respect to the set eq S.

Suppose Σ is an arbitrary s.i. By means of the following procedure we construct the identities μ(Σ) with respect to Σ.

Step 1. Remove from Σ all identities such that the first or last letters of the right and left sides do not coincicde, and all identities such that the length of at least one of the sides is even. The set of remaining identities is denoted by Σ_1.

Step 2. Suppose u = v is an identity in Σ_1, and $L_1(L_2)$ is the set of all letters which are encountered in an odd (respectively even) number of places in u or v. We remove from Σ_1 all identities u = v for which $L_1 \cap L_2 \neq \emptyset$, and we denote the set of remaining identities by Σ_2.

Step 3. Suppose u = v is an identity in Σ_2, z is a letter not appearing in the word uv. We remove from Σ_2 all identities u = v such that there exist two occurrences of some letter x in the word uzv in an even number of places which the neighbors on the right or left of the letter at the first occurrence of x are different from the corresponding neighbors of the letter at its second occurrence. The set of remaining identities is denoted by Σ_3.

Step 4. For each identity u = v in Σ_3, remove all occurrences of letters which in u or v occur at even number of places. The set obtained is $\mu(\Sigma)$.

Proposition 20.1 (Sukhanov [88]). The set of all identities satisfied in all possible r.s.m.t. over a given semigroup S coincides with the set of identities $\mu(\text{eq } S)$.

b) Completely simple semigroups. Each completely simple semigroup is isomorphic to some r.s.m.t. over a group. This shows a priori that the analysis of identities of completely simple semigroups cannot bypass the consideration of identities of groups (another exhibition of the often cited relation between the theories of varieties of semigroups and groups). As is known, for the representation of a completely simple semigroup in the form of an r.s.m.t. the sandwich matrix can be assumed to be normalized, i.e., containing in some column and some row only elements equal to the identity of the structural group.

Suppose u = v is an identity. The following two conditions are obviously necessary for it to be satisfied in the semigroup $M(G;I, \Lambda, P)$.

(C1). The identity u = v is satisfied in the group G.

(C2). If the set I (respectively Λ) is not a singleton, then the first (respectively last) letters of the words u and v coincide.

Suppose t and w are words. We denote the number of occurrences of the word t in the word w as a subword by $l_t(w)$. For the case of a normalized matrix P a simpler necessary condition for satisfaction of the identity u = v in the semigroup $M(G;I, \Lambda, P)$ was found in [63]. The condition is the following.

(C3). If both the sets I and Λ are not singletons, then for each word t of length 2 and for each element a of the matrix P in G the identity $a^{l_t(u)} = a^{l_t(v)}$ holds.

We note that if we do not require that P be normalized, then (C3) ceases to be necessary in general. In the general case conditions (C1)-(C3) are not sufficient. Indeed, consider the r.s.m.t. over the alternating group of fourth order with sandwich-matrix $\begin{pmatrix} e & e & e \\ e & (123) & (124) \end{pmatrix}$ (e is the identity of the group). Then as is easily verified, the identity $(x^6y^6z^6t^6)^3 = (xt)^6$ satisfies conditions (C1)-(C3) in relation to this semigroup, but the identity is not satisfied in the semigroup (for example, when

$$x = (e;\ 2,\ 1),\ z = (e;\ 3,\ 1),\ y = t = (e;\ 1,\ 2)).$$

On the other hand, a direct computation shows that if conditions (C1)-(C3) are satisfied for the identity with respect to the semigroup M(G;I, Λ, P) such that elements of the (not necessarily normalized) sandwich matrix P lie in the center of G (not necessarily a group!), then this identity is true in M(G;I, Λ, P). We therefore obtain the following

Proposition 20.2 (Mashevitskii [63]). In order that the identity u = v be true in the semigroup M(G;I, Λ, P), where P is a normalized matrix over the center of the group G, it is necessary and sufficient that conditions (C1)-(C3) be satisfied.

We note that the semigroup M(G;I, Λ, P) such that the elements of P lie in the center of G is a central Clifford semigroup in the sense of §8, and conversely, if the central completely simple semigroup is isomorphic to the r.s.m.t. over the group G with normalized sandwich matrix P, then the elemenets of P will be central in G. From Proposition 20.2 it follows easily that if the elements of the normalized sandwich matrix lie in the center C of the group G then

$$\operatorname{var} M(G;\ I,\ \Lambda,\ P) = \operatorname{var} G \vee \operatorname{var} M(C;\ I,\ \Lambda,\ P).$$

This reduces the problem of analyzing identities of central completely simple semigroups to the analogous problems for groups and for completely simple semigroups over Abelian groups. The latter problem was considered by Fortunatov [99], Mashevitskii [62,63], Rasin [177,176], and Kim and Roush [140]. In order to compactly describe the corresponding results, we introduce notation. We denote the supremum of the orders of elements of the matrix P (of the group G) by exp P (exp G) if such a number exists, and by zero otherwise.

