Deterministically Isolating a Perfect Matching in Bipartite Planar Graphs

Joint work with

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Chennai Mathematical Institute

Outline

- Introduction

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- Introduction
- Historical Perspective

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- New Results

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- Open Questions

Perfect Matching

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- Min-Weight Perfect Matching: pm with min weight

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- Uniqueness: Is #pm's = 1?

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- Small Genus: Embeddable on surface of small genus

Known Results

	General	Bipartite	Planar	Bip Planar
Decision	$P \cap RNC$	$P \cap RNC$	NC	NC (SPL)
Search	$P \cap RNC$	$P \cap RNC$	$P \cap RNC$	NC (SPL)
Counting	#P	#P	NC	NC
Uniqueness	P	NC	NC	NC (SPL)

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- Parallel (Uniqueness): Hoang-Mahajan-Thierauf

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[Mahajan-Varadarajan]

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- Constant fraction of faces processed in parallel

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  - $e \in$ unique min-wt matching iff W increases in $G - e$

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  - Double Counting : extension of Inductive Counting.

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  - Clever use of [Mahajan-Vinay] algo for determinant

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- Deterministic Isolation possible in planar graphs (reachability).
- Can one deterministically isolate a perfect matching in planar graphs?

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- ([Allender-Reinhardt-Zhou]) Extract isolated matching

Reformulation of Problem

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Problem Give a weighing scheme such that $circ(C) \neq 0$ for every ("nice") cycle in grid graphs.

Signs of Blocks

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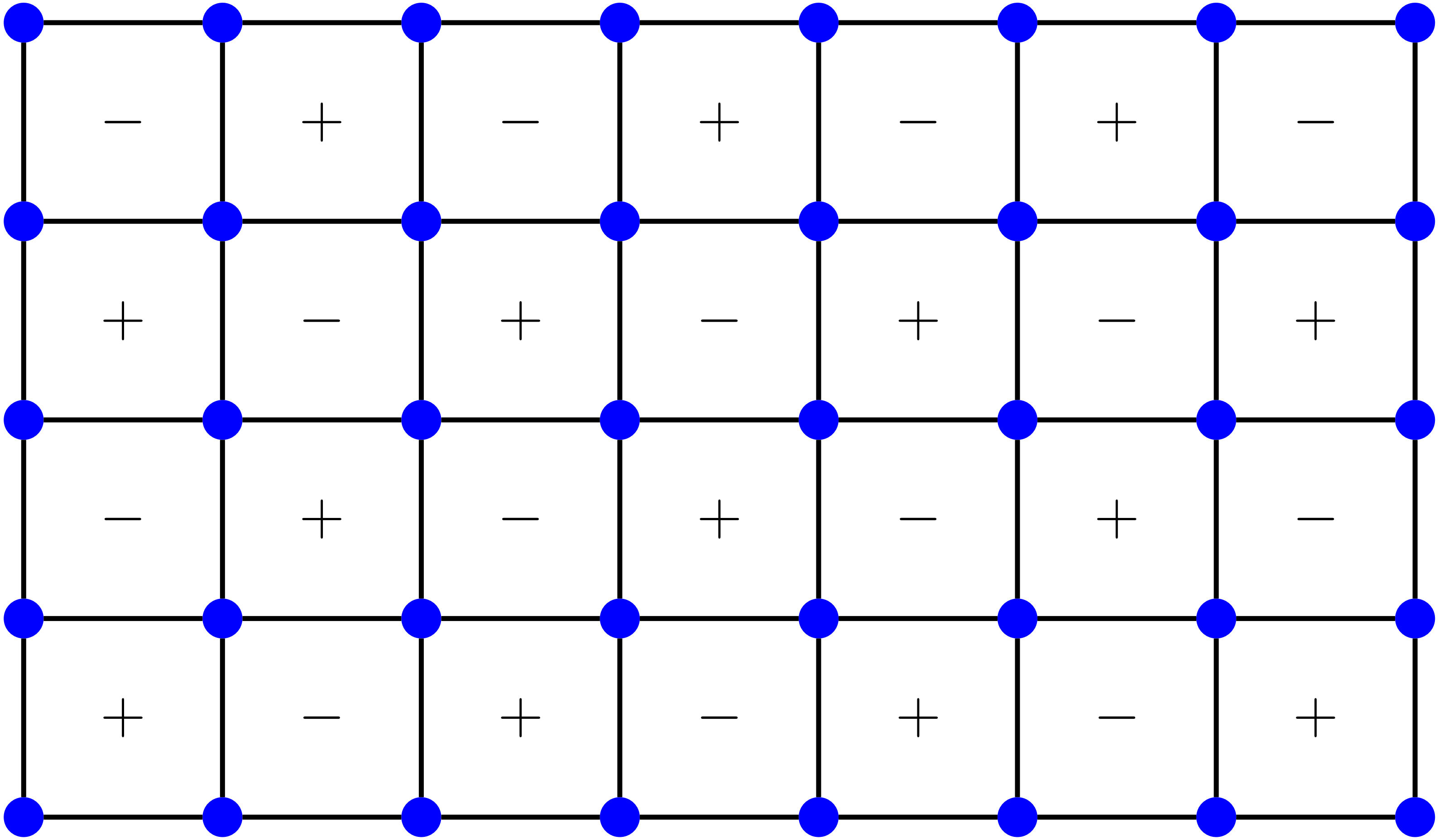