The most complete description of bases of identities of completely simple semigroups over Abelian groups exists in the "Clifford" signature $\langle\cdot, {}^{-1}\rangle$. For convenience, for an arbitrary element x of the Clifford semigroup we set $x^0 = xx^{-1}$.

Theorem 20.1 (Rasin [77,176]). Suppose G is an Abelian group, I and Λ are nonsingleton sets, P is a normalized Λ × I-matrix over G, exp G = n, exp P = m, where n and m are nonnegative integers. Then the identity $(xy)^n x = x$, $x^2yx = xyx^2$, $(x^ny^n)^m = (xy)^n$ forms a basis of identities in the semigroup M(G;I, Λ, P).

If the number n is different from zero, then the indicated identities can be considered as ordinary s.i. of the signature $\langle\cdot\rangle$. Theorem 20.1 remains true also in this case, i.e., these identities form a basis for s.i. of the semigroup M(G;I, Λ, P). However the case when exp G = 0 needs a separate treatment for this signature. If exp P = 0, then the answer is given by the following

Theorem 20.2 (Kim and Roush [140]). A basis of identities of the semigroup M(G; I, Λ, P), where G is an Abelian group, P is a normalized matrix over G and exp P = 0, is formed by the identities

$$xz_1yz_2xz_3y = xz_3yz_2xz_1y,\quad xz_1yxz_3y = xz_3yxz_1y,\quad xz_1yz_2xy = xyz_2xz_1y,$$
$$xz_1yxy = xyxz_1y,\quad xz_1xz_2x = xz_2xz_1x,\quad x^2z_1x = xz_1x^2.$$

The case when exp G = 0 and exp P ≠ 0 remains open except for the trivial case when exp P = 1 (i.e., when the corresponding completely semigroup is a square group).

We shall now show how it is possible to find bases of identities of central completely simple semigroups. Suppose G is a group, $\exp G = n \neq 0$ and $\{w_\gamma = e\}_{\gamma\in\Gamma}$ is a basis of identities of the group G. Since $x^{n-1} = x^{-1}$ for any $x \in G$, it is possible to assume that in the list of the words w_γ there are only positive powers of letters, i.e., in other words w_γ are semigroup words. We fix an arbitrary letter x and to each word w_γ we associate the word w_γ^x obtained by substituting the word x^nzx^n for each letter z of the word w_γ.

Theorem 20.3. Suppose $\{w_\gamma = e\}_{\gamma\in\Gamma}$ is a basis of identities of the group G, I and Λ are nonsingleton sets, P is a normalized Λ × I-matrix over the center of the group G, $\exp G = n \neq 0$, $\exp P = m$. Then the identities

$$w_\gamma^x = x^n\ (\gamma\in\Gamma),\quad (xy)^n x = x,\quad (x^ny^n)^m = (xy)^n,\quad x^ny^nx = xy^nx^n$$

form a basis of identities of the semigroup M(G;I, Λ, P).

Theorem 20.3 is obtained by a simple combination of results of Mashevitskii [67] and Jones [133] (or Petrich and Reilly [165]).

c) Completely 0-simple semigroups with adjoined zero.

The simplest type of completely 0-simple semigroups is the semigroup of the form S^0, where S is a completely simple semigroup. From results of Mel'nik [68] it follows that $\mathrm{var}\, S^0 = \mathrm{var}\, S \vee \mathfrak{S}$, where $\mathfrak{S}$ is the variety of all semilattices. In the language of identities this means that eq S^0 consists precisely of all homotypic identities in eq S. In [68] it was shown how a basis of identities of the variety $\mathfrak{B} \vee \mathfrak{S}$ is constructed from a basis of identities of an arbitrary heterotypic variety $\mathfrak{B}$. However in the particular case when $\mathfrak{B}$ is a variety of completely simple semigroups, a more elegant construction given by Petrich in [162] can be used.

d) R.s.m.t. with 0. The semigroups A_2 and B_2. In what follows we will consider completely 0-simple semigroups such that zero is not adjoined. The least possible number of elements in a completely 0-simple semigroup with this property is five. To within isomorphism there exist two five-element completely 0-simple semigroups with divisors of zero, in particular the semigroups A_2 and B_2 (see §3). Bases of identities of these semigroups are described by the following

Theorem 20.4 (Trakhtman [94,95]). A) The identities

$$x^2 = x^3,\ xyx = xyxyx,\ xyxzx = xzxyx$$

form a basis of identities of the semigroup A_2. B) The identities

$$x^2 = x^3,\ xyx = xyxyx,\ x^2y^2 = y^2x^2$$

form a basis of identities of the semigroup B_2.