Signs of Edges

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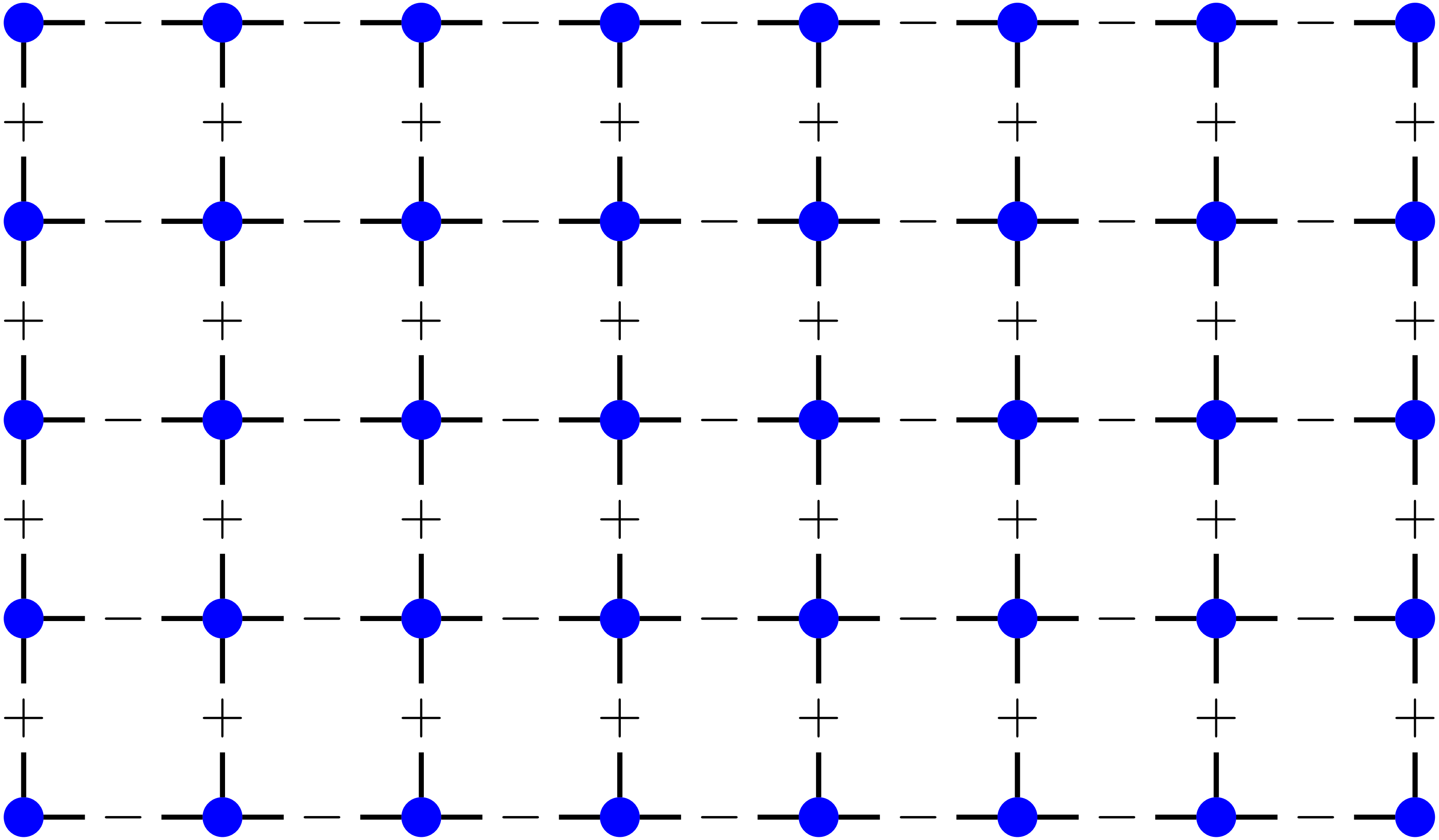