We note that earlier in [26] it was proved that the identities introduced in B) form a basis in the set of all identities of not more than two letters satisfied in B_2.

The following characterizes the system of all identities of the semigroup A_2.

Proposition 20.3 (Mashevitskii [63]). The identity u = v is satisfied in the semigroup A_2 if and only if the words u and v have the same first letters, the same last letters and satisfy the condition

(C4). The words u and v have the same set of subwords of length 2.

The description of all identities of the semigroup B_2 can be extracted from results of [64]; however it is cumbersome and will not be done here.

The semigroup B_2 is inverse. Bases of its inverse identities were found by Kleiman in [143], where the following general result was proved.

Theorem 20.5. Suppose $u_1 = u_2$ is an identity satisfied in B_2 but not satisfied in B_2^1, and $v_1 = v_2$ is an identity satisfied in B_2 but not satisfied in any nontrivial group. Then the identities $u_1 = u_2$, $v_1 = v_2$ form a basis of inverse identities for the semigroup B_2.

From Theorem 20.5 is follows in particular that the identities in assertion B) of Theorem 20.4 will also be a basis for inverse identities of the semigroup B_2. Among others, this fact is very essential for the proof of Theorem 20.4.

Another, evidently the simplest, basis found in [143] consists of the one identity $xy^2x^{-1} = xyx^{-1}$.

It was very recently found that var B_2 is a proper subvariety in var A_2 (Karnofski [138]). This clearly follows from a comparison of assertions A) and B) of Theorem 20.4.

e) B_2-semigroups. We call a completely 0-simple semigroup a B_2-semigroup if it contains divisors of zero but does not contain subsemigroups isomorphic to A_2. (In [64] such semigroups were considered under the name "semigroups of type 7.") A $\Lambda \times I$-matrix P over a group with zero is called cell-diagonal if the sets Λ and I can be divided into nonempty subsets $\Lambda_1, \ldots, \Lambda_k, I_1, \ldots, I_k$ respectively so that for all $\lambda \in \Lambda_m$, $i \in I_n$, where $m \neq n$, $p_{\lambda i} = 0$. The submatrices of P formed by intersection of rows with numbers in Λ_m and columns with numbers in I_m are called cells of P. Clearly, the semigroup $M^0(G; I, \Lambda, P)$ will be a B_2-semigroup if and only if P is a cell-diagonal matrix such that the cells do not contain zeros. An r.s.m.t. over G with these cells as sandwich matrices are completely simple semigroups, and in those cases when a description of identities of completely simple semigroups is known the identities of B_2-semigroups are described in a similar form. For the case when nonzero elements of P lie in the center of G such a description is given in [64].

The most important class of B_2-semigroups for the theory of semigroups consists of Brandt semigroups, i.e., semigroups of the form $M^0(G, I, I, \Delta)$, where Δ is the I × I identity matrix. From [26] or [143] it follows that the variety generated by an arbitrary Brandt semigroup with divisors of zero over the group G^0 coincides with var $G \vee$ var B_2. We deal with the question of a basis of identities for this variety. By d) we need to consider only the case when the group G is nontrivial.

Brandt semigroups are inverse semigroups, moreover they are completely 0-simple inverse semigroups. A basis of inverse identities of an arbitrary Brandt semigroup was found in [143], where the following general result was proved.

Theorem 20.6. Suppose B is a Brandt semigroup with divisors of zero over the group G. If the identity $w_1 = w_2$ is satisfied in B but not in B_2^1, and the collection of identities $(u_\gamma = v_\gamma)_{\gamma \in \Gamma}$ is satisfied in B_2 and defines the variety var G in the class of groups, then

$$\text{var } B = \text{var } \{w_1 = w_2,\ u_\gamma = v_\gamma\ (\gamma \in \Gamma)\}.$$

From Theorem 20.6 it follows in particular that if the system of identities $(w_\gamma = e)_{\gamma \in \Gamma}$ forms a basis of identities of the group G, then the following is a basis of identities for the semigroup B:

$$(xy^2x^{-1})(xy^2x^{-1})^{-1} = (xyx^{-1})(xyx^{-1})^{-1}, \quad w_\gamma^2 = w_\gamma \ (\gamma \in \Gamma).$$

Using the idea of [143] it is possible to obtain a description of a basis of s.i. of the variety $\mathrm{var}\, G \vee \mathrm{var}\, B_2$ in the case when exp G ≠ 0.

Theorem 20.7 (Volkov [21]). Suppose B is a Brandt semigroup with divisors of zero over the group G and exp G = n ≠ 0, 1. If the set of identities $\{w_\gamma = e\}_{\gamma\in\Gamma}$, where w_γ are semigroup words, forms a basis of identities of G, then the following is a basis of identities for the semigroup B:

$$w_\gamma^2 = w_\gamma \ (\gamma \in \Gamma), \ x^n = x^{n+2}, \ x^n y^n = y^n x^n, \ xyx = (xy)^{n+1} x.$$

A basis of s.i. of the variety $\mathrm{var}\, G \vee \mathrm{var}\, B_2$ for exp G = 0 is unknown even in the simplest special case, when G is an infinite cyclic group. We formulate the corresponding question

Question 20.1. What is a basis of identities of the Brandt semigroup with divisors of zero over an infinite cyclic group?

f) A_2-semigroups. A completely 0-simple semigroup containing a subsemigroup isomorphic to A_2 is called an A_2-semigroup. Clearly, each identity u = v satisfied in an A_2-semigroup $\mathcal{M}^0(G; I, \Lambda, P)$, satisfies the conditions (C1), (C2) and (C4). However (C3) in the general case is not necessary even for a normalized matrix P. If the matrix P is locally standard, i.e., if for any $i \in I$, $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$ there exist $j \in I$ and $\mu \in \Lambda$ such that $p_{\lambda j} = p_{\mu i} = p_{\mu j} = e$ (e is the identity in G), then condition (C3) is necessary. In this case the identities of r.s.m.t. with 0 and with central sandwich matrices were described by Mashevitskii in [65]: the identity u = v is satisfied in the A_2-semigroup $\mathcal{M}^0(G; I, \Lambda, P)$ with locally standard matrix P over the center of G if and only if it satisfies conditions (C1)-(C4). We note that not every completely 0-simple A_2-semigroup is isomorphic to an r.s.m.t. with 0 with a locally standard sandwich matrix. In [65] a condition was presented which is a weakening of (C3) and which is necessary for the identity u = v to be satisfied in a given A_2-semigroup. However, as shown in the same paper, this condition together with (C1), (C2) and (C4) is not in general sufficient even for semigroups with Abelian maximal semigroups. Thus the following remains unsolved.

Problem 20.1. Describe the set of all identities of an arbitrary completely 0-simple semigroup over a nontrivial Abelian group.

We proceed to an examination of bases of identities of A_2-semigroups. Here the following general result holds, originally published here.

Theorem 20.8 (Volkov). Suppose G is a group of finite exponent, S is an r.s.m.t. with 0 and with sandwich matrix P, H is a subgroup in G generated by the nonzero elements of P. If $A_2 \in \operatorname{var} S$, and $G \notin \operatorname{var} H$, then the semigroup S does not have a finite basis of identities.

Theorem 20.8 is obtained by an application of Theorem 3.2 to the semigroup S.

If (in the notation of Theorem 20.8) the matrix P is normalized and $G \in \operatorname{var} H$, then the question of finiteness of a basis of identities for S still remains open even under the assumption that S is finite (with the exception of the case when the group C is trivial: in this case S is equationally equivalent to A_2 and is therefore finitely based). We distinguish the case when G = H, i.e., when the group G is generated by the nonzero elements of the matrix P. As is known, for a completely 0-simple semigroup S with normalized sandwich matrix the given condition is equivalent to the fact that S is generated by the set of its own idempotents. Therefore the previous question can be reformulated in this case as follows:

Question 20.2. Does a finite completely 0-simple idempotent generated A_2 semigroup over a nontrivial group have a finite basis of identities?

g) Varieties of n-testable semigroups. The identity u = v is called n-testable if the initial intervals of length n, the final intervals of length n and the sets of all subwords of length n coincide in the words u and v. For each n we consider the variety $\mathfrak{T}_n$, defined by all n-testable identities. This is called the variety of n-testable semigroups. This class of varieties was introduced in [190,118,151] in connection with several problems in the theory of formal languages./

By comparing the definition of $\mathfrak{T}_n$ and Proposition 20.3 we see that the variety $\mathfrak{T}_n$ for n ≥ 2 in a known sense is related to the variety var A_2. This a priori analogy is also verified a posteriori: thus, for example, it turns out that each semigroup in $\mathfrak{T}_n$ $(n > 2)$ is an ideal extension of some semigroup in var A_2 by means of an n-nilpotent semigroup [183]. The problem of finding a basis of identities for the variety $\mathfrak{T}_n$ when n ≥ 2 was posed by the first author in 1976 in the author's lectures on periodic semigroups at Ural University. The solution to the problem is as follows.