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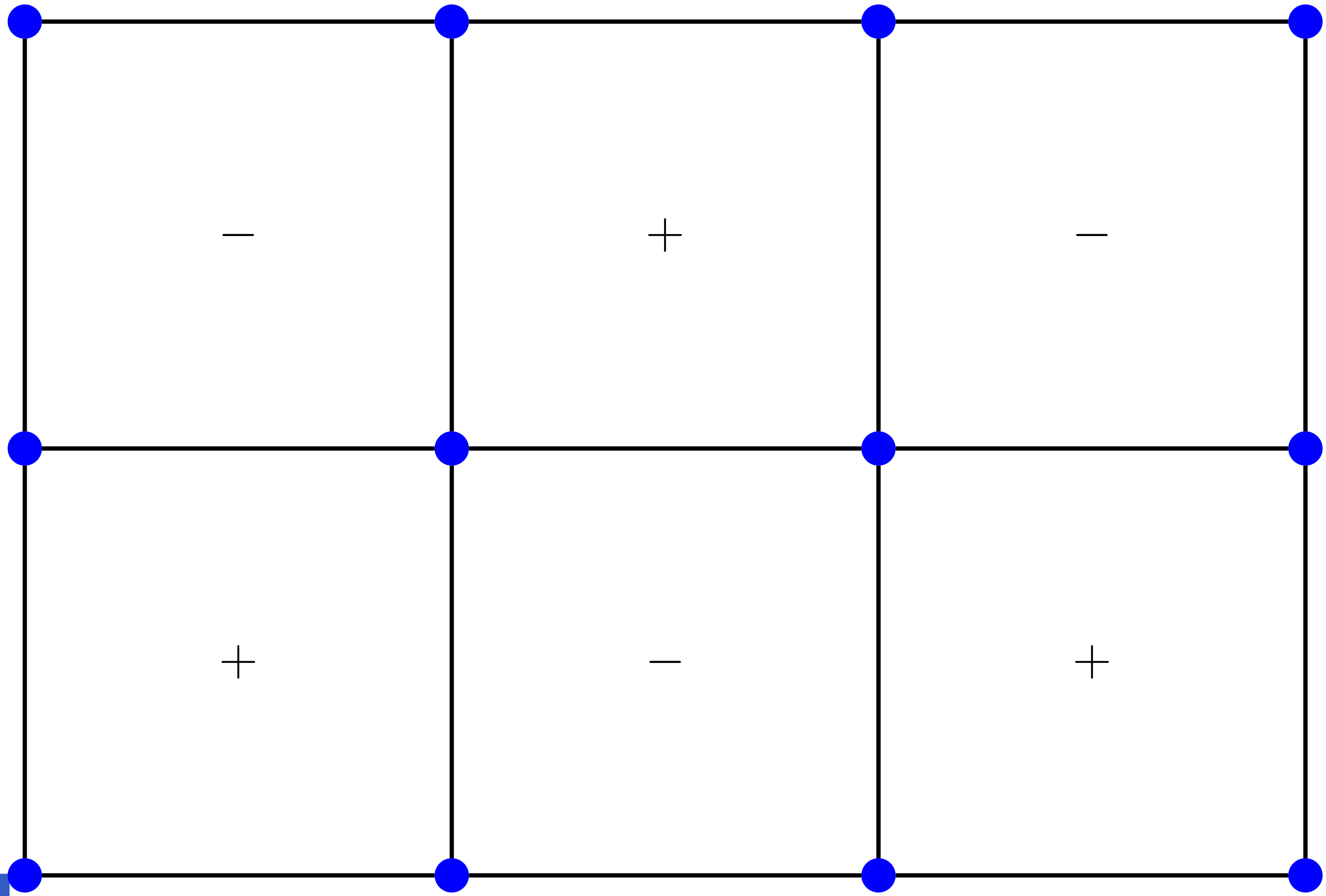
- For a cycle, $C = (e_0, e_1, \ldots, e_l)$ (e_0 leftmost-topmost edge)

$$circ(C) = \sum_{i=0}^{l} (-1)^i weight(e_i)$$

Block Circulations : Example



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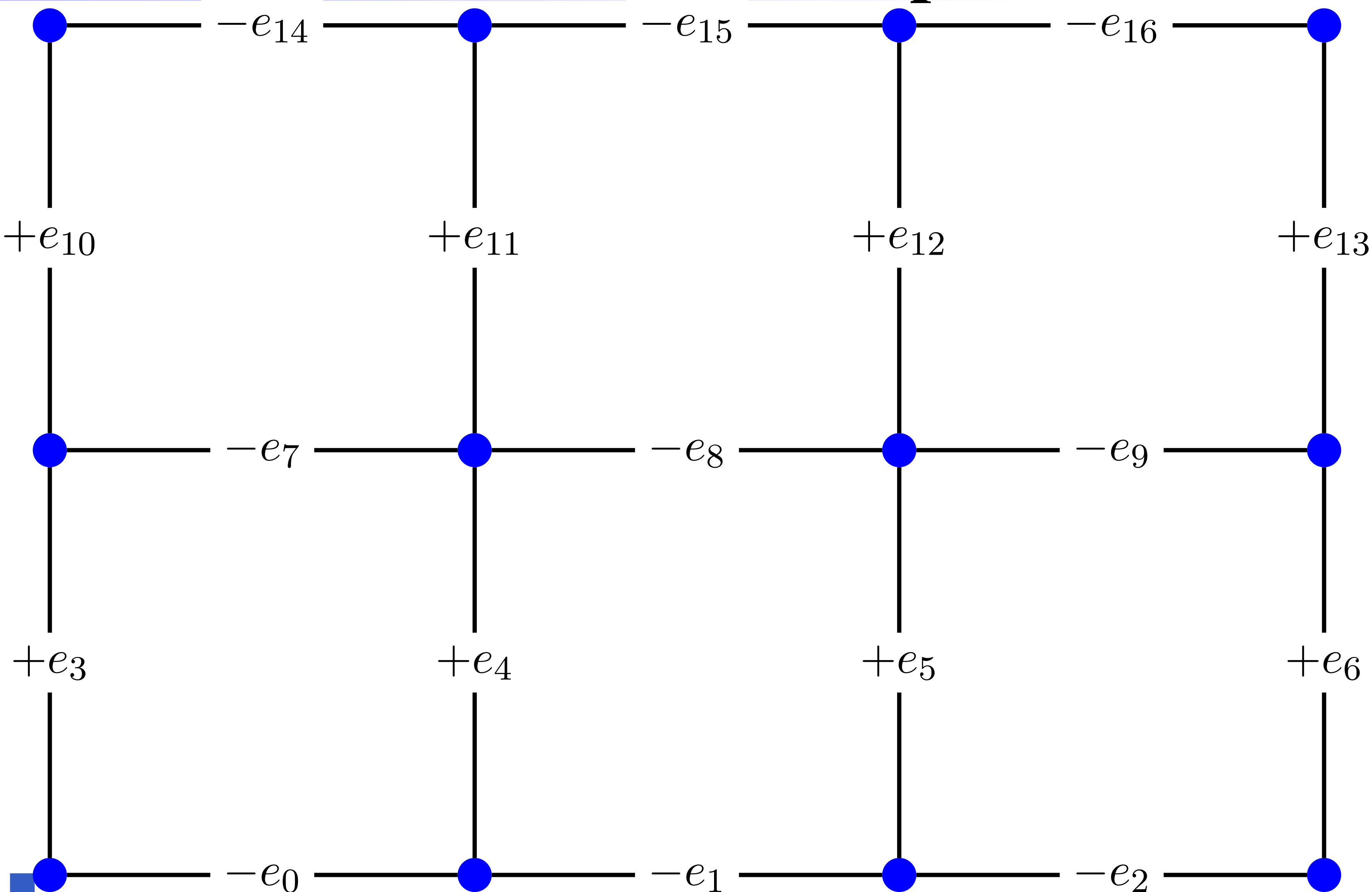

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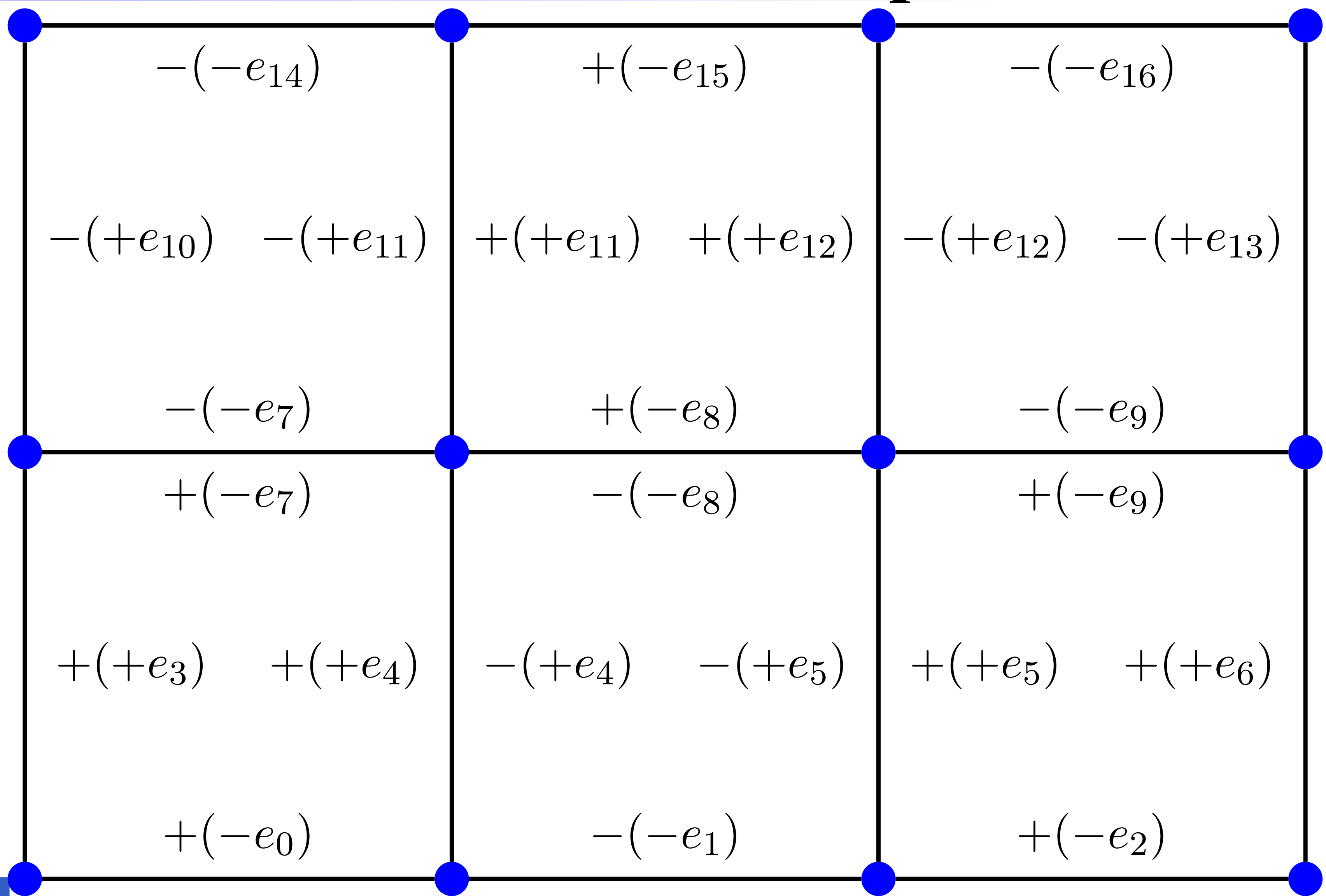