Theorem 20.9 (Trakhtman [183]). A basis of identities for the variety $\mathfrak{T}_n$ when n ≥ 2 is formed by the identities $t_1x_1 \dots x_{n-1}yx_1 \dots x_{n-1}zx_1 \dots x_{n-1}t_2 = t_1x_1 \dots x_{n-1}zx_1 \dots$ $\dots x_{n-1}yx_1 \dots x_{n-1}t_2$, $(x_1 \dots x_r)^{m+1}x_1 \dots x_p = (x_1 \dots x_r)^{m+2}x_1 \dots x_p$, where $r \in \{1, \dots, n\}$, m is the quotient and p is the remainder of n - 1 upon division by r.

In conclusion we note that a basis of identities for the variety $\mathfrak{T}_1$ was found earlier in [191]: it is formed by the identities $x^2 = x$, $xyzx = xzyx$.

21. FINITELY DEFINED SEMIGROUPS

a) The free semigroup problem. The first question which arises in the analysis of f.d. semigroups from the point of view of the theory of varieties is the question of in what semigroups are there (nontrivial) s.i. As follows from results of Markov [57], in the general case this question is algorithmically unsolvable, i.e., it is impossible to give an answer in terms of "recognizable" combinatoric properties of defining relations. More urgent is the problem of finding algebraically equivalents to the property of "having s.i." To date, perhaps the only candidate for the role for such an equivalent is the property "does not contain noncylcical free subsemigroups." This property is obviously necessary, but its sufficiency is a very intriguing open question.

Question 21.1. Does each f.d. semigroup not satisfying any s.i. contain a noncyclical free subsemigroup?

It is known that under certain restrictions on the defining relations the answer to this question is positive. Below we present results of this type, and we then establish conditions which guarantee the presence of noncyclical free subsemigroups, and consequently the absence of s.i., for a given f.d. semigroup.

Proposition 21.1 (Shneerson [108]). If the semigroup S with n generators is defined by m defining relations and $n - m \geq 2$ then S contains a noncyclic free subsemigroup.

Suppose the semigroup S is defined by some system of defining relations $u_i = v_i$ $(i = 1, \ldots, m)$. The word w is called multiple with respect to this system if it appears as a subword in a defining word at least two times, i.e., if words $u, v \in \{u_1, v_1, \ldots, u_m, v_m\}$ (possibly $u \equiv v$) can be found such that $u \equiv u'wu''$, $v \equiv v'wv''$, where $u' \not\equiv v'$ or $u'' \not\equiv v''$.

Proposition 21.2 (Aleksandrov [8]). If the system of defining relations of a noncyclic f.d. semigroup S is such that no defining word is a product of two multiple (with respect to this system) words, then S contains a noncyclic free subsemigroup.

A particular case of Proposition 21.1 is the earlier result of [25].

In conclusion we note that for finitely generated semigroups the answer to the question analogous to Question 21.1 is negative. The corresponding example is found in [111].

b) F.d. semigroups with nontrivial identities. In this point we present a description of f.d. semigroups with nontrivial identities in certain important classes of f.d. semigroups. We present notions and notation needed in what follows. A left (right) graph of a system of defining relations $u_1 = v_1, \ldots, u_m = v_m$ over the alphabet $\{a_1, \ldots, a_n\}$ is a graph with the set of vertices $\{a_1, \ldots, a_n\}$, in which the vertices a_i and a_j are joined by an edge if and only if for some k a_i and a_j are the first letters on the left (right) of the words u_k and v_k respectively. If the semigroup S can be defined by a system of defining relations such that the left (right) graph has no cycles, we shall say that S has no left (right) cycles. This class of f.d. semigroups was first introduced and analyzed by Adyan in [1], see also his book [3]. If w is a word over the alphabet $\{a_1, \ldots, \ldots, a_n\}$, then $w^{(i)}$ denotes the word obtained from w by deleting the letters $a_i, a_{i+1}, \ldots, a_n$.

Theorem 21.1. The following are equivalent:

(1) the semigroup S can be defined by n generators and m defining relations, where n > m, and satisfies a nontrivial s.i.;

(2) the semigroup S has no left or right cycles and satisfies a nontrivial s.i.;