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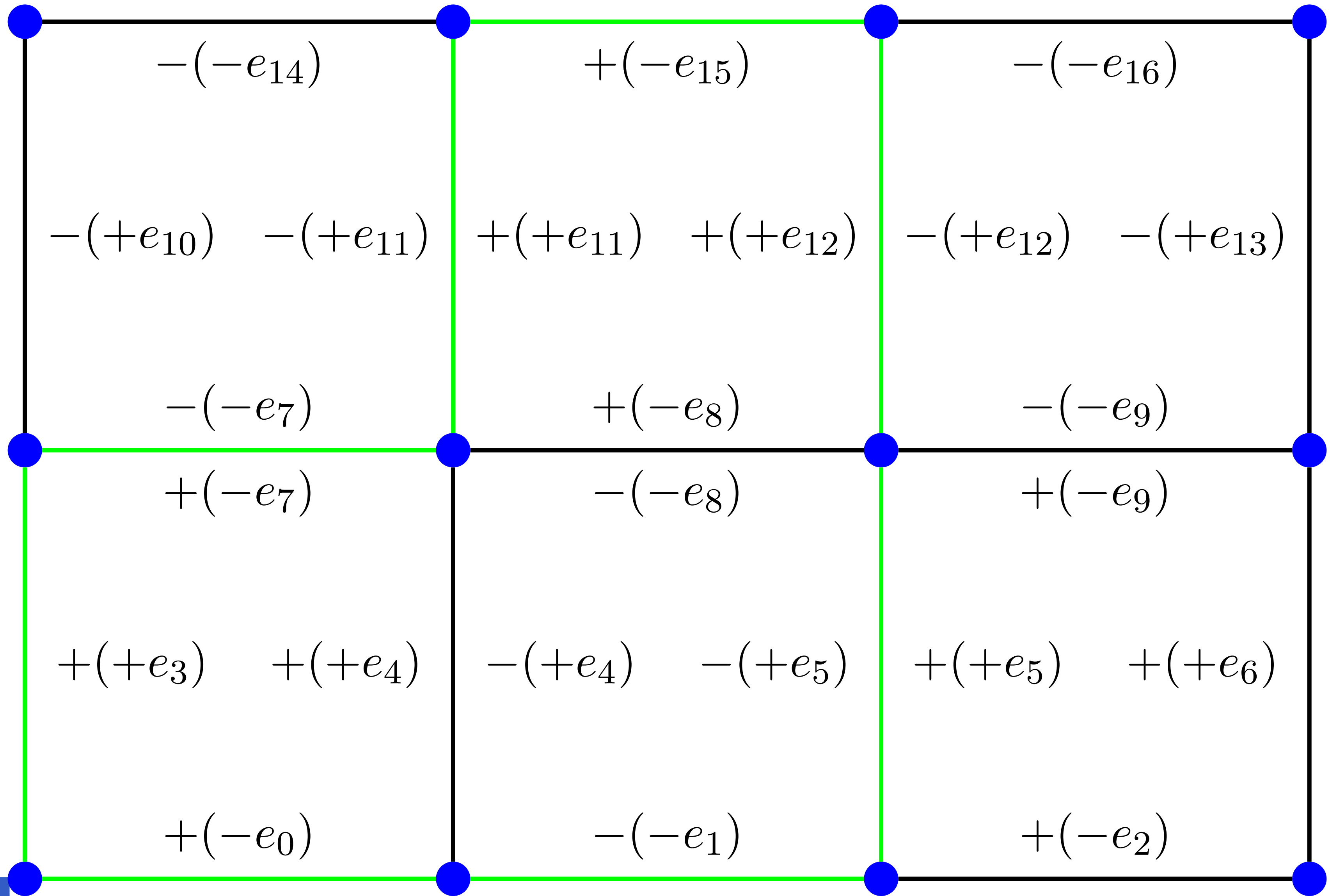

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Observation If $circ(B) = sign(B)$ for each B, $circ(C) \neq 0$.

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Hence Done!

Weighing Scheme

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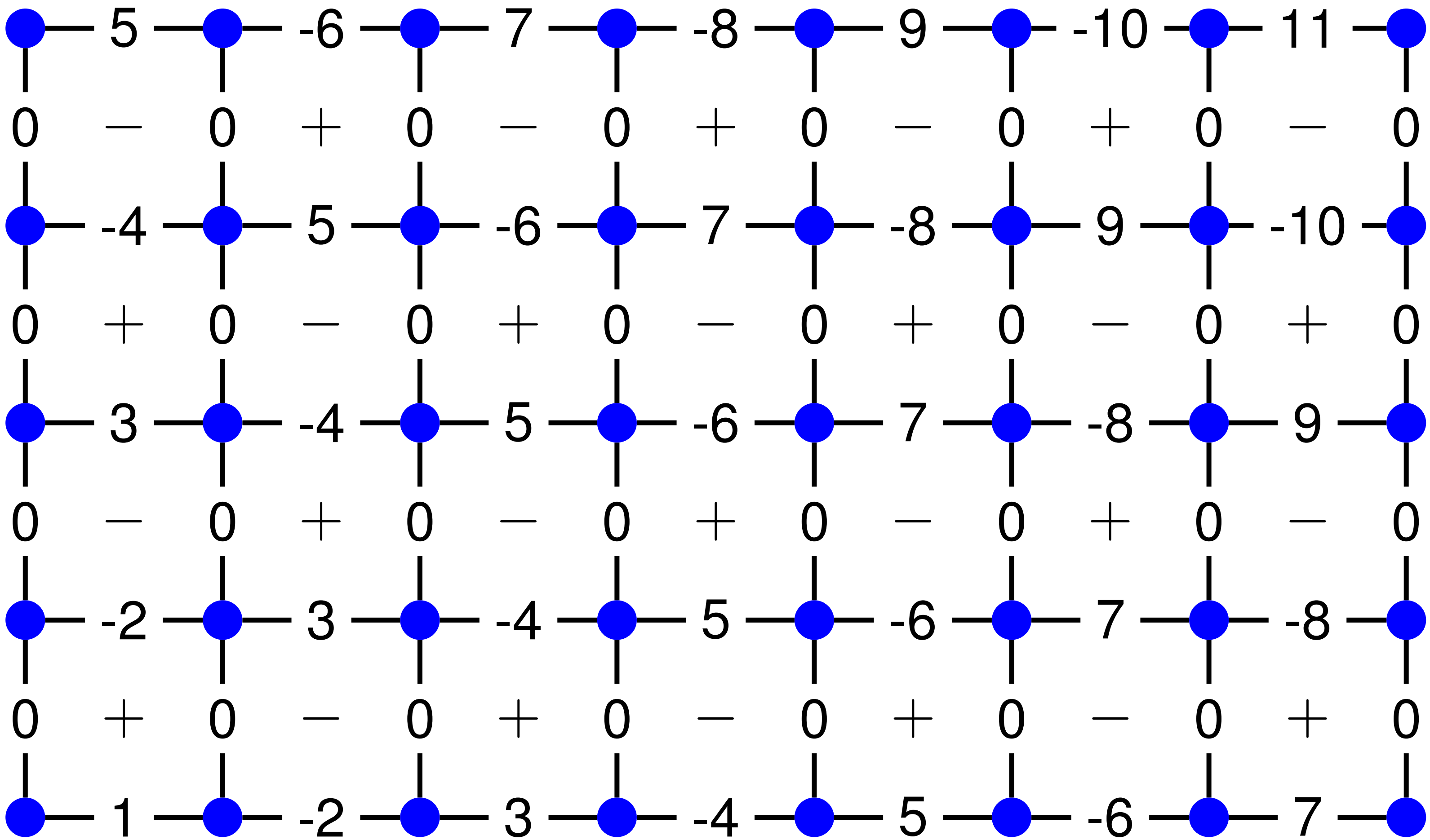