(3) the semigroup S is isomorphic or anti-isomorphic to a semigroup of one of the following types: 1) $\langle a \rangle$; 2) $\langle a, b \mid ab = ba \rangle$; 3) $\langle a, b \mid ab = b^k \rangle$, $k = 1, 2, \ldots$; 4) $\langle a, b \mid aba = ba \rangle$; 5) $\langle a, b \mid aba_1 = _1 b \rangle$; 6) $\langle a, b \mid a^2 = b^2 \rangle$; 7) $\langle a, b \mid aba^2 = ba \rangle$; 8) $\langle a, b, c \mid a = cab, bc = c^2 \rangle$; 9) $\langle a, b, c \mid a = v, bc = w \rangle$, where $v \equiv ca$ or $v \equiv cac$, and w is an arbitrary word over the alphabet $\{a, c\}$; 10) $\langle a_1, \ldots, a_n \mid a_1 = v_1, \ldots, a_{n-1} = v_{n-1} \rangle$, where $v_i^{(i+2)} \equiv a_{i+1} a_i$ for $i = 2, \ldots, n-1$, $v_1^{(3)} \equiv a_2 a_1$ or $v_1^{(3)} \equiv a_2 a_1 a_2$; 11) $\langle a_1, \ldots, a_n \mid a_1 = v_1, \ldots, a_{n-2} = v_{n-2}, a_{n-1} a_n = v_{n-1} \rangle$, where $v_{n-1} \equiv a_n a_{n-1} a_n$ or $v_{n-2} \equiv a_n a_{n-2}$, the semigroup $\langle a_1, \ldots, a_{n-1} \mid a_1 = v_1^{(n)}, \ldots, a_{n-2} = v_{n-2}^{(n)}, a_{n-1} = v_{n-1}^{(n)} \rangle$ is isomorphic to some semigroup of type 10), and the system of relations

$$a_1 = v'_1, \ldots, a_{n-2} = v'_{n-2}, a_n = v'_{n-1},$$

where the word v'_i is obtained from v_i by deleting the letter a_{n-1}, has a connected left graph.

The implications (2) → (3) and (3) → (1) were proved in [107] (the case of semigroups with one defining relation was analyzed earlier in [105] and in [86]). The proof of the implication (1) → (2) is the subject of [110]. From results of [106,107,110] it follows that in addition condition (3) of Theorem 21.1 can be verified effectively, and consequently there exists an algorithm determining with respect to a given semigroup of n generators and m defining relations, where n > m, whether a nontrivial s.i. is satisfied in this semigroup. It was mentioned earlier that an algorithm deciding the same question for an arbitrary f.d. semigroup does not exist. In connection with Question

21.1 we also note that from the proof of Theorem 21.1 it follows that for semigroups with a number of generators greater than the number of defining relations the absence of nontrivial identities implies the presence of free noncyclic subsemigroups in them.

Another type of restriction on the defining relations was considered in [2] (see also [3]). A (not necessarily finite) semigroup with 1 is called special if in some (not necessarily finite) alphabet it can be defined by defining relations of the form w = 1.

Theorem 21.2 (Adyan [2,3]). A special semigroup which is not a group satisfies a nontrivial identity if and only if it is isomorphic to an infinite cyclic semigroup or the bicyclic semigroup $\langle a, b \mid ab = 1\rangle$.

For the particular case of semigroups with one defining relation it was shown in [106] that a special semigroup with one relation which is a group satisfies an s.i. if and only if it is isomorphic either to a cyclic group or the group $\langle a, b \mid ab^2a = 1\rangle$.

We recall the papers of Oganesyan [76] and Kolesnikova [41] where f.d. semigroups with certain other restrictions on the defining relations are studied from the point of view of the presence of nontrivial s.i.

c) Identities of noncyclic semigroups with one defining relation. By Theorem 21.1 a noncyclic semigroup with one defining relation admitting an s.i. is isomorphic or anti-isomorphic to one of the semigroups

$$S_1 = \langle a, b \mid ab = ba\rangle,\ S_{2,k} = \langle a, b \mid ab = b^k\rangle,\ k = 1, 2, \ldots,$$
$$S_3 = \langle a, b \mid aba = ba\rangle,\ S_4 = \langle a, b \mid aba = b\rangle,\ S_5 = \langle a, b \mid a^2 = b^2\rangle,$$
$$S_6 = \langle a, b \mid aba^2 = ba\rangle.$$

The question of bases of identities of these semigroups is answered by the following.

Theorem 21.3 (Shneerson [112]).* The identities

$$yx^2y = (xy)^2,\ yx^2zy = xyxzy,\ yxzxy = xyzxy,\ yxzxty = xyzxty$$

form a basis of identities of the semigroup $S_{2,k}$ for any k. The semigroups S_3, S_4, S_5, S_6 do not have finite bases of identities, and moreover S_4 and S_6 are equationally equivalent.

An analogous question is solved in [112] for noncylcic monoids with one defining relation. If such a monoid satisfies a nontrivial s.i. it is isomorphic or anti-isomorphic to one of the monoids

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*The original report on this result in [109] contained an inaccuracy.

$$S_1^1,\ S_{2,k}^1,\ S_3^1,\ S_4^1,\ S_5^1,\ S_6^1,\ S_7=\langle a,\ b\mid ab=1\rangle,\ S_8=\langle a,\ b\mid ab^2a=1\rangle$$

(see Theorem 21.2 and the remark following it).