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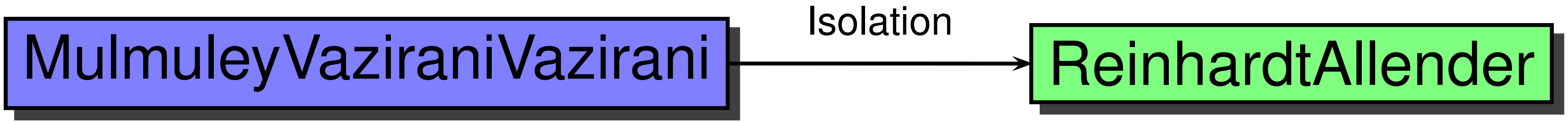
From Matching to Reachability and Back

MulmuleyVaziraniVazirani

From Matching to Reachability and Back



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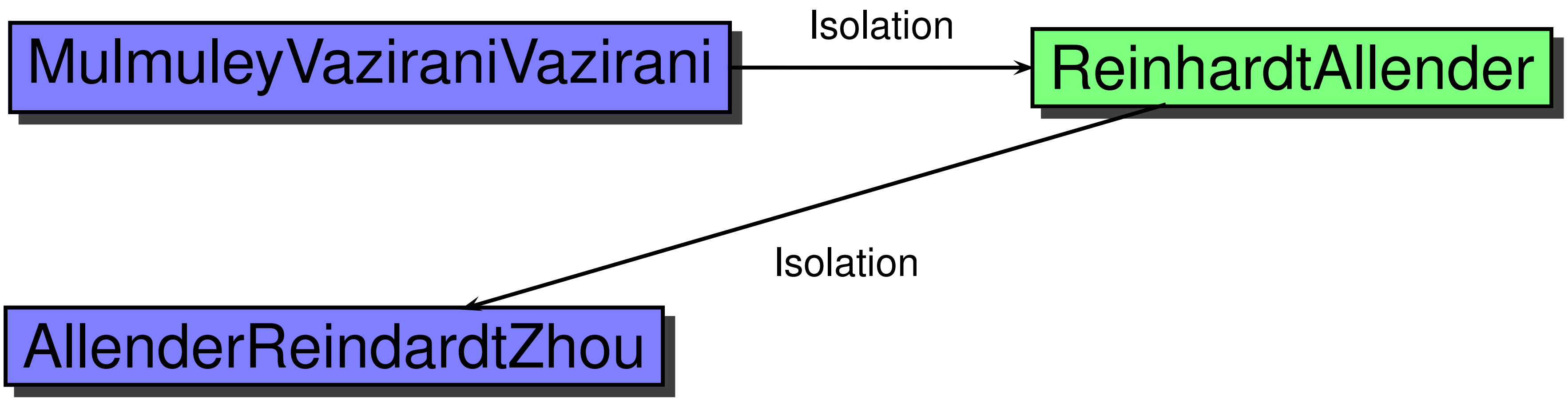

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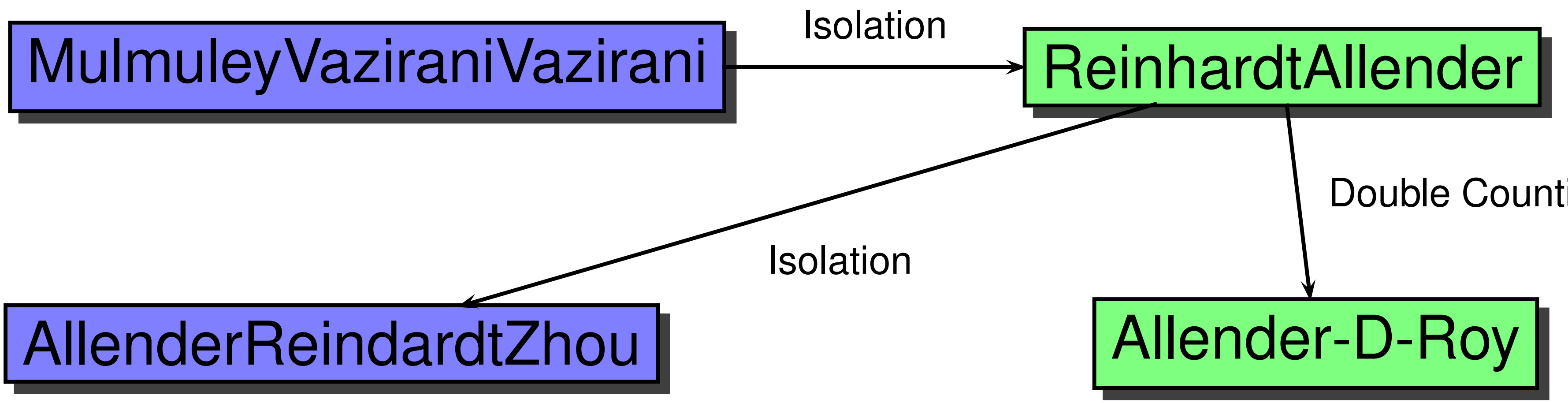

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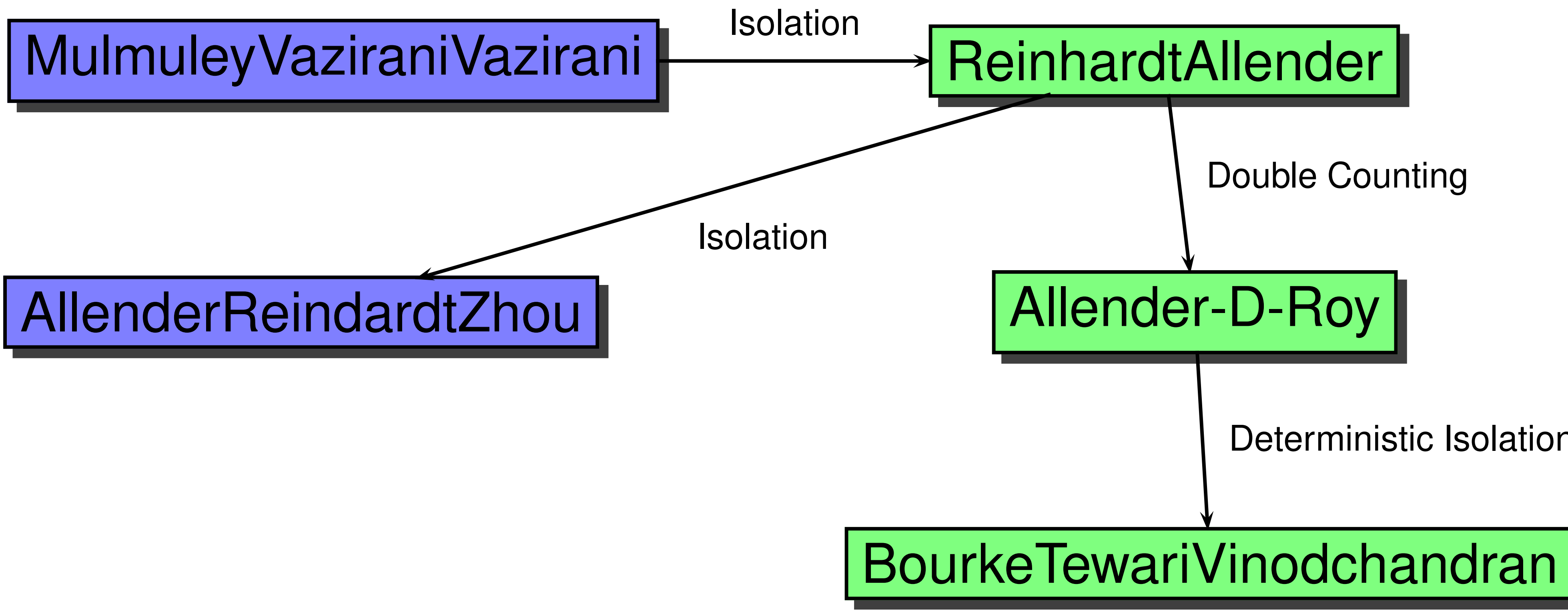