Theorem 21.4 (Shneerson [112]). The semigroups S_1 and S_1^1, $S_{2,k}$ and $S_{2,k}^1$, S_4 and S_4^1, S_5 and S_5^1, S_6 and S_6^1 are equationally equivalent. The identity

$$x_1x_2x_3x_1x_4x_2x_5x_1=x_2x_1x_3x_1x_4x_2x_5x_1$$

forms a basis of identities for the semigroup S_3^1. The semigroups S_7 and S_8 do not have a finite basis of identities, and moreover S_8 is a group equationally equivalent to the semigroups S_4 and S_5.

We see in particular that a bicyclic semigroup does not have a finite basis of identities. In view of the large role which this semigroup plays in the structural theory, the following is of interest.

Question 21.2 (Petrich). Is there a "good" basis of identities for a bicyclic semigroup? In particular, is the axiomatic rank of the variety generated by it finite?

We note that by Proposition 4.1 a bicyclic semigroup has an irreducible basis of identities.

d) Identities of cyclic semigroups. For the sake of completeness we touch on identities of cyclic semigroups. Obviously, a basis of identitites for an infinite cyclic semigroup is the identity $xy = yx$.

Proposition 21.3. Suppose $C_{h,d}=\langle a\mid a^h=a^{h+d}\rangle$ is a finite cyclic semigroup of index h and period d. A basis of identities of the semigroup $C_{h,d}$ is formed by the identities $xy=yx$, $x^dx_1\ldots x_h=x_1\ldots x_h$, $x^d=y^d$, if $h\le d$ and

$$xy=yx,\ x^dx_1\ldots x_h=x_1\ldots x_h,\ x^ry^{r+d}x_1\ldots x_{h-3r-d}=x^{r+d}y^rx_1\ldots x_{h-3r-d},$$
$$r=0,\ 1,\ \ldots,\ \left[\frac{h-d}{3}\right],$$

if $d < h$.

Proposition 21.3 was obtained by the second author and is originally published here. Its proof is based on the description of a system of all identities of the semigroup $C_{h,d}$, following results of Lyapin [53].

Identities of monogenic inverse semigroups were studied in [143] and [31]. Monogenic inverse semigroups were first described by Gluskin in [24], using the improved classification of Ershova [28] by which an arbitrary monogenic inverse semigroup belongs to one of the following four types:

$$1)\ \langle a\mid a^n=a^{n+m}\rangle;\ 2)\ \langle a\mid a^{-1}a^n=a^na^{-1}\rangle;\ 3)\ \langle a\mid a^n=a^{-1}a^{n+1}\rangle;\ 4)\ \langle a\rangle$$

(the free monogenic inverse semigroup).

Proposition 21.4 (Kleiman [143,31]). An inverse semigroup of type 1) has a basis of identities $xy = yx,\ x = x^{m+1},\ xx^{-1} = yy^{-1}$ for n = 1 and $n = 1$ u $(x^{-1}y^{-1}xy)^2 = x^{-1}y^{-1}xy,\ x^2 = x^{2m},$ $xy^2x^{-1}(xy^2x^{-1})^{-1} = xyx^{-1}(xyx)^{-1}$ for n = 2. An inverse semigroup of type 2) has a basis of identities $xy = yx,\ xx^{-1} = yy^{-1}$ for n = 1 and

$$(x^{-1}y^{-1}xy)^2 = x^{-1}y^{-1}xy,\ xy^2x^{-1}(xy^2x^{-1})^{-1} = xyx^{-1}(xyx)^{-1}$$

for n = 2. Inverse semigroups of types 1) and 2) for n > 2, as well as types 3) and 4) do not have a finite basis of identities.

From Proposition 21.4 it follows in particular that the description of bases of identities of monogenic inverse semigroups given in [27] is false.

e) Identities of semigroups f.d. in varieties. Suppose $\mathfrak{B}$ is a variety of semigroups, S is a semigroup defined inside $\mathfrak{B}$ by a finite number of generators and defining relations. It is possible to pose a question of under what conditions does S have a nontrivial identity in $\mathfrak{B}$. Such a problem was solved for semigroups defined by one defining relation in the varieties $\mathrm{var}\{x^2 = xyx = 0\}$, $\mathrm{var}\{x^2 = x,\ xyz = xzy\}$ and $\mathrm{var}\{x_1 \ldots x_n = 0\}$ (Kolesnikova [38-40]).

§22. SEMIGROUPS OF TRANSFORMATIONS

Semigroups of transformations are a classical and at the same time in a certain sense the universal example of semigroups. In view of this the analysis of their abstract properties holds an important place in the general theory of semigroups. It is natural to consider these semigroups also from the point of view of the theory of varieties. Up to the most recent time, however, such consideration was not carried out in practice;* in particular, only recently was an answer to the question of finiteness of a basis of identities for complete semigroups of transformations given.