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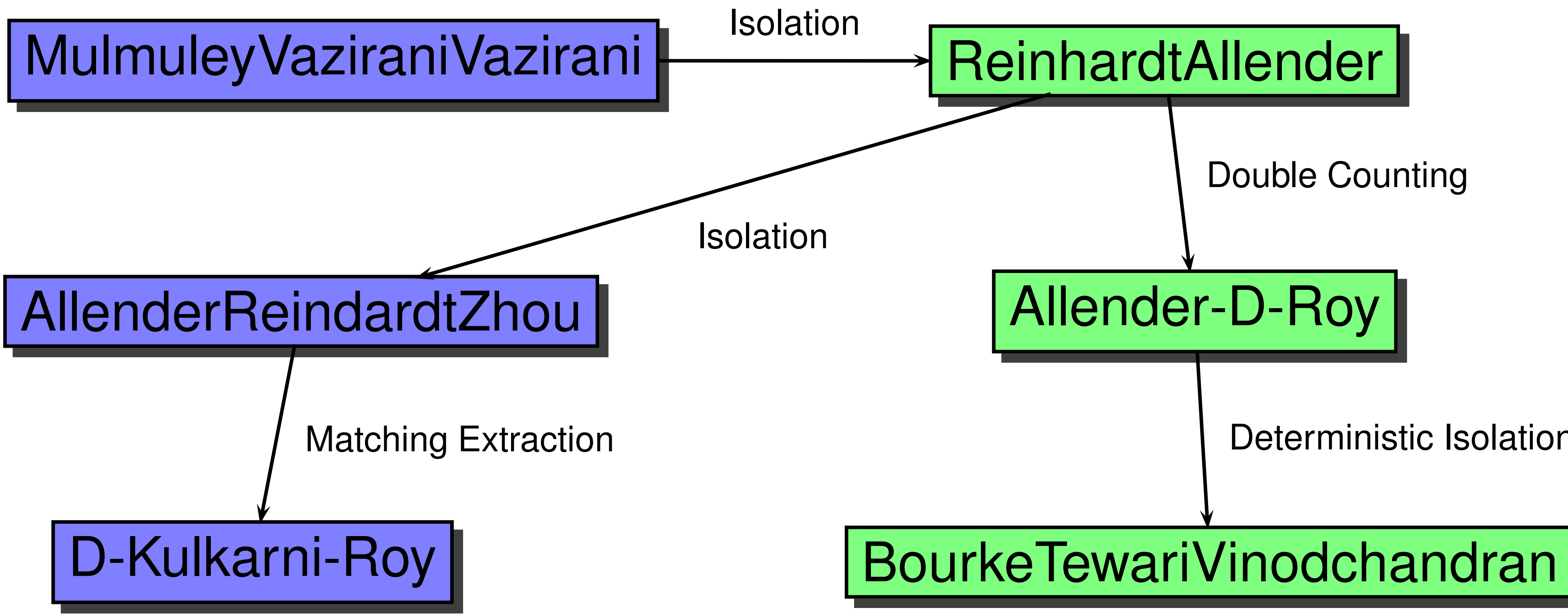

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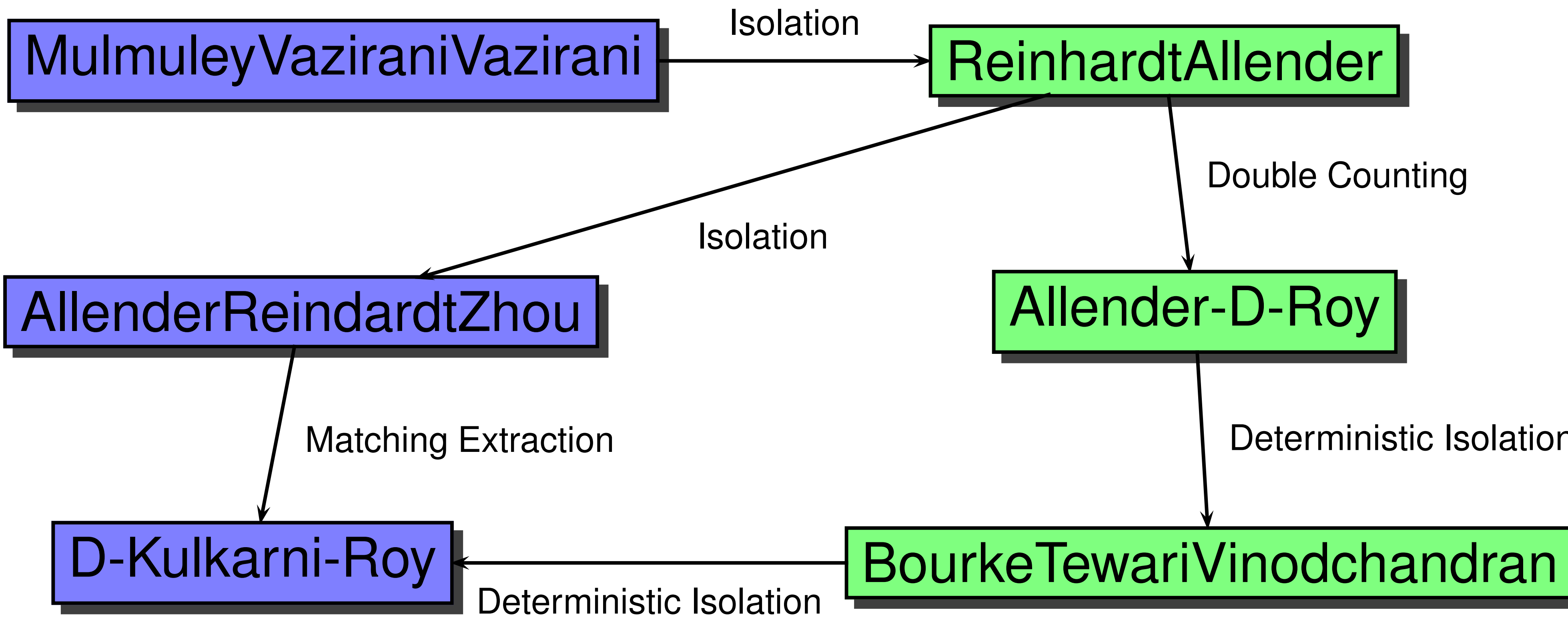

Thank You!