Suppose n is a natural number. We introduce the following notation: T_n is the semigroup of all transformations of an n-element set, MT_n is the semigroup of all multivalued transformations of an n-element set, PT_n is the semigroup of all partial transformations of an n-element set, B_n is the semigroup of all partial multivalued transformations (= binary relations) of an n-element set, I_n is the semigroup of all one-to-one partial transformations of an n-element

*In [83] an attempt was made to characterize the set of all identities of the semigroup T_n. However, the fundamental theorem of this paper is false. The identity

$$x_1x_2yzt_1t_2 = x_1x_2zyt_1t_2$$

satisfies all its conditions but is not satisfied in T_n for n > 1.

set, $LT_n(q)$ is the semigroup of all linear transformations of an n-dimensional vector space over a field of q elements.

Theorem 22.1 (Volkov [22]). The semigroup T_n for $n \geq 3$ and the semigroup MT_n, PT_n, B_n, $LT_n(q)$ for $n \geq 2$ do not have a finite basis of identities.

For the remaining n in Theorem 22.1 the semigroups have finite bases of identities: the semigroups T_1 and MT_1 are singletons, the semigroups B_1 and PT_1 are two-element, the semigroup $PT_1(q)$ is commutative and finitely based by Proposition 12.2, the semigroup T_2 is four-element and finitely based by Theorem 10.1.*

The following result is originally published here.

Theorem 22.2 (Sapir). The semigroup I_n for $n > 1$ does not have a finite basis of identities as a semigroup (in the signature $\langle\cdot\rangle$) nor as an inverse semigroup (in the signature $\langle \cdot, {}^{-1} \rangle$).

Obviously, for n = 1 the finite basis property holds: the semigroup I_1 is two-element.

Finally, we note that infinite semigroups of all transformation of any of the above indicated types either do not have nontrivial s.i. or are commutative. Thus, in an exhaustive manner Theorems 22.1 and 22.2 solve the finite basis question for the semigroups considered.

Theorem 22.1 is obtained by means of Theorem 3.2. We note that it follows also from Theorem 3.3. Theorem 22.2 is a consequence of Theorem 3.3.

In complete semigroups of transformations it is possible to distinguish the important subsemigroups consisting of all transformations formed by some or other property. For example, for each $m \leq n$ in the semigroup T_n it is possible to distinguish the subsemigroup $T_{m,n}$ consisting of all transformations of rank $\leq m$. Using Theorem 3.3 it is not difficult to deduce that for $m \geq 3$ the identities of the semigroup $T_{n,m}$ do not have a finite basis. On the other hand, for any n the semigroup $T_{n,1}$ is a semigroup of left zeros, and in particular is an f.b. semigroup. The following question naturally arises.

Question 22.1. Does the semigroup $T_{n,2}$ of all transformations of an n-element set of rank ≤ 2 have a finite basis of identities?

Torlopova [89] gave a characterization of the set of all identities of the semigroup $T_{n,2}$ for $n \geq 5$. Because of cumbersome details we do not present

*A concrete basis of identities for T_2 was found in [85]; it consists of the identities $x = x^3$, $(xy)^2 x = xy^2$. In [83] there is a characterization of the system of all identities of this semigroup.

it here. We only note that in particular it implies that this set does not depend on n, i.e., $\mathrm{eq}\,T_{5,2}=\mathrm{eq}\,T_{6,2}=\ldots$

Among subsemigroups of the semigroup $LT_n(q)$ (which it is convenient to think of as a semigroup of all n × n-matrices over a q-element field) the greatest interest evidently lies in the semigroup of all upper triangular matrices. At present almost nothing is known about its identities. In particular, the following is an open question:

Question 22.2. Does the semigroup of all upper triangular matrices over a finite field have a finite basis of identitites?

§23. MISCELLANY

In Sukhanov [87], for each variety of semigroups $\mathfrak{B}$ the set of all identities satisfied in the class of all possible bundles of some or other type (arbitrary, commutative, rectangular, ordinal sums) with components in $\mathfrak{B}$ were described. Identities of ordinal sums were studied also by Lyapin [53], where a method for construction is found for the set of all identities of a given ordinal sum with respect to known systems of identities for its components. By means of this result the sets of all identities of semigroups such that each subset is a subsemigroup, as well as finite holoidal semigroups were described. In [146] Lyapin characterized those identities of semigroups which are inherited in passing to a global subsemigroup.

The papers [98] and [97] are devoted to identities of semigroups of directed and strongly directed transformations of ordered sets. A basis of identitites of one very special semigroup of transformations of an ordered set was enumerated in [84].

A basis of identities for an arbitrary semigroup in which each subsemigroup is a one-sided ideal was found by Shutov in [113].

Added in proof: A (negative) answer to Question 22.1 is contained in [193].

